FORMULATING BASIC NOTIONS OF FINITE GROUP THEORY VIA THE LIFTING PROPERTY

by

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Abstract. — We reformulate several basic notions of notions in finite group theory in terms of iterations of the lifting property (orthogonality) with respect to particular morphisms. Our examples include the notions being nilpotent, solvable, perfect, torsion-free; p-groups and prime-to-p-groups; Fitting subgroup, perfect core, p-core, and prime-to-p core.

We also reformulate as in similar terms the conjecture that a localisation of a (transfinitely) nilpotent group is (transfinitely) nilpotent.

1. Introduction.

We observe that several standard elementary notions of finite group theory can be defined by iteratively applying the same diagram chasing "trick", namely the lifting property (orthogonality of morphisms), to simple classes of homomorphisms of finite groups.

The notions include a finite group being nilpotent, solvable, perfect, torsion-free; p-groups, and prime-to-p groups; p-core, the Fitting subgroup, cf.§2.2-2.3.

In §2.5 we reformulate as a labelled commutative diagram the conjecture that a localisation of a transfinitely nilpotent group is transfinitely nilpotent; this suggests a variety of related questions and is inspired by the conjecture of Farjoun that a localisation of a nilpotent group is nilpotent.

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This paper commemorates the centennial of the birth of N.A. Shanin, the teacher of S.Yu.Maslov and G.E.Mints, who was my teacher. I hope the motivation behind this paper is in spirit of the Shanin's group ТРЭПЛО (теоритическая разработка эвристического поиска логических обоснований, theoretical development of heuristic search for logical evidence/arguments).

The goal of this paper to present a collection of examples which show the lifting property is all that's needed to be able to define a number of notions from simplest (counter)examples of interest.

Curiously, our observations lead to a concise and uniform notation (Theorem 2.2, Corollary 2.3 and 2.5), e.g.

$$(\mathbb{Z}/p\mathbb{Z} \longrightarrow 0)^{rr}, \quad (AbKer)^{lr}, \quad \text{and} \quad (0 \longrightarrow *)^{lr}$$

denote the classes of homomorphisms (of finite groups) whose kernel is a p-group, resp. soluble, subgroup, and those corresponding to subnormal subgroups. One might hope that a notation so concise and uniform might be of use in computer algebra and automated theorem provers.

Deciphering this notation can be used as an elementary exercise in a first course of group theory or category theory on basic definitions and diagram chasing.

Such reformulations lead one to the following questions:

– Can one extend this notation to capture more of finite group theory?
– Is this a hint towards category theoretic point of view on finite group theory?

If one believes the evidence provided by our examples is strong enough to demand an explanation, then one should perhaps start by trying to find more examples defined in this way, and by calculating the classes of homomorphisms obtained by iteratively applying the Quillen lifting property to simple classes of morphisms of finite groups.

Motivation. — Our motivation was to formulate part of finite group theory in a form amenable to automated theorem proving while remaining human readable; [GP] tried to do the same thing for the basics of general topology.

Little attempt has been made to go beyond these examples. Hence open questions remain: are there other interesting examples of lifting properties in the category of (finite) groups? Can a complete group-theoretic argument be reformulated in terms of diagram chasing, say the classification of CA-groups or pq-groups, or elementary properties of subgroup series; can category theory notation be used to make expositions easier to read? Can these reformulations be used in automatic theorem proving? Is there a decidable fragment of (finite) group theory based on the Quillen lifting property and, more generally, diagram chasing, cf. [GLS]? Can the Sylow theorems (only existence and uniqueness of Sylow subgroups) be proven using this characterization of p-groups? Could the components of a finite group, and their properties (commute pairwise, commute with normal p-subgroups) be characterized and proven to exist with these methods?

2. Definitions and examples of reformulations

2.1. Key definition: the Quillen lifting property (negation/orthogonal).

— The Quillen lifting property, also known as orthogonality of morphisms, is a property of a pair of morphisms in a category. It appears in a prominent way in the theory of model categories, an axiomatic framework for homotopy theory introduced by Daniel Quillen, and is used to define properties of morphisms starting from an explicitly given class of morphisms.

Definition 2.1. — We say that two morphisms $A \xrightarrow{f} B$ and $X \xrightarrow{g} Y$ in a category C are *orthogonal* and write $f \rightthreetimes g$ iff for each $i : A \longrightarrow X$ and $j : B \longrightarrow Y$ making the square commutative, i.e. $f \circ j = i \circ g$ there is a diagonal arrow $\tilde{j} : B \longrightarrow X$ making the total diagram $A \xrightarrow{f} B \xrightarrow{\tilde{j}} X \xrightarrow{g} Y, A \xrightarrow{i} X, B \xrightarrow{j} Y$ commutative, i.e. $f \circ \tilde{j} = i$ and $\tilde{j} \circ g = j$ (see Figure 1a).

We may also say that f *lifts wrt* g, f *left-lifts wrt* g, or g *right-lifts wrt* f, or that f *antagonizes* g.

By analogy with orthogonal complement of a non-symmetric bilinear form, define *left/right Quillen negation* or *left/right Quillen orthogonal* of a class P of morphisms:

$$P^{\rightthreetimes l} := \{ f \ : \ f \rightthreetimes g \text{ for each } g \in P \}$$
$$P^{\rightthreetimes r} := \{ g \ : \ f \rightthreetimes g \text{ for each } g \in P \}$$

We have

$$P^{\rightthreetimes l} = P^{\rightthreetimes lrl},\ P^{\rightthreetimes r} = P^{\rightthreetimes rlr},\ P \subset P^{\rightthreetimes lr},\ P \subset P^{\rightthreetimes rl}$$
$$P \subset Q \text{ implies } Q^{\rightthreetimes l} \subset P^{\rightthreetimes l},\ Q^{\rightthreetimes r} \subset P^{\rightthreetimes r},\ P^{\rightthreetimes lr} \subset Q^{\rightthreetimes lr},\ P^{\rightthreetimes rl} \subset Q^{\rightthreetimes rl}$$
$$P \cap P^{\rightthreetimes l} \subset (Isom), \quad P \cap P^{\rightthreetimes r} \subset (Isom)$$

Under certain assumptions on the category and property P Quillen small object argument shows that each morphism $G \longrightarrow H$ decomposes both as

$$G \xrightarrow{(P)^{\rightthreetimes lr}} \cdot \xrightarrow{(P)^{\rightthreetimes r}} H \text{ and } G \xrightarrow{(P)^{\rightthreetimes r}} \cdot \xrightarrow{(P)^{\rightthreetimes lr}} H.$$

Using the Quillen lifting property is perhaps the simplest way to define a class of morphisms *without* a given property in a manner useful in a category theoretic diagram chasing computation.

2.2. A list of iterated Quillen negations of simple classes of morphisms.

— Let $(0 \longrightarrow *)$, resp. $(0 \longrightarrow Ab)$, denote the class of morphisms from the trivial groups to an arbitrary group, resp. Abelian group. Let $(* \longrightarrow 0)$, resp. $(Ab \longrightarrow 0)$, denote the class of morphisms to the trivial groups from an arbitrary group, resp. Abelian group. Let $(AbKer)$ denote the class of homomorphisms with an Abelian kernel.

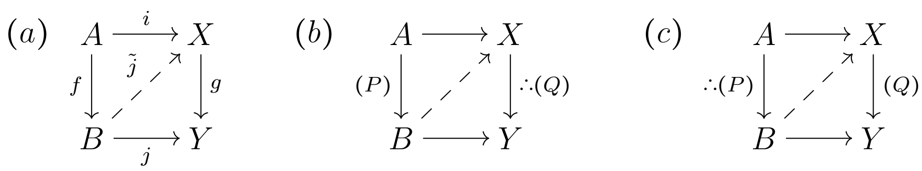

FIGURE 1. (a) The definition of a lifting property $f \rightthreetimes g$: for each $i : A \longrightarrow X$ and $j : B \longrightarrow Y$ making the square commutative, i.e. $f \circ j = i \circ g$, there is a diagonal arrow $\tilde{j} : B \longrightarrow X$ making the total diagram $A \xrightarrow{f} B \xrightarrow{\tilde{j}} X \xrightarrow{g} Y, A \xrightarrow{i} X, B \xrightarrow{j} Y$ commutative, i.e. $f \circ \tilde{j} = i$ and $\tilde{j} \circ g = j$. We say that f lifts wrt g, f left-lifts wrt g, or g right-lifts wrt f. (b) Right Quillen negation. The diagram defines a property Q of morphisms in terms of a property P; a morphism has property (label) Q iff it right-lifts wrt any morphism with property P, i.e. $Q = \{ q : p \rightthreetimes q \text{ for each } p \in P \}$ (c) Left Quillen negation. The diagram defines a property P of morphisms in terms of a property Q; a morphism has property (label) P iff it left-lifts wrt any morphism with property Q, i.e. $P = \{ p : p \rightthreetimes q \text{ for each } q \in Q \}$

Theorem 2.2. — *In the category of Finite Groups,*

1. $(AbKer)^{\rightthreetimes lr}$ *is the class of homomorphisms whose kernel is solvable*
2. $(0 \longrightarrow *)^{\rightthreetimes lr}$ *is the class of subnormal subgroups*
3. $(0 \longrightarrow Ab)^{\rightthreetimes lr} = (0 \longrightarrow Ab)^{\rightthreetimes lr} = \{[G,G] \longrightarrow G : G \text{ is arbitrary}\}^{\rightthreetimes lr}$ *is the class of subgroups* $H < G$ *such that there is a chain of subnormal subgroups* $H = G_0 \triangleleft G_1 \triangleleft \ldots \triangleleft G_n = G$ *such that* G_{i+1}/G_i *is Abelian, for* $i = 0, ..., n-1$.
4. $(0 \longrightarrow S)^{\rightthreetimes lr}$ *is the class of subgroups* $H < G$ *such that there is a chain of subnormal subgroups* $H = G_0 \triangleleft G_1 \triangleleft \ldots \triangleleft G_n = G$ *such that* G_{i+1}/G_i *embeds into* S, *for* $i = 0, ..., n-1$.
5. $(\mathbb{Z}/p\mathbb{Z} \longrightarrow 0)^{r}$ *is the class of homomorphisms whose kernel has no elements of order* p
6. $(\mathbb{Z}/p\mathbb{Z} \longrightarrow 0)^{\rightthreetimes rr}$ *is the class of surjective homomorphisms whose kernel is a* p*-group*

In the category of Groups,

1. $(* \longrightarrow 0)^{\rightthreetimes l}$ *is the class of retracts*
2. $(0 \longrightarrow *)^{\rightthreetimes r}$ *is the class of split homomorphisms*
3. $(0 \longrightarrow \mathbb{Z})^{r}$ *is the class of surjections*
4. $(\mathbb{Z} \longrightarrow 0)^{r}$ *is the class of injections*
5. *a group* F *is free iff* $0 \longrightarrow F$ *is in* $(0 \longrightarrow \mathbb{Z})^{rl}$
6. *a group* A *is Abelian iff* $A \longrightarrow 0$ *is in* $(\mathbb{F}_2 \longrightarrow \mathbb{Z} \times \mathbb{Z})^{\rightthreetimes r}$

7. *group G can be obtained from H by adding commutation relations, i.e. the kernel of $H \longrightarrow G$ is generated by commutators $[h_1, h_2]$, $h_1, h_2 \in H$, iff $H \longrightarrow G$ is in $(\mathbb{F}_2 \longrightarrow \mathbb{Z} \times \mathbb{Z})^{rl}$*
8. *subgroup H of G is a normal span of substitutions in words $w_1, .., w_i$ of the free group $\mathbb{F}_n$ iff $G \longrightarrow G/H$ is in $(\mathbb{F}_n \longrightarrow \mathbb{F}_n / < w_1, ..., w_i >)^{rl}$*
9. *$(AbKer)^{\rightthreetimes l}$ is the class of homomorphisms whose kernel is perfect*

Proof. — The proof is a matter of deciphering notation.

Proof of item 1. First note that $P \longrightarrow 0 \in (AbKer)^{\rightthreetimes l}$ for a perfect group, and $P \longrightarrow 0 \rightthreetimes H \longrightarrow G$ implies $\mathrm{Ker}\,(H \longrightarrow G)$ is soluble. This means that $(AbKer)^{\rightthreetimes lr}$ is contained in the class of maps whose kernel is soluble. On the other hand, any such map is a composition of maps $H/[S_n, S_n] \longrightarrow H/S_n$, ..., $H/[S_0, S_0] \longrightarrow H/S_0$, and $\mathrm{Im}\, H \longrightarrow G$ where $S_{n+1} = [S_n, S_n], S_0 = \mathrm{Ker}\,(H \longrightarrow G)$ is the descending derived series.

Now let us prove item 2. By definition, $A \longrightarrow B$ is in $(0 \longrightarrow *)^{\rightthreetimes l}$ iff $A \longrightarrow B \rightthreetimes 0 \longrightarrow G$ for any group G. Take $G = B/A^B$ to be the quotient of B by the normal closure of A, and $B \longrightarrow G$ to be the quotient map. This shows that if $G = B/A^B$ is non-trivial, then the lifting property fails. On the other hand, it is easy to check the lifting property holds that in a commutative square, the map to G factors via B/A^B, hence the lifting property holds if B/A^B is trivial.

Let $C \triangleleft D$ be a normal subgroup. The lifting property $A \longrightarrow B \rightthreetimes 0 \longrightarrow D/C$ implies $A \longrightarrow B \rightthreetimes C \longrightarrow D$. Orthogonals are necessarily closed under composition, hence this implies that if C is a subnormal subgroup of D, i.e. there exists a series if $C \triangleleft D_n \triangleleft D_{n-1} \triangleleft ... \triangleleft D_1 \triangleleft D$, then the lifting property holds and $C \longrightarrow D$ is in $(0 \longrightarrow *)^{lr}$.

Now assume that C is not subnormal in D and let $C < C'$ be a minimal subnormal subgroup of D containing C. Then $C \longrightarrow C'$ is in $(0 \longrightarrow *)^{\rightthreetimes l}$ and the lifting property $C \longrightarrow C' \rightthreetimes C \longrightarrow D$ fails, as required.

Items 5 and 6 use Cauchy theorem that a prime p divides the order of a group iff the group has an element of order p.

□

2.3. Concise reformulations in terms of iterated Quillen negation.

— We use the observations above to concisely reformulate several elementary notions in finite group theory.

Corollary 2.3. — *In the category of Finite Groups,*

1. *a finite group S is soluble iff either of the following equivalent conditions holds:*
   – $S \longrightarrow 0$ *is in* $(AbKer)^{\rightthreetimes lr}$
   – $0 \longrightarrow S$ *is in* $(0 \longrightarrow Ab)^{\rightthreetimes lr}$

2. *a finite group H is nilpotent iff*
   – *the diagonal map* $H \longrightarrow H \times H$, $x \mapsto (x,x)$, *is in* $(0 \longrightarrow *)^{\measuredangle lr}$
3. *the Fitting subgroup F of G is the largest subgroup such that*
   – *the diagonal map* $F \longrightarrow G \times G$, $x \mapsto (x,x)$, *is in* $(0 \longrightarrow *)^{\measuredangle lr}$
4. *a finite group H is a p-group iff one of the following equivalent conditions hold:*
   – $H \longrightarrow 0$ *is in* $(\mathbb{Z}/p\mathbb{Z} \longrightarrow 0)^{\measuredangle rr}$
   – $0 \longrightarrow H$ *is in* $(0 \longrightarrow \mathbb{Z}/p\mathbb{Z})^{\measuredangle lr}$

In the category of Groups,

1. *a group G is torsion-free iff iff* $0 \longrightarrow G$ *is in* $\{n\mathbb{Z} \longrightarrow \mathbb{Z} : n > 0\}^{\measuredangle r}$
2. *a subgroup* $H < G$ *contains torsion and is pure iff* $H \longrightarrow G$ *is in* $\{n\mathbb{Z} \longrightarrow \mathbb{Z} : n > 0\}^{\measuredangle r}$
3. *H is a verbal subgroup of G generated by substitutions in words* $w_1, .., w_i$ *in the free group* $\mathbb{F}_n$ *iff H fits into an exact sequence*
$$H \longrightarrow G \xrightarrow{(\mathbb{F}_n \longrightarrow \mathbb{F}_n/<w_1,..,w_i>)^{\measuredangle rl}} \cdot \xrightarrow{(\mathbb{F}_n \longrightarrow \mathbb{F}_n/<w_1,..,w_i>)^{\measuredangle r}} 0,$$
   or, equivalently, is the kernel of the corresponding homomorphism
4. *a free Burnside group of power n is a group B which fits into a decomposition of form*
$$0 \xrightarrow{(0 \longrightarrow \mathbb{Z})^{rl}} \cdot \xrightarrow{(\mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z})^{\measuredangle rl}} B \xrightarrow{(\mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z})^{\measuredangle r}} 0,$$
5. *a group S is transfinitely soluble, i.e. there exists an ordinal* α *such that* $G^\alpha = 0$, *where* $G^{\beta+1} = [G, G^\beta]$, *and* $G^\beta = \cap_{\gamma<\beta} G^\gamma$ *whenever* $\beta \neq \gamma + 1$ *for* $\gamma < \beta$, *iff*
   – $S \longrightarrow 0$ *is in* $(AbKer)^{\measuredangle lr}$
6. *a group G is transfinitely nilpotent, i.e. there exists an ordinal* α *such that* $G^\alpha = 0$, *where* $G^{\beta+1} = [G, G^\beta]$, *and* $G^\beta = \cap_{\gamma<\beta} G^\gamma$ *whenever* $\beta \neq \gamma + 1$ *for* $\gamma < \beta$, *iff*
   – *the diagonal map* $H \longrightarrow H \times H$, $x \mapsto (x,x)$, *is in* $(0 \longrightarrow *)^{\measuredangle lr}$

Corollary 2.4. — *The statement that a group of odd order is necessarily soluble is represented by either of the following inclusions*
$$(\mathbb{Z}/2\mathbb{Z} \longrightarrow 0)^{\measuredangle l} \subset (AbKer)^{\measuredangle lr}$$
$$(2\mathbb{Z} \longrightarrow \mathbb{Z})^{\measuredangle r} \cap (0 \longrightarrow *)^{\measuredangle lr} \subset (0 \longrightarrow \mathbb{Q}/\mathbb{Z})^{\measuredangle lr}$$
calculated in the category of Finite Groups.

2.4. p-, p'-, and p,p'-core as an example of a weak factorisation system. — Axiom M2 of a Quillen model category requires that each morphism $A \longrightarrow B$ decomposes as
$$A \xrightarrow{(c)} \cdot \xrightarrow{(f)} B$$

where (c) and (f) are orthogonal to each other. These decomposition give rise to weak factorisation systems whose existence is proven by the Quillen small object argument.

There are somewhat similar decompositions in group theory.

That "each group admits a surjection from a free group" can be denoted as follows; each morphism $0 \longrightarrow G$ admits a decomposition

$$0 \xrightarrow{(0\longrightarrow\mathbb{Z})^{rl}} \cdot \xrightarrow{(0\longrightarrow\mathbb{Z})^{r}} G$$

in this notation, we think of the Quillen orthogonals as *labels* put on arrows, hence the notation means that the homomorphisms belong to the corresponding Quillen orthogonals.

In a finite group, the descending derived series stabilises at a perfect subgroup $P = [P, P]$ (its perfect core) which is characteristic, corresponds to the unique decomposition of form

$$H \xrightarrow{(AbKer)^{\rightthreetimes l}} \cdot \xrightarrow{(AbKer)^{\rightthreetimes lr}} G$$

of a morphism into a map with a perfect kernel P, and a map with a soluble kernel.

Note these decompositions are analogous to decompositions appearing in weak factorisation systems proved by the Quillen small object argument.

Corollary 2.5 **(p-core, p'-core, p,p'-core).** — *In the category of Finite Groups,*

- *the p-core $O_p(G)$ of G, i.e. the largest normal p-subgroup of G, is the group appearing in the unique decomposition of form*

$$G \xrightarrow{(\mathbb{Z}/p\mathbb{Z}\longrightarrow 0)^{\rightthreetimes rr}} G/O_p(G) \xrightarrow{(\mathbb{Z}/p\mathbb{Z}\longrightarrow 0)^{\rightthreetimes rrl}} 0$$

- *the p'-core $O_{p'}(G)$ of G, i.e. the largest normal p'-subgroup of G, is the group appearing in the unique decomposition of form*

$$G \xrightarrow{(\mathbb{Z}/p\mathbb{Z}\longrightarrow 0)^{\rightthreetimes r}} G/O_{p'}(G) \xrightarrow{(\mathbb{Z}/p\mathbb{Z}\longrightarrow 0)^{\rightthreetimes rl}} 0$$

- *the p,p'-core $O_{p,p'}(G) = O_p(G/O_{p'}(G))$ of G is the group appearing in the unique decomposition of form*

$$G \xrightarrow{(\mathbb{Z}/p\mathbb{Z}\longrightarrow 0)^{\rightthreetimes r}} G/O_{p'}(G) \xrightarrow{(\mathbb{Z}/p\mathbb{Z}\longrightarrow 0)^{\rightthreetimes rr}} G/O_p(G/O_{p'}(G)) \xrightarrow{(\mathbb{Z}/p\mathbb{Z}\longrightarrow 0)^{\rightthreetimes rl}} 0$$

We end with a couple of test questions suggested by Bob Oliver (private communication); see §3.1 for some suggestions.

Question 2.6 **(Bob Oliver).** — *Can the Sylow theorems (only existence and uniqueness of Sylow subgroups) be proven using the characterization of p-groups by Corollary 2.3(4) ?*

Could the components of a finite group, and their properties (commute pairwise, commute with normal p-subgroups) be characterized and proven to exist with help of our reformulations?

2.5. f-local groups, localisations and nilpotent groups.— [1]

Let $f \rightthreetimes g$ denote the *unique* lifting property. For a morphism f of groups, a group A is called f-local iff $f \rightthreetimes A \longrightarrow 0$. Under some assumptions, each morphisms $H \xrightarrow{g} G$ of groups decomposes as

$$H \xrightarrow{(f)^{\rightthreetimes rl}} \cdot \xrightarrow{(f)^{\rightthreetimes r}} G$$

A diagram chasing argument shows that whenever such a decomposition always exists, there is a functor $L = L_f$: Groups $\longrightarrow$ Groups defined by

$$H \xrightarrow{(f)^{\rightthreetimes rl}} L(H) \xrightarrow{(f)^{\rightthreetimes r}} 0,$$

a natural transformation η : Id $\longrightarrow L$: Groups $\longrightarrow$ Groups which induces isomorphisms $\eta_G : LG \xrightarrow{(iso)} LLG$, $\eta_{LG} = L(G \xrightarrow{\eta_X} LG) : LG \longrightarrow LLG$. A functor with these data is called an idempotent monad or a *localisation*, and by [CSS] Vopenka principle implies that any localisation is of this form. See [AIP] for details and references.

Our notation allows to express a property closely related to the conjecture of Farjoun that the localisation of a nilpotent group is nilpotent, as follows; see [AIP] and references therein for a discussion of this conjectures.

Note the diagram has a symmetry: it mentions the diagonal map $H \longrightarrow H \times H$.

Conjecture 2.7 **(Farjoun).** — *The following diagram holds for any property (class) of homomorphisms L.*

$$\begin{array}{ccccc} H & \xrightarrow{(L)^{\rightthreetimes rl}} & H_L & \xrightarrow{(L)^{\rightthreetimes r}} & 0 \\ \downarrow{\scriptstyle (0\longrightarrow *)^{\rightthreetimes lr}} & & \downarrow{\scriptstyle \therefore(0\longrightarrow *)^{\rightthreetimes lr}} & & \downarrow \\ H \times H & \xrightarrow{(L)^{\rightthreetimes rl}} & H_L \times H_L & \xrightarrow{(L)^{\rightthreetimes r}} & 0 \end{array} \qquad \begin{array}{ccccc} H & \xrightarrow{(L)^{\rightthreetimes rl}} & H_L & \xrightarrow{(L)^{\rightthreetimes r}} & 0 \\ \downarrow{\scriptstyle (0\longrightarrow Ab)^{\rightthreetimes lr}} & & \downarrow{\scriptstyle \therefore(0\longrightarrow Ab)^{\rightthreetimes lr}} & & \downarrow \\ H \times H & \xrightarrow{(L)^{\rightthreetimes rl}} & H_L \times H_L & \xrightarrow{(L)^{\rightthreetimes r}} & 0 \end{array}$$

In the diagram above, "$\therefore$(label)" reads as: given a (valid) diagram whose arrows have properties indicate by their labels, the arrow marked by $\therefore$ has the property indicated by its label. See Fig. 1 and Corollary 2.3(2) for explanations and details.

Our notation suggests the following modifications of the conjecture.

[1] We thank S.O.Ivanov for pointing out the notion of f-local groups and the conjecture of Farjoun that a localisation of a nilpotent group is nilpotent [AIP].

Question 2.8. — *Does it hold for each morphism $H \longrightarrow G$ of groups and any homomorphism f:*

$$\begin{array}{ccccc}
H & \xrightarrow{(f)^{!\rightthreetimes rl}} & H_f & \xrightarrow{(f)^{!\rightthreetimes r}} & 0 \\
\downarrow{\scriptstyle (0\longrightarrow Ab)^{!\rightthreetimes lr}} & & \downarrow{\scriptstyle \therefore(0\longrightarrow Ab)^{!\rightthreetimes lr}} & & \downarrow \\
G & \xrightarrow{(f)^{!\rightthreetimes rl}} & G_f & \xrightarrow{(f)^{!\rightthreetimes r}} & 0
\end{array}$$

Question 2.9. — *Does it hold for any diagonal morphism $H \longrightarrow H \times H$ of groups, any properties (classes) L and P of homomorphisms:*

$$\begin{array}{ccccc}
H & \xrightarrow{(L)^{!\rightthreetimes rl}} & H_L & \xrightarrow{(L)^{!\rightthreetimes r}} & 0 \\
\downarrow{\scriptstyle (P)^{!\rightthreetimes lr}} & & \downarrow{\scriptstyle \therefore(P)^{!\rightthreetimes lr}} & & \downarrow \\
H & \xrightarrow{(L)^{!\rightthreetimes rl}} & H \times H_L & \xrightarrow{(L)^{!\rightthreetimes r}} & 0
\end{array}
\qquad
\begin{array}{ccccc}
H & \xrightarrow{(L)^{rl}} & H_L & \xrightarrow{(L)^{r}} & 0 \\
\downarrow{\scriptstyle (P)^{lr}} & & \vdots{\scriptstyle \exists(P)^{lr}} & & \downarrow \\
H & \xrightarrow{(L)^{rl}} & H \times H_L & \xrightarrow{(L)^{r}} & 0
\end{array}$$

Question 2.10. — *Under what assumptions on morphism $f : H \longrightarrow G$, properties L and P of homomorphisms it holds:*

$$\begin{array}{ccccc}
H & \xrightarrow{(L)^{!\rightthreetimes rl}} & H_L & \xrightarrow{(L)^{!\rightthreetimes r}} & 0 \\
\downarrow{\scriptstyle (P)^{!\rightthreetimes lr}} & & \downarrow{\scriptstyle \therefore(P)^{!\rightthreetimes lr}} & & \downarrow \\
G & \xrightarrow{(L)^{!\rightthreetimes rl}} & G_L & \xrightarrow{(P)^{!\rightthreetimes r}} & 0
\end{array}
\qquad
\begin{array}{ccccc}
H & \xrightarrow{(L)^{\rightthreetimes rl}} & H_L & \xrightarrow{(L)^{\rightthreetimes r}} & 0 \\
\downarrow{\scriptstyle (P)^{\rightthreetimes lr}} & & \vdots{\scriptstyle \exists(P)^{\rightthreetimes lr}} & & \downarrow \\
G & \xrightarrow{(L)^{\rightthreetimes rl}} & G_L & \xrightarrow{(P)^{\rightthreetimes r}} & 0
\end{array}$$

In an obvious way the notation suggests a large number of similar questions. The following is only an example, there is little motivation for this particular choice. We use this example as an opportunity to use shortened notation.

Question 2.11. — *Under what assumptions on properties Δ, L and P of homomorphisms it holds:*

$$\begin{array}{ccccc}
\cdot & \xrightarrow{(L)^{!\rightthreetimes r..rl}} & \cdot & \xrightarrow{(L)^{!\rightthreetimes r..r}} & \cdot \\
\downarrow{\scriptstyle (P)^{!\rightthreetimes l..lr}\,(\Delta)} & & \downarrow{\scriptstyle \therefore(P)^{!\rightthreetimes l..lr}} & & \downarrow{\scriptstyle (\Delta)} \\
\cdot & \xrightarrow{(L)^{!\rightthreetimes r..rl}} & \cdot & \xrightarrow{(P)^{!\rightthreetimes r..r}} & \cdot
\end{array}
\qquad
\begin{array}{ccccc}
\cdot & \xrightarrow{(L)^{\rightthreetimes r..rl}} & \cdot & \xrightarrow{(L)^{\rightthreetimes r..r}} & \cdot \\
\downarrow{\scriptstyle (P)^{\rightthreetimes l..lr}\,(\Delta)} & & \vdots{\scriptstyle \exists(P)^{\rightthreetimes l..lr}} & & \downarrow{\scriptstyle (\Delta)} \\
\cdot & \xrightarrow{(L)^{\rightthreetimes r..rl}} & \cdot & \xrightarrow{(P)^{\rightthreetimes r..r}} & \cdot
\end{array}$$

In particular, when does it hold for $\Delta = \{H \longrightarrow H \times H : H \text{ a group}\}$ the class of diagonal embeddings?

2.6. Reformulations with less notation.

— In this subsection in a verbose manner we decipher the notation of Quillen negation of the examples below. Fig. 3 represents considerations below as diagrams.

There is no non-trivial homomorphism from a group F to G, write $F \not\longrightarrow G$, iff

$$0 \longrightarrow F \rightthreetimes 0 \longrightarrow G \text{ or equivalently } F \longrightarrow 0 \rightthreetimes G \longrightarrow 0.$$

A group A is *Abelian* iff

$$\langle a,b\rangle \longrightarrow \langle a,b : ab = ba\rangle \rightthreetimes A \longrightarrow 0$$

where $\langle a,b\rangle \longrightarrow \langle a,b : ab = ba\rangle$ is the abelianisation morphism sending the free group into the Abelian free group on two generators; a group G is *perfect*, $G = [G,G]$, iff $G \not\rightthreetimes A$ for any Abelian group A, i.e.

$$\langle a,b\rangle \longrightarrow \langle a,b : ab = ba\rangle \rightthreetimes A \longrightarrow 0 \implies G \longrightarrow 0 \rightthreetimes A \longrightarrow 0$$

equivalently, for an arbitrary homomorphism g,

$$\langle a,b\rangle \longrightarrow \langle a,b : ab = ba\rangle \rightthreetimes g \implies G \longrightarrow 0 \rightthreetimes g$$

Yet another reformulation is that, for each group S,

$$0 \longrightarrow G \rightthreetimes [S,S] \longrightarrow S.$$

In the category of finite or algebraic groups, a group H is *soluble* iff $G \not\rightthreetimes H$ for each perfect group G, i.e.

$$0 \longrightarrow G \rightthreetimes 0 \longrightarrow H \text{ or equivalently } G \longrightarrow 0 \rightthreetimes H \longrightarrow 0.$$

Alternatively, a group H is *soluble* iff for every homomorphism f it holds

$$f \rightthreetimes [G,G] \longrightarrow G \text{ for each group } G \implies f \rightthreetimes 0 \longrightarrow H.$$

A prime number p does not divide the number elements of a finite group G iff G has no element of order p, i.e. no element $x \in G$ such that $x^p = 1_G$ yet $x^1 \neq 1_G, ..., x^{p-1} \neq 1_G$, equivalently $\mathbb{Z}/p\mathbb{Z} \not\rightthreetimes G$, i.e.

$$0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \rightthreetimes 0 \longrightarrow G \text{ or equivalently } \mathbb{Z}/p\mathbb{Z} \longrightarrow 0 \rightthreetimes G \longrightarrow 0.$$

A finite group G is a p-group, i.e. the number of its elements is a power of a prime number p, iff in the category of finite groups

$$0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \rightthreetimes 0 \longrightarrow H \implies 0 \longrightarrow H \rightthreetimes 0 \longrightarrow G.$$

A group H is the normal closure of the image of N, i.e. no proper normal subgroup of H contains the image of N, iff for an arbitrary group G

$$N \longrightarrow H \rightthreetimes 0 \longrightarrow G.$$

A group D is a subnormal subgroup of a finite group G iff

$$N \longrightarrow H \rightthreetimes 0 \longrightarrow B \text{ for each group } B \implies N \longrightarrow H \rightthreetimes D \longrightarrow G$$

i.e. $D \longrightarrow G$ right-lifts wrt any map $N \longrightarrow H$ such that H is the normal closure of the image of N; the lifting property implies that $D \longrightarrow G$ is injective. Recall that D is a subnormal subgroup of a finite group G iff there is a finite series of subgroups

$$D = G_0 \triangleleft G_1 \triangleleft \ldots \triangleleft G_n = G$$

such that G_i is normal in G_{i+1}, $i = 0, \ldots, n-1$. This is probably the only claim which requires a proof. First notice that if D is normal in G then the lifting property holds. Given a square corresponding to $N \longrightarrow H \rightthreetimes D \longrightarrow G$, the preimage of D in H is a normal subgroup of H containing the image of N, hence the preimage of D contains H and the lifting property holds. The lifting property is closed under composition, hence it holds for subnormal subgroups as well. Now assume D is not subnormal in G. As G is finite, there is a minimal subnormal subgroup $D' > D$ of G. By construction no proper normal subgroup of D' contains D but the lifting property $D \longrightarrow D' \rightthreetimes D \longrightarrow G$ fails.

Finally, a finite group G is nilpotent iff the diagonal group G is subnormal in $G \times G$ [Nilp], i.e. iff the diagonal map $G \xrightarrow{\Delta} G \times G$, $g \mapsto (g, g)$ right-lifts wrt any $N \longrightarrow H$ such that H is the normal closure of the image of N,

$$N \longrightarrow H \rightthreetimes 0 \longrightarrow B \text{ for each group } B \implies N \longrightarrow H \rightthreetimes G \xrightarrow{\Delta} G \times G.$$

3. Speculations. Extending the notation: Sylow subgroups and normalisers.

In this section we make some speculation and remarks on ways to extend our notation to capture the notions of *Sylow subgroup* and *normaliser* of a subgroup.

3.1. Sylow subgroups and diagrams commuting up to conjugations.—

It is useful to consider diagrams which commute *up to conjugation*. Inner automorphisms have the following properties which are useful in a diagram chasing computation, and which in fact characterise inner automorphisms among all automorphisms [Inn, Sch]:

- An inner automorphism of a group G extends along any group homomorphism $\iota : G \longrightarrow H$, i.e. for any $g \in G$, the inner automorphism $G \longrightarrow G, x \mapsto gxg^{-1}$ extends to an inner automorphism $H \longrightarrow H, x \mapsto \iota(g)x\iota(g^{-1})$
- An inner automorphism of a group G lifts along any surjective group homomorphism $\iota : G \longrightarrow H$, i.e. for any $g \in G$, the inner automorphism $G \longrightarrow G, x \mapsto gxg^{-1}$ extends to an inner automorphism $H \longrightarrow H$, $x \mapsto \iota^{-1}(g)x\iota^{-1}(g^{-1})$

***Definition 3.1* (orthogonal up to conjugation).** — Say that two morphisms $A \xrightarrow{f} B$ and $X \xrightarrow{g} Y$ in a category C are *orthogonal up to conjugation* and write $f \underline{\rightthreetimes} g$ iff for each $i : A \longrightarrow X$ and $j : B \longrightarrow Y$ making the square commutative, i.e. $f \circ j = i \circ g$, there is a diagonal arrow $\tilde{j} : B \longrightarrow X$ and an element $y \in Y$ making the total diagram $A \xrightarrow{f} B \xrightarrow{\tilde{j}} X \xrightarrow{ygy^{-1}} Y, A \xrightarrow{i} X, B \xrightarrow{j} Y$ commutative,

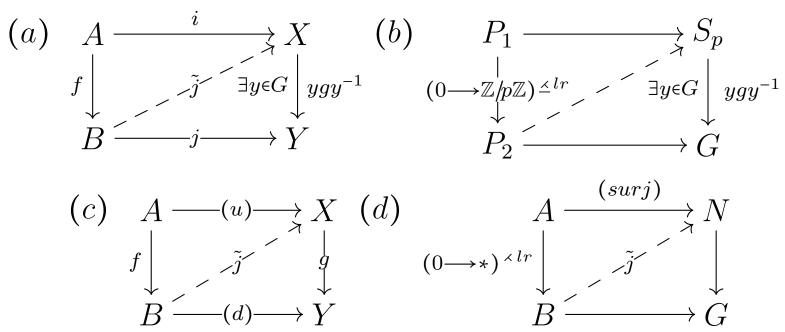

FIGURE 2. (a) The definition of a lifting property $f \preceq g$ *up to conjugation.*
(b) A corollary of Sylow theorem: any p-subgroup is contained in the Sylow subgroup S_p up to conjugation. To see this, take P_1 to be the trivial group, and note that P_2 in $(0 \longrightarrow \mathbb{Z}/p\mathbb{Z})^{\preceq lr}$ means P_2 is a p-group. To see that this property holds for the Sylow subgroup, note that $P_1 \longrightarrow P_2$ in $(0 \longrightarrow \mathbb{Z}/p\mathbb{Z})^{\preceq lr}$ implies there is a subgroup subgroup series with $\mathbb{Z}/p\mathbb{Z}$ quotients connecting P_1 and P_2, hence $Card\, P_2/Card\, P_1$ is a power of p, hence $Card\,\mathrm{Im}\, P_2$ is a power of p, hence maps to S_p up to conjugation.
(c) The definition of orthogonality with properties/labels.
(d) The diagram expresses that N is self-normalising in G. The diagram says that if A is subnormal in B, then any extension of a surjection of A on N to B is necessarily trivial, if the map takes value inside of G.

i.e. $f \circ \tilde{j} = i$ and $\tilde{j} \circ (ygy^{-1}) = j$ (see Figure 3a). Define *left/right Quillen negation* or *left/right Quillen orthogonal* $P^{\preceq l}$, $P^{\preceq r}$ *up to conjugation* in the obvious way.

Then the corollary of the Sylow theorem that there is a p-subgroup which contains any other p-subgroup up to conjugation, and such a p-subgroup is unique up to conjugation, can be expressed as: each morphism $0 \longrightarrow G$ decomposes as

$$0 \xrightarrow{(0\longrightarrow\mathbb{Z}/p\mathbb{Z})^{\preceq lr}} S_p \xrightarrow{(0\longrightarrow\mathbb{Z}/p\mathbb{Z})^{\preceq lrr}} G,$$

and such decomposition is unique up to conjugation.

3.2. Normalisers and orthogonality with properties.—

***Definition 3.2* (orthogonality with properties).** — Let (u) and (d) be two classes of morphisms; we think of them, and write them as, labels on arrows.

Say that two morphisms $A \xrightarrow{f} B$ and $X \xrightarrow{g} Y$ in a category C are $(u) \rightthreetimes (d)$-*orthogonal* and write $f((u)\underline{\rightthreetimes}(d))g$ iff for each $i : A \longrightarrow X$ with property (u) and $j : B \longrightarrow Y$ with properby (d) making the square commutative, i.e. $f \circ j = i \circ g$ there is a diagonal arrow $\tilde{j} : B \longrightarrow X$ making the total diagram $A \xrightarrow{f} B \xrightarrow{\tilde{j}} X \xrightarrow{g} Y, A \xrightarrow{i} X \xrightarrow{X}, B \xrightarrow{j} Y \xrightarrow{Y}$ commutative, i.e. $f \circ \tilde{j} = i$ and $\tilde{j} \circ g = j$ (see Figure 3a). Define *left/right Quillen negation* or *left/right Quillen orthogonal* $P^{((u)\rightthreetimes(d))l}$, $P^{((u)\rightthreetimes(d))r}$ *up to conjugation* in the obvious way.

A subgroup N of G is self-normalising iff

$$N \longrightarrow G \text{ is in } \left((0 \longrightarrow *)^{\rightthreetimes lr}\right)^{((surj)\rightthreetimes(all))r}$$

Acknowledgments and historical remarks.

This work is a continuation of [DMG]; early history is given there.

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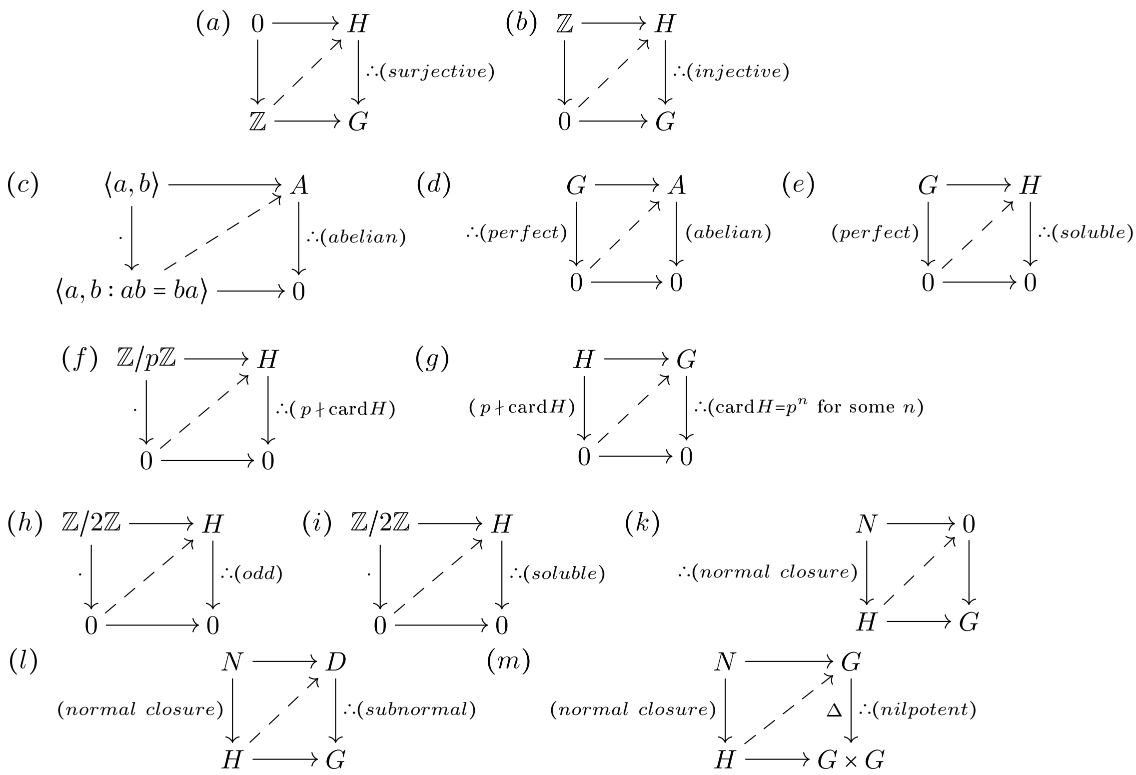

FIGURE 3. Lifting properties/Quillen negations. Dots ∴ indicate free variables. Recall these diagrams represent rules in a diagram chasing calculation and "∴(*label*)" reads as: given a (valid) diagram, add label (*label*) to the corresponding arrow. A diagram is valid iff for every commutative square of solid arrows with properties indicated by labels, there is a diagonal (dashed) arrow making the total diagram commutative. A single dot indicates that the morphism is a constant.
(a) a homomorphism $H \longrightarrow G$ is surjective, i.e. for each $g \in G$ there is $h \in H$ sent to g
(b) a homomorphism $H \longrightarrow G$ is injective, i.e. the kernel of $H \longrightarrow G$ is the trivial group
(c) a group is abelian iff each morphism from the free group of two generators factors through its abelianisation $\mathbb{Z} \times \mathbb{Z}$.
(d) a group G is perfect, $G = [G, G]$, iff it admits no non-trivial homomorphism to an abelian group
(e) a finite group is soluble iff it admits no non-trivial homomorphism from a perfect group; more generally, this is true in any category of groups with a good enough dimension theory.
(f) by Cauchy's theorem, a prime p divides the number of elements of a finite group G iff the group contains an element $e, e^p = 1, e \neq 1$ of order p
(f) a group has order p^n for some n iff iff the group contains no element $e, e^l = 1, e \neq 1$ of order l prime to p
(h) by Cauchy's theorem, a finite group has an odd number of elements iff it contains no involution $e, e^2 = 1, e \neq 1$
(i) The Feit-Thompson theorem says that each group of odd order is soluble, i.e. it says that this diagram chasing rule is valid in the category of finite groups. Note that it is not a definition of the label unlike the other lifting properties.
(k) a group H is the normal closure of the image of N iff $N \longrightarrow H \rlap{\,/}\times 0 \longrightarrow G$ for an arbitrary group G
(l) $D \longrightarrow G$ is injective and the subgroup D is a subnormal subgroup of a finite group G iff $D \longrightarrow G$ right-lifts wrt any map $N \longrightarrow H$ such that H is the normal closure of the image of N
(m) a group G is nilpotent iff the diagonal map $G \xrightarrow{\Delta} G \times G$, $g \mapsto (g, g)$ right-lifts wrt any inclusion of a subnormal subgroup $N \longrightarrow H$

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