SIMPLICIAL SETS WITH A NOTION OF SMALLNESS

preliminary notes on a draft of a research program

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Abstract. — We consider simplicial sets equipped with a notion of smallness, and observe that this slight "topological" extension of the "algebraic" simplicial language allows a concise reformulation of a number of classical notions in topology, e.g. continuity, limit of a map or a sequence along a filter, various notions of equicontinuity and uniform convergence of a sequence of functions; completeness and compactness; in algebraic topology, locally trivial bundles as a direct product after base-change and geometric realisation as a space of discontinuous paths.

In model theory, we observe that indiscernible sequences in a model form a simplicial set with a notion of smallness which can be seen as an analogue of the Stone space of types.

These reformulations are presented as a series of exercises, to emphasise their elementary nature and that they indeed can be used as exercises to make a student familiar with computations in basic simplicial and topological language. (Formally, we consider the category of simplicial objects in the category of filters in the sense of Bourbaki.)

This work is unfinished and is likely to remain such for a while, hence we release it as is, in the small hope that our reformulations may provide interesting examples of computations in basic simplicial and topological language on material familiar to a student in a first course of topology or category theory.

These preliminary notes are intended as an invitation to the topic, and are released in the hope of generating further activity on the subject.

Warning: Unfortunately, the notes are likely to contain misprints and perhaps mistakes. We hope the elementary nature of the material makes them easy to ignore. The notes are likely to remain in current state for a while. I will be grateful for corrections of mistakes and inaccuracies and generally help in proofreading but may not be in a position to make substantial changes. `mishap.sdf.org/6a6ywke/` Corrections to be sent to either here or `mishap@sdf.org` .

Contents

1. Introduction

We consider simplicial sets equipped with an additional *neighbourhood* or *being small enough* structure, which (paraphrasing [Bourbaki, General Topology, §Introduction] on topological structure) 'enables us to give a precise meaning to the phrase "such and such a property holds for all [simplices sufficiently small] (orig. points sufficiently near a)" : by definition this means that the set of [simplices] (orig. points) which have this property is a neighbourhood for the [neighbourhood] (orig. topological) structure in question.' In presence of a metric, a 'small' simplex would mean a simplex being wholly contained in a small ball.

We observe this leads to a notion (category) of spaces allowing a concise reformulation in category-theoretic terms of a number of classical notions in topology, e.g. continuity, limit of a map or a sequence along a filter, various notions of equicontinuity and uniform convergence of a sequence of functions; completeness and compactness; in algebraic topology, locally trivial bundles as a direct product after base-change, and geometric realisation as a space of discontinuous paths.

The notion of smallness on a simplicial set allows to discuss topology as follows. In a topological or metric space X, a point x is near a point

y iff the simplex $(x, y) \in E \times E$ is small in the simplicial set of Cartesian powers of X equipped with an appropriate smallness structure..

Formally, we consider the category of simplicial objects in the category of filters in the sense of Bourbaki, and the intuitive description above is formalised as follows. In a topological space X, a neighbourhood of a point x consists of all points y such that "the simplex (x, y) is ε-small" for some ε a neighbourhood according to the filter on $E \times E$. Yet more formally, $U \ni x$ is a neighbourhood iff there is a subset $\varepsilon \subset E \times E$ big according to the filter on 1-simplicies $E \times E$ such that $y \in U$ whenever $(x, y) \in \varepsilon$.

For a metric space X, the filters on the Cartesian powers can be described explicitly as follows: a subset $\varepsilon \subset E \times ... \times E$ is big, or a neighbourhood, iff there is $\epsilon > 0$ such that a tuple belongs to ε whenever it consists of points at distance at most ϵ apart.

The structure of the paper is as follows. In §2 we define the category ɀφ and give a number of reformulations. Its purpose is give the reader a feeling of expressive power of ɀφ. Notably, following the approach of [Besser],[Grayson],[Drinfeld], we show(1) that the geometric realisation of a simplicial set can be interpreted as an mapping space of discontinuous paths. In §3 we discuss the intuition behind the new notion of space. In §3.1 in a verbose manner we argue that the description by [Bourbaki, Introduction] of the intuition of (basic general) topology transfers to ɀφ almost verbatim. In §3.2 we offer several vague speculations about intuition of algebraic topology. In §3.3 we say that ɀφ has objects originating in Ramsey theory and model theory.

In §4 we mostly repeat §2 somewhat more formally and with more details. There we present our reformulations as a series of exercises, to emphasise their elementary nature and that they indeed can be used as exercises to make a student familiar with computations in basic simplicial and topological language.

We end in §5 by a discussion of Ramsey theory and indiscernibles in model theory.

2. A sample of definitions and reformulations

2.1. Constructions in general topology. —

(1)We stress, again, the preliminary nature of these notes

2.1.1. The definition of the category of simplicial sets with a notion of smallness. — Here we state the key definition of the paper. In this section we introduce notation as we go; see §4.1.2 for explanation if necessary. Here we just note that a version of this paper uses type-theory notation for Hom-sets, which the author finds more readable for nested formulae: in a category C, the set of all maps from an object X to Y is denoted by either $\left\{X \underset{C}{\rightarrow} Y\right\}$ or $\mathrm{Hom}_C(X,Y)$, and the space of maps from X to Y, usually an object of C, if defined, is denoted by either $\left\{X \underset{C}{\Rightarrow} Y\right\}$ or $\underline{\mathrm{Hom}}_C(X,Y)$.

The definition of our main category uses the following definition [Bourbaki, I§6.1, Def.I] which is given more formally later in Definition 4.1.

Definition I. ***A filter on a set*** **X** ***is a set*** $\mathfrak{F}$ ***of subsets of*** **X** ***which has the following properties:***

(F$_{\mathrm{I}}$) ***Every subset of*** **X** ***which contains a set of*** $\mathfrak{F}$ ***belongs to*** $\mathfrak{F}$.

(F$_{\mathrm{II}}$) ***Every finite intersection of sets of*** $\mathfrak{F}$ ***belongs to*** $\mathfrak{F}$.

(F$_{\mathrm{III}}$) ***The empty set is not in*** $\mathfrak{F}$.

Subsets in $\mathfrak{F}$ are called *neighbourhoods* or $\mathfrak{F}$*-big*. Unlike [Bourbaki], we do *not* require $(\mathrm{F_{III}})$ and allow both $X = \varnothing$ and $\varnothing \in \mathfrak{F}$; necessarily $X \in \mathfrak{F}$.

A *morphism of filters* is a function of underlying sets such that the preimage of a neighbourhood is necessarily a neighbourhood; we call such maps of filters *continuous*.

Let $\boldsymbol{\varphi}$ denote the category of filters.

***Definition 2.1.1.1* (Simplicial filters $\boldsymbol{\iota\varphi}$).** — Let $\boldsymbol{\iota\varphi} = Func(\Delta^{\mathrm{op}}, \boldsymbol{\varphi})$ be the category of functors from Δ^{op}, the category opposite to the category Δ of finite linear orders, to the category $\boldsymbol{\varphi}$ of filters. We refer to its objects as either *simplicial filters*, *simplicial neighbourhoods*, or *situses*, for lack of a good name.

2.1.2. Neighbourhood structures associated with a metric space. — See 4.2.1 for a precise definition.

With the set M of points of a metric space associate the simplicial sets represented by M:

$$\Delta^{\mathrm{op}} \longrightarrow Sets,\ n \longmapsto \mathrm{Hom}_{Sets}(n, M) = M^n$$

$$\Delta^{\mathrm{op}} \longrightarrow Sets,\ n \longmapsto \mathrm{Hom}_{Sets}(n+1, M) = M \times M^n$$

Call a subset $\varepsilon \subset M^n = \mathrm{Hom}_{Sets}(n, M)$ a *neighbourhood (of the diagonal)* iff there is $\epsilon > 0$ such that $(x_1, .., x_n) \in \varepsilon$ whenever $\mathrm{dist}(x_i, x_j) < \epsilon$ for all

$0 < i < j \leq n$. In this way we associate two simplicial neighbourhood with a metric space denoted by $M\sigma\eta$ and $M\sigma\eta[+1]$, resp. There is an $\zeta\varphi$-morphism $[-1] : M\sigma\eta[+1] \longrightarrow M\sigma\eta$ projecting each $M \times M^n$ to M^n.

A $\zeta\varphi$-morphism $f : L\sigma\eta \longrightarrow M\sigma\eta$ is a uniformly continuous map $L \longrightarrow M$.

2.1.3. Limits and $[+1]$ ==> $id : \Delta^{\mathrm{op}} \longrightarrow \Delta^{\mathrm{op}}$. *Uniform convergence.*— Let $\mathbb{N}_{\mathrm{cof}}$ be the filter of cofinite subsets, and let $\left(n \mapsto \mathbb{N}^n_{\mathrm{cof}}\right)$ denote the simplicial object $\Delta^{\mathrm{op}} \longrightarrow \varphi$, $n_\leq \mapsto \mathbb{N}^n_{\mathrm{cof}}$ of Cartesian powers of $\mathbb{N}_{\mathrm{cof}}$. A *Cauchy sequence* $(a_n)_{n\in\mathbb{N}}$ in a metric space M is a morphism $\bar{a} : \left(n \mapsto \mathbb{N}^n_{\mathrm{cof}}\right) \longrightarrow M\sigma\eta$, $(i_1, .., i_n) \mapsto (a_{i_1}, ..., a_{i_n})$: for each $\epsilon > 0$ the preimage of $\varepsilon := \{(x, y) : \mathrm{dist}(x, y) < \epsilon\}$ contains $\delta := \{(n, m) : n, m > N\}\}$ for some N large enough, i.e. $\mathrm{dist}(a_n, a_m) < \epsilon$ for $n, m > N$. The sequence $(a_n)_{n\in\mathbb{N}}$ *converges* iff the morphism $\bar{a} : \left(n \mapsto \mathbb{N}^n_{\mathrm{cof}}\right) \longrightarrow M\sigma\eta$ factors as $\bar{a} : \left(n \mapsto \mathbb{N}^n_{\mathrm{cof}}\right) \xrightarrow{\bar{a}_\infty} M\sigma\eta[+1] \xrightarrow{[-1]} M\sigma\eta$. Moreover, the morphism $\bar{a}_\infty : \left(n \mapsto \mathbb{N}^n_{\mathrm{cof}}\right) \xrightarrow{\bar{a}_\infty} M\sigma\eta[+1]$ is necessarily of form $M^n \longrightarrow M^{n+1}$, $(i_0, i_1, ..., i_n) \mapsto (a_\infty, a_{i_1}, ..., a_{i_n})$ where a_∞ is the *limit* of sequence $(a_n)_{n\in\mathbb{N}}$.

To see this, first note that the underlying sset of $M\sigma\eta[+1]$ is a disjoint union $M\sigma\eta[+1] = \sqcup_{a\in M}\{a\}\times M\sigma\eta$ of copies of $M\sigma\eta$, and that the underlying sset of $\left(n \mapsto \mathbb{N}^n_{\mathrm{cof}}\right)$ is connected. Hence, to pick a factorisation of the underlying ssets is to pick an $a \in M$. Now, the map $\left(n \mapsto \mathbb{N}^n_{\mathrm{cof}}\right) \longrightarrow \{a\}\times M\sigma\eta$ is continuous iff for each $\epsilon > 0$ the preimage of $\varepsilon := \{(a, x) : \mathrm{dist}(a, x) < \epsilon\}$ contains $\delta := \{(n, m) : n, m > N\}$, i.e. for $m > N$ $\mathrm{dist}(a, a_m) < \epsilon$.

A *uniformly continuous function* $f : L \longrightarrow M$ is a morphism $L\sigma\eta \longrightarrow M$, cf. §4.11.3. Indeed, for every $\epsilon > 0$ the preimage of $\varepsilon := \{(u, v) : \mathrm{dist}(u, v) < \epsilon\} \subset M \times M$ contains $\mathbf{\delta} := \{(x, y) : \mathrm{dist}(x, y) < \delta\} \subset L \times L$ for some $\delta > 0$, i.e. $\mathrm{dist}(f(x), f(y)) < \varepsilon$ whenever $\mathrm{dist}(x, y) < \delta\}$.

A *uniformly equicontinuous sequence* $(f_i : L \longrightarrow M)_{i\in\mathbb{N}}$ *of uniformly continuous functions* is a morphism $(n \mapsto \mathbb{N}_{\mathrm{cof}}) \times L\sigma\eta \longrightarrow M\sigma\eta$,

$$\mathbb{N} \times L^n \longrightarrow M^n,\ (i, x_1, ..., x_n) \mapsto (f_i(x_1), ..., f_i(x_n)),$$

or, equivalently, is a morphism $\left(n \mapsto (\mathbb{N}^n_{\mathrm{cof}})_{\mathrm{diag}}\right) \times L\sigma\eta \longrightarrow M\sigma\eta$,

$$\mathbb{N}^n \times L^n \longrightarrow M^n,\ (i_1, ..., i_n, x_1, ..., x_n) \mapsto (f_{i_1}(x_1), ..., f_{i_n}(x_n))$$

where $\left(n \mapsto (\mathbb{N}^n_{\mathrm{cof}})_{\mathrm{diag}}\right)$ denotes the simplicial object $\Delta^{\mathrm{op}} \longrightarrow \varphi$, $n_\leq \mapsto (\mathbb{N}^n_{\mathrm{cof}})_{\mathrm{diag}}$ of Cartesian powers equipped with "the filter of cofinite diagonals", i.e. a subset of $\mathbb{N}^n$ is a neighbourhood of $(\mathbb{N}^n_{\mathrm{cof}})_{\mathrm{diag}}$ iff it contains the set $\{(i, i, .., i) : i > N\}$ for some $N > 0$.

Indeed, for every $\epsilon > 0$ the preimage of $\varepsilon := \{(u,v) : \mathrm{dist}(u,v) < \epsilon\} \subset M \times M$ contains $\text{δ} := \{(n,x,y) : n > N, \mathrm{dist}(x,y) < \delta\} \subset \mathbb{N}\times\mathbb{N}\times L\times L$, resp., $\text{δ} := \{(n,n,x,y) : n > N, \mathrm{dist}(x,y) < \delta\} \subset \mathbb{N}\times\mathbb{N}\times L\times L$, for some $\delta > 0$ and $N > 0$, i.e. $\mathrm{dist}(f_n(x), f_n(y)) < \varepsilon$ whenever $n > N$ and $\mathrm{dist}(x,y) < \delta$.

A *uniformly Cauchy sequence* $(f_i : L \longrightarrow M)_{i\in\mathbb{N}}$ *of uniformly continuous functions* is a morphism $\big(n \mapsto \mathbb{N}^n_{\mathrm{cof}}\big) \times L_{\sigma\eta} \longrightarrow M_{\sigma\eta}$,

$$\mathbb{N}^n \times L^n \longrightarrow M^n,\ (i_1, ..., i_n, x_1, ..., x_n) \mapsto (f_{i_1}(x_1), ..., f_{i_n}(x_n)).$$

Indeed, for every $\epsilon > 0$ the preimage of $\varepsilon := \{(u,v) : \mathrm{dist}(u,v) < \epsilon\} \subset M \times M$ contains $\text{δ} := \{(n,m,x,y) : n,m > N, \mathrm{dist}(x,y) < \delta\} \subset \mathbb{N}\times\mathbb{N}\times L\times L$ for some $\delta > 0$ and $N > 0$, i.e. $\mathrm{dist}(f_n(x), f_m(y)) < \varepsilon$ whenever $n,m > N$ and $\mathrm{dist}(x,y) < \delta$.

The uniformly equicontinuous sequence $(f_i : L \longrightarrow M)_{i\in\mathbb{N}}$ *uniformly converges to a uniformly continuous function* $f_\infty : L \longrightarrow M$ iff this morphism "lifts by $[-1]$", i.e. fits into a commutative diagram

$$\begin{array}{ccc} L_{\sigma\eta}[+1] \times \big(n \mapsto (\mathbb{N}^n_{\mathrm{cof}})_{\mathrm{diag}}\big) & \dashrightarrow & M_{\sigma\eta}[+1] \\ \downarrow{\scriptstyle [-1]\times\mathrm{id}} & & \downarrow{\scriptstyle [-1]\times\mathrm{id}} \\ L_{\sigma\eta} \times \big(n \mapsto (\mathbb{N}^n_{\mathrm{cof}})_{\mathrm{diag}}\big) & \xrightarrow{(f_1, f_2, \dots)} & M_{\sigma\eta} \end{array}$$

where the top row morphism is, necessarily, of form

$$\mathbb{N}^n \times L^{n+1} \longrightarrow M^{n+1},\ (i_1, ..., i_n, x_0, x_1, ..., x_n) \mapsto (\,f_\infty(x_0), f_{i_1}(x_1), ..., f_{i_n}(x_n)\,)$$

To see this, use that $(n \mapsto (\mathbb{N}^n_{\mathrm{cof}})_{\mathrm{diag}})$ is connected and therefore maps into a connected component of $M_{\sigma\eta}[+1]$.

2.1.4. Complete metric spaces as a lifting property. — A metric space M is *complete* iff every Cauchy sequence converges, i.e. the following lifting property[2] holds:

$$\varnothing \longrightarrow (n \mapsto \mathbb{N}^n_{\mathrm{cof}}) \rightthreetimes M_{\sigma\eta}[+1] \longrightarrow M_{\sigma\eta}$$

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[2] A morphism $i : A \to B$ in a category has *the left lifting property* with respect to a morphism $p : X \to Y$, and $p : X \to Y$ also has *the right lifting property* with respect to $i : A \to B$, denoted $i \rightthreetimes p$, iff for each $f : A \to X$ and $g : B \to Y$ such that $p \circ f = g \circ i$ there exists $h : B \to X$ such that $h \circ i = f$ and $p \circ h = g$. This notion is used to define properties of morphisms starting from an explicitly given class of morphisms, often a list of (counter)examples, and a useful intuition is to think that the property of left-lifting against a class C is a kind of negation of the property of being in C, and that right-lifting is also a kind of negation.

or, in another notation,(3)

$\varnothing \longrightarrow (n \mapsto \mathbb{N}^n_{\mathrm{cof}}) \in \{M\sigma\eta[+1] \longrightarrow M\sigma\eta \,:\, M \text{ is a complete metric space}\}^{\mathrm{l}}$

Hence, $\varnothing \longrightarrow \big(n \mapsto \mathbb{N}^n_{\mathrm{cof}}\big) \in \{\mathbb{R}\sigma\eta[+1] \longrightarrow \mathbb{R}\sigma\eta\}^{\mathrm{l}}$, and therefore

$M\sigma\eta[+1] \longrightarrow M\sigma\eta \in \{\mathbb{R}\sigma\eta[+1] \longrightarrow \mathbb{R}\sigma\eta\}^{\mathrm{lr}}$ implies M is complete,

and a little argument shows the converse holds for precompact metric spaces.

Compactness can also be reformulated as a lifting property, see §4.11 for this and other examples.

2.2. Elementary constructions in homotopy theory. —

2.2.1. The unit interval. — With the unit interval $[0,1]$ associate the simplicial set

$$\Delta^{\mathrm{op}} \longrightarrow Sets,\ n_{\leq} \longmapsto \mathrm{Hom}_{preorders}(n_{\leq}, [0,1]_{\leq})$$

Equip it with a neighbourhood structure using the metric: $\varepsilon \subset \mathrm{Hom}_{preorders}(n_{\leq}, [0,1]_{\leq})$ is a neighbourhood iff there is $\epsilon > 0$ such that $(t_1 \leq ... \leq t_n) \in \varepsilon$ whenever $t_n < t_1 + \epsilon$. This neighbourhood structure can be defined entirely in terms of the simplicial set itself, cf. §4.4 for details: $\varepsilon \subset \mathrm{Hom}_{preorders}(n_{\leq}, [0,1]_{\leq})$ is a neighbourhood iff for any $\tau > 0$ there is $T > \tau > n$ and a simplex $s = (s_1 \leq .. \leq s_T)$ such that $t[i_1 \leq .. \leq i_n] = (t_{i_1} \leq .. \leq t_{i_n}) \in \varepsilon$ whenever $T' > 0$, the simplex s is a face of a simplex $t = (t_1 < t_2 < .. < t_{T'})$ and $i_1 \leq .. \leq i_n < i_1 + \tau$. Denote this simplicial neighbourhood by $[0.1]_{\leq}$.

A *path* $\gamma : [0,1] \longrightarrow M$ in a metric space M is same as a morphism $[0,1]_{\leq} \longrightarrow M\sigma\eta$. An automorphism $[0,1]_{\leq} \longrightarrow [0,1]_{\leq}$ is a non-decreasing (necessarily uniformly) continuous automorphism $[0,1] \longrightarrow [0,1]$ of the unit interval.

2.2.2. Simplicies as ε-discretised homotopies.— A map f is homotopic to a map g iff there is a sequence $f = f_0,\ f_1,\ ...,\ f_t, ..., f_T = g$ where f_t is as near as we please to f_{t+1}, $0 \leq t \leq t+1 \leq T$. In $\boldsymbol{\sigma\eta}$ this is readily formalised by saying that the simplex (f,g) is a face of simplex $\overrightarrow{f}$ with

(3)Denote by P^{l} and P^{r} the classes (properties) of morphisms having the left, resp. right, lifting property with respect to all morphisms with property P:

$$P^{\mathrm{l}} := \{f \rightthreetimes g : g \in P\} \qquad P^{\mathrm{r}} := \{f \rightthreetimes g : f \in P\}$$

It is convenient to refer to P^{l} and P^{r} as the property of *left, resp. right, Quillen negation of property P.*

consecutive faces as small as we please, i.e. for each neighbourhood ε in the set of 1-simplices there is a simplex $\overrightarrow{f} = \overrightarrow{f_\varepsilon}$ in the space of maps from X to Y such that $(f,g) = \overrightarrow{f}[0 \le T]$ for some $t \ge 0$, and $\overrightarrow{f}[t \le t+1] \in \varepsilon$ for $0 \le t < T = \dim \overrightarrow{f}$; here $\overrightarrow{f}[0 \le T]$ denotes the face of simplex $\overrightarrow{f}$ corresponding to Δ-morphism $0 \le T : 2_\le \longrightarrow T_\le$, $0 \mapsto 0$, $1 \mapsto T$, and similarly for $\overrightarrow{f}[t \le t+1]$. This formalisation immediately suggests we should let ε vary among neighbourhoods of arbitrary dimension T' and rather require that $\overrightarrow{f} = \overrightarrow{f_{\varepsilon,n}}$ and $\overrightarrow{f}[t_0 \le t_1 \le ... \le t_{T'}] \in \varepsilon$ whenever $t_{T'} \le t_0 + n$ (where $\varepsilon \subset X_{T'}$ and $n > 0$).

This leads to the following definition.

For a neighbourhood $\varepsilon \subset M_T$ and $n > 0$, a simplex $s : M_{T'}$ is *ε/n-fine* iff $s[t_0 \le ... \le t_k] \in \varepsilon$ whenever $0 \le t_0 \le ... \le t_k \le t_0 + n \le T'$. A simplex s is *Archimedean* iff it can be split into finitely many arbitrarily small parts, i.e. is a face of some ε/n-fine simplex for every neighbourhood $\varepsilon \subset X_k$ and every $T, n > 0$.[(4)] For example, a pair of points $(x,y) \in M \times M$ in a metric space M is an Archimedean simplex in $M_{\mathfrak{M}}$ iff for each $\epsilon > 0$ there is an *ϵ-discretised homotopy* $x = x_0, x_1, ..., x_l = y$, $\operatorname{dist}(x_t, x_{t+1}) < \epsilon$ for $0 \le t < l$, *from* $x = (x,y)[0]$ *to* $y = (x,y)[1]$.

Archimedean simplices of a simplicial filter $X : \mathfrak{sF}$ form a subobject (subfunctor) X_{Arch}, as the definition is invariant.

A well-known lemma says that two functions $f, g : A \longrightarrow M$ from an arbitrary topological space A to a metric space M are homotopic iff there is a ϵ-discretised homotopy $f = f_2, ..., f_n = g$ such that for any $x \in A$ $\operatorname{dist}(f_t(x), f_{t+1}(x)) < \epsilon$, under some assumptions on the metric space M; it is enough to assume that for every $\epsilon > 0$ there is $\delta > 0$ such that every ϵ-ball contains a contractible δ-ball. We reformulate this by saying that two functions $f, g : A \longrightarrow M$ are homotopic iff (f,g) is an Archimedean simplex of the mapping space $Func(A, M)_{\mathfrak{M}}$ with the sup-metric, or, equivalently, a 1-simplex of $(Func(A, M)_{\mathfrak{M}})_{\text{Arch}}$.

2.2.3. Topological spaces as simplicial filters. — See §3.1 for the intuition and §4.2.1 for a precise definition.

As with metric spaces, with the set X of points of a topological space associate the simplicial set

$$\Delta^{\text{op}} \longrightarrow Sets, \; n_\le \longmapsto \operatorname{Hom}_{Sets}(n, X) = X^n$$

(4) This definition applies to any object of $\mathfrak{sF}$ but should likely be modified even for the metric spaces. For explanation see the footnote in §4.6.1.

Define the filters of neighbourhoods (of the diagonal) as follows. The filter on X is antidiscrete, as X is the diagonal of itself and thus every neighbourhood has to contain X. A subset $\varepsilon \subset X \times X$ is a neighbourhood iff ε contains a set of the form

$$\bigcup_{x \in X} \{x\} \times U_x$$

where $U_x \ni x$ is an open neighbourhood of x. The filter on X^n, $n > 2$, is the coarsest filter compatible with all the face maps $X^n \longrightarrow X \times X$. Let $X_{\mathbb{T}}$ denote the simplicial neighbourhood obtained in this way.

2.2.4. A forgetful functor to topological spaces. — The embedding of topological spaces admits an inverse $-_{\mathbb{T}^{-1}} : \text{ɀ}\boldsymbol{\varphi} \longrightarrow \mathrm{Top}$ defined similarly to the definition of the topology associated with a uniform structure [Bourbaki,II§1.2,Prop.1,Def.3], as follows.

The set of points of $X_{\mathbb{T}^{-1}}$ is the set of points which are ε-small for each neighbourhood $\varepsilon \subset X_0$, i.e. $X_{\text{points}} := \bigcap_{\varepsilon \subset X_0 \text{ is a neighbourhood}} \varepsilon$. The topology is generated by the subsets that together with each point contain all ε-near points for some $\varepsilon \subset X_1$, i.e. subsets U with the following property: $U_x \in x$ is a neighbourhood iff there is a neighbourhood $\varepsilon \subset X_1$ such that $y \in U_x$ whenever $y \in X_{\text{points}}$ and $(x, y) \in \varepsilon$ or $(y, x) \in \varepsilon$.

It is easy to check that for a topological space X, $(X_{\mathbb{T}})_{\mathbb{T}^{-1}} = X$, and that $([0.1]_{\leq})_{\mathbb{T}^{-1}} = [0, 1]$ as a topological space.

2.2.5. Locally trivial bundles. — Let X, B, F be topological spaces. A map $X \xrightarrow{p} B$ is locally trivial iff it becomes a direct product after pull back to the “local base” $[-1] : B_{\mathbb{T}}[+1] \longrightarrow B_{\mathbb{T}}$, i.e. it fits into the following commutative diagram:

$$\begin{array}{ccccc}
B_{\mathbb{T}}[+1] \times F_{\mathbb{T}} & \xrightarrow{(iso)} & B_{\mathbb{T}}[+1] \times_{B_{\mathbb{T}}} X_{\mathbb{T}} & \longrightarrow & X_{\mathbb{T}} \\
& \searrow & \downarrow & & \downarrow p \\
& & B_{\mathbb{T}}[+1] & \longrightarrow & B_{\mathbb{T}}
\end{array}$$

That is, a map $X \xrightarrow{p} B$ is *locally trivial* iff there is an $\text{ɀ}\boldsymbol{\varphi}$-isomorphism $B_{\mathbb{T}}[+1] \times F_{\mathbb{T}} \xrightarrow{(iso)} B_{\mathbb{T}}[+1] \times_{B_{\mathbb{T}}} X_{\mathbb{T}}$ over $B_{\mathbb{T}}[+1]$.

Let us verify that this diagram represents the usual definition of local triviality. To give a morphism of sSets

$$B_{\mathbb{T}}[+1] \times_{B_{\mathbb{T}}} X_{\mathbb{T}} = \bigsqcup_{b \in B} \{b\} \times X_{\mathbb{T}} \xrightarrow{(iso)} B_{\mathbb{T}}[+1] \times F_{\mathbb{T}} = \bigsqcup_{b \in B} \{b\} \times B_{\mathbb{T}} \times F_{\mathbb{T}}$$

over $B_{\mathbb{T}}[+1]$ is to give for each $b \in B$ a morphism $f_b : X \longrightarrow B \times F$; to check this, use that ssets $X_{\mathbb{T}}$ and $B_{\mathbb{T}} \times F_{\mathbb{T}}$ are connected. The morphism of ssets is an isomorphism of ssets iff each f_b is an isomorphism of sets, i.e. a bijection.

Let us now prove that each f_b is a homeomorphism with a neighbourhood of form $U_b \times F$ where $b \in U_b \subset B$ is open.

For each $(b', y') \in B \times F$ pick a neighbourhood $W_{(b',y')} \subset B \times B \ni (b', y')$ which is a counterexample to continuity of f_b at the unique preimage of (b', y') if it is indeed not continuous at that point. The following is a neighbourhood at the set of 2-simplicies of $B_{\mathbb{T}}[+1] \times F_{\mathbb{T}}$:

$$\varepsilon := \left\{(b,\ (b', y'),\ (b'', y''))\ :\ (b', y') \in B \times F,\ (b'', y'') \in W_{(b',y')}\right\} \cup \bigsqcup_{b' \neq b} \{b'\} \times (B \times F) \times (B \times F)$$

By continuity its preimage $\delta_b := \overset{-1}{f_b}(\varepsilon)$ contains a set of the form

$$\{(b, x', x'') \in B \times X \times X\ :\ p(x') \in U_b,\ x'' \in V_{x'}\}$$

where $U_b \ni b, V_{x'} \ni x'$ are open. Hence $f(V_{x'}) \subset W_{f_b(x')}$ for all x' such that $p(x') \in U_b \subset B$, and, by choice of the neighbourhoods $W_{(b',x')}$, the function f_b is continuous over the preimage of $U_b \times F \subset B \times F$.

A similar argument establishes continuity of $\overset{-1}{f_b}$.

2.3. Geometric realisation as path mapping spaces: the approach of Besser, Grayson and Drinfeld. — [Grayson, Remark 2.4.1-2] interprets the geometric realisation of a simplex Δ_N as a space of non-decreasing maps $[0,1] \longrightarrow (N+1)_\leq$.

$$|\Delta_N| = \{(s_1, .., s_N) \in \mathbb{R}^N : 0 \leq s_1 \leq ... \leq s_N \leq 1\} \approx \{s : [0,1]_\leq \longrightarrow (N+1)_\leq\}$$

$$0 \leq s_1 \leq ... \leq s_N \leq 1\ \approx ([0, s_1) \mapsto 0, ..., [s_{N-1}, s_N) \mapsto N-1, [s_N, 1] \mapsto N)$$

with a metric analogous to Levi-Prokhorov or Skorokhod metrics on the spaces of discontinuous functions used in probability theory; roughly, two functions are close in such a metric iff one can be obtained from the other by a small perturbation of both values and arguments; in other words, a small neighbourhood of the graph of one function contains the other one.

We use this observation and the construction of geometric realisation by [Drinfeld] to define, for a simplicial set X, a ɀƍ-structure on the inner Hom in sSets

$$\underline{\mathrm{Hom}}_{\mathrm{sSets}}(\mathrm{Hom}_{\mathrm{preorders}}(-, [0,1]_\leq), X)$$

analogous to the Skorokhod metric. We then argue that the metric space associated with this $\boldsymbol{\zeta\varphi}$-object is the geometric realisation of X, under some assumptions.

2.3.1. Drinfeld construction of geometric realisation as a space of paths with Skorokhod metric. — As a warm-up, for the reader familiar with [Grayson, §2.4] and [Drinfeld], we sketch a construction of a metric on

$$\mathrm{Hom}_{\mathrm{sSets}}(\mathrm{Hom}_{\mathrm{preorders}}(-,[0,1]_\leq),X)$$

analogous to the Skorokhod metric in probability theory.

A finite subset $F \subset [0,1]$ and an $x \in X(\pi_0([0,1] \setminus F))$ determines a morphism of sSets $\mathrm{Hom}_{\mathrm{preorders}}(-,[0,1]_\leq) \longrightarrow X$ as follows:

$$\mathrm{Hom}_{\mathrm{preorders}}(n_\leq,[0,1]_\leq) \longrightarrow X(n_\leq)$$

$$\vec{t} \longmapsto x\left[\xrightarrow{n_\leq \xrightarrow{\vec{t}} [0,1] \longrightarrow \pi_0([0,1]\setminus F)}\right] \in X(n_\leq)$$

where $[0,1] \longrightarrow \pi_0([0,1] \setminus F)$ is the obvious map contracting the connected components (we need to make a convention where to send points of F).

A verification shows that this defines, moreover, a map of sets

$$|X| \coloneqq \varinjlim_{F\subset[0,1] \text{ finite}} X(\pi_0([0,1]\setminus F)) \longrightarrow \mathrm{Hom}_{sSets}(\mathrm{Hom}_{\mathrm{preorders}}(-,[0,1]_\leq),X)$$

Conversely, a map $\pi : \mathrm{Hom}_{\mathrm{preorders}}(-,[0,1]_\leq) \longrightarrow X$ of ssets determines a system of points as follows:

$$\pi\left(\xrightarrow{\theta : n_\leq \longrightarrow [0,1]}\right) \in X([0,1] \setminus \{\theta(0),..,\theta(n-1)\})$$

and thereby a point of $|X|$.

Define the following pseudometric analogous to *Levi-Prokhorov or Skorokhod*(5) metric (here we allow distance to be 0):

$$f, g : \mathrm{Hom}_{\text{preorders}}(-, [0,1]_{\leq}) \longrightarrow X$$

$$\mathrm{dist}(f,g) := \inf\{\epsilon > 0 \,:\, \forall n > 0\, \forall \overrightarrow{t} = (t_1 \leq \ldots \leq t_n)\, \exists \overrightarrow{t}' = (t'_1 \leq \ldots \leq t'_n)$$
$$\Big(f(\overrightarrow{t}) = g(\overrightarrow{t}') \;\&\; |t_1 - t'_1| < \epsilon \;\&\; \ldots \;\&\; |t_n - t'_n| < \epsilon \Big)$$

Let us now compare this construction with [Grayson, Remark 2.4.1-2] for $X = \mathrm{Hom}_{\text{preorders}}(\cdot, N_{\leq})$ the $(N-1)$-simplex Δ_{N-1}. In this case the Yoneda lemma gives us that a map $[0,1]_{\leq} = \mathrm{Hom}_{\text{preorders}}(\cdot, [0,1]_{\leq}) \longrightarrow \mathrm{Hom}_{\text{preorders}}(\cdot, N_{\leq})$ is the same as a map $[0,1]_{\leq} \longrightarrow N_{\leq}$, which, in turn, is essentially the same as a sequence $0 \leq s_1 \leq \ldots \leq s_{N-1} \leq 1$, i.e. a point of the geometric realisation

$$|\Delta_{N-1}| = \{(s_1, .., s_{N-1}) \,:\, 0 \leq s_1 \leq \ldots \leq s_{N-1} \leq 1\} \subset \mathbb{R}^{N-1}.$$

We also see that the metric coincides with the metric defined by [Grayson, Remark 2.4.1-2].

2.3.2. The Skorokhod filter on a Hom-set. — For $N > 2n$, $\delta \subset X_N$ and $\varepsilon \subset Y_n$, a *$\varepsilon\delta$-Skorokhod neighbourhood of Hom-set Hom*$_{\text{sSets}}(X_{as\ sSet}, Y_{as\ sSet})$ of the underlying simplicial sets of X and Y is the subset consisting of all the function $f : X_{\text{as sSet}} \longrightarrow Y_{\text{as sSet}}$ with the following property:

(5) This definition is similar to the definition of Skorokhod metric as phrased by [Kolmogorov, §2, Def.1 of ε-equivalence], particularly if we consider functions taking values in a discrete metric space $0, 1, .., N$, $\rho(n,m) := |n-m|$: two functions f, g are called ε-equivalent iff there exists r and

$$0 = t_0 < t_1 < \ldots < t_r = 1,$$
$$0 = t'_0 < t'_1 < \ldots < t'_r = 1,$$

such that for $k = 1, ..., r$ the following inequalities hold:

$$|t_k - t'_k| \leqslant \varepsilon,$$
$$\sup_{\substack{\theta \in [t_{k-1}+0,\ t_k - 0] \\ \theta' \in [t'_{k-1}+0,\ t'_k - 0]}} \rho(f(\theta), g(\theta')) \leqslant \varepsilon.$$

The original goal of the definition was to define a distance or convergence for (distributions of) stochastic processes such that a small distortion of either timings of events or their values results in a small distance.

there is a neighbourhood $\delta_0 \subset X_n$ such that each "δ_0-small" $x \in \delta_0$ has a "δ-small" "continuation" $x' \in X_N$, $x = x'[1..N]$ such that its "tail" maps into something "ε-small", i.e. $f(x'[N-n+1..N]) \in \varepsilon$.
As a formula, this is

$$\{f : X \longrightarrow Y \,:\, \exists \delta_0 \subset X_n \,\forall x \in \delta_0 \,\exists x' \in \delta (x = x'[1...n] \;\&\; f(x'[N-n+1, ..., N]) \in \varepsilon)\}$$

The *Skorokhod filter on* $Hom_{\mathrm{sSets}}(X_{as\ sSet}, Y_{as\ sSet})$ is the filter generated by all the Skorokhod $\varepsilon\delta$-neighbourhoods for $N \geq 2n > 0$ (sic!), neighbourhoods $\delta \subset X_N$ and $\varepsilon \subset Y_n$.

Let $\underline{\mathrm{Hom}}^{sSets}_{\wr\varphi}(X, Y)$ denote $\mathrm{Hom}_{\mathrm{sSets}}(X_{\text{as sSet}}, Y_{\text{as sSet}})$ equipped with the Skorokhod neighbourhood structure. This allows to define mapping spaces in $\wr\varphi$ by equipping the inner Hom of ssets with the Skorokhod filters.

Definition 2.3.2.1 **(Mapping space)**. — The *Skorokhod mapping space* $\underline{Hom}^{sSets}_{\wr\varphi}(X, Y)$ is the inner Hom $\underline{\mathrm{Hom}}_{\mathrm{sSets}}(X_{\text{as sSet}}, Y_{\text{as sSet}})$ of the underlying simplicial sets of X and Y equipped with the neighbourhood structure as follows. Equip $\mathrm{Hom}_{\mathrm{preorders}}(-, n_{\leq})$ with the antidiscrete filter, equip $X \times \mathrm{Hom}_{\mathrm{preorders}}(-, n_{\leq})$ with the product filter, and, finally, equip the set of $(n{-}1)$-simplicies $\mathrm{Hom}_{\mathrm{sSets}}(X_{\text{as sSet}} \times \mathrm{Hom}_{\mathrm{preorders}}(-, n_{\leq}), Y_{\text{as sSet}})$ with the resulting Skorokhod neighbourhood structure.

The *mapping space* $\underline{Hom}_{\wr\varphi}(X, Y)$ is the subspace of the Skorokhod mapping space consisting of "continuous functions", i.e. the simplicies "over" 0-simplicies in $\mathrm{Hom}_{\wr\varphi}(X, Y)$. That is, it is formed by the simplicies $s \in \underline{\mathrm{Hom}}_{\mathrm{sSets}}(X_{\text{as sSet}}, Y_{\text{as sSet}})$ such that all its 0-dim faces $s[t] : X \longrightarrow Y$, $0 \leq t \leq \dim s$, are $\wr\varphi$-morphisms.

2.3.3. The geometric realisation of a simplex and its Skorokhod space of paths. — We now rephrase [Grayson, Remark 2.4.1-2] in terms of $\wr\varphi$. Let $(\Delta_N)_{\mathrm{diag}}$ denote the standard simplex $\Delta_N = \mathrm{Hom}_{\mathrm{sSets}}(-, N+1_{\leq})$ equipped with the *filter of diagonals*, i.e. the filter on $(\Delta_N)_0$ is antidiscrete and for $n > 0$ the filter on $(\Delta_N)_n$ is the coarsest filter such that the diagonal degeneracy map $(\Delta_N)_0 \longrightarrow (\Delta_N)_n$ is continuous.

Let us now follow [Grayson, Remark 2.4.1-2] and see that the Hausdorffisation of the topological space corresponding to the Skorokhod space of maps from $[0,1]_{\leq}$ to $(\Delta_N)_{\mathrm{diag}}$ is the geometric realisation of Δ_N:

$$|\Delta_N| = \left(\underline{\mathrm{Hom}}^{sSets}_{\wr\varphi}([0,1]_{\leq}, (\Delta_N)_{\mathrm{diag}})_{\mathbb{T}^{-1}}\right)_{\mathrm{Hausdorff}}$$

Let us explicitly describe the underlying simplicial set. By Yoneda Lemma

$$\mathrm{Hom}_{\mathrm{sSets}}\left([0,1]_{\leq}, \Delta_N\right) = \mathrm{Hom}_{\leq}\left([0,1]_{\leq}, (N+1)_{\leq}\right)$$

For $M > 0$ and $n > 0$, a map $\overrightarrow{f} : [0,1]_{\leq} \times \Delta_M \longrightarrow \Delta_N$ of sSets is necessarily of form

$$(t_1 \leq ... \leq t_n, n_{\leq} \xrightarrow{\theta} M_{\leq}) \longmapsto (f_{\theta(1)}(t_1), f_{\theta(2)}(t_2), .., f_{\theta(n)}(t_n))$$

where $f_i : [0,1]_{\leq} \longrightarrow (N+1)_{\leq}$, $1 \leq i \leq M$.

Let us now verify the following. The Skorokhod filter on 0-simplicies is antidiscrete. A subset U of the set of 1-simplicies is a Skorokhod neighbourhood iff a 1-simplex $\overrightarrow{f} = (f,g) \in U$ whenever $\mathrm{dist}(f,g) < \delta$ where Skorokhod distance $\mathrm{dist}(f,g)$ is defined in §2.3.1.

Let us check this. In dimension 0, in the definition of the $\varepsilon\delta$-Skorokhod neighbourhood necessarily each 0-simplex of $[0,1]_{\leq}$, resp. Δ_N, is δ-small, resp. ε-small, hence each function is $\varepsilon\delta$-Skorokhod-small. In dimension 1, we may assume that ε is as small as possible, i.e. the diagonal, and that $\delta = \{(t_0 \leq t_1 \leq t_2 \leq t_3) : t_3 \leq t_0 + \delta\}$ and $\delta_0 = \{(t_0 \leq t_1) : t_1 \leq t_0 + \delta_0\}$ for some $\delta > 0$ and $\delta_0 > 0$. "Each $x \in \delta_0$" means we take arbitrary $t_0 \leq t_1 \leq t_0 + \delta$. Choosing an δ-small "continuation" x' of x amounts to choosing $(t_2 \leq t_3)$ such that $t_0 \leq t_1 \leq t_2 \leq t_3 \leq t_0 + \delta$, and that its "tail" maps into something ε-small means that $f(t_2) = g(t_3)$.

3. A convenient category for topology

3.1. The intuition of general topology. — Here we argue that the category ¿φ of simplicial filters (see §2.1.1 and §4.1 for a definition) is one of the " structures which give a mathematical content to the intuitive notions of *limit, continuity and neigh- bourhood*" and that the intuition of general topology as described by [Bourbaki] applies to ¿φ almost verbatim.

We do so by paraphrasing the Introduction of [Bourbaki], which we quote in full for reader's convenience.

In order to bring out what is essential in the ideas of limit, continuity and neighbourhood, we shall begin by analysing the notion of *neighbourhood* (although historically it appeared later than the other two). If we start from the physical concept of approximation, it is natural to say that a subset A of a set E is a neighbourhood of an element *a* of A if, whenever we replace *a* by an element that "approximates" *a*, this new element will also belong to A, provided of course that the "error" involved

is small enough; or, in other words, if all the points of E which are "sufficiently near" *a* belong to A. This definition is meaningful whenever precision can be given to the concept of sufficiently small error or of an element sufficiently near another. In this direction, the first idea was to suppose that the "distance" between two elements can be measured by a (positive) real number. Once the "distance" between any two elements of a set has been defined, it is clear how the "neighbourhoods" of an element *a* should be defined : a subset will be a neighbourhood of *a* if it contains all elements whose distance from *a* is less than some preassigned strictly positive number. Of course, we cannot expect to develop an interesting theory from this definition unless we impose certain conditions or axioms on the "distance" (for example, the inequalities relating the distances between the three vertices of a triangle which hold in Euclidean geometry should continue to hold for our generalized distance). In this way we arrive at a vast generalization of Euclidean geometry. It is convenient to continue to use the language of geometry : thus the elements the set itself is called a *space*. We shall study such spaces in Chapter IX.

So far we have not succeeded in freeing ourselves from the real numbers. Nevertheless, the spaces so defined have a great many properties which can be stated without reference to the "distance" which gave rise to them. For example, every subset which contains a neighbourhood of *a* is again a neighbourhood of *a*, and the intersection of two neighbourhoods of *a* is a neighbourhood of *a*. These properties and others have a multitude of consequences which can be deduced without any further recourse to the "distance" which originally enabled us to define neighbourhoods. We obtain statements in which there is no mention of magnitude or distance.

We are thus led at last to the general concept of a topological space, which does not depend on any preliminary theory of the real numbers. We shall say that a set E carries a *topological structure* whenever we have associated with each element of E, by some means or other, a family of subsets of E which are called *neighbourhoods* of this element — provided of course that these neighbourhoods satisfy certain conditions (the *axioms* of topological structures). Evidently the choice of axioms to be imposed is to some extent arbitrary, and historically has been the subject of a great deal of experiment (see the Historical Note to Chapter I). The system of axioms finally arrived at is broad enough for the present needs of mathematics, without falling into excessive and pointless generality.

As we have already said, a topological structure on a set enables one to give an exact meaning to the phrase "whenever x is sufficiently near a, x has the property $\mathrm{P}\{x\}$". But, apart from the situation in which a "distance" has been defined, it is not clear what meaning ought to be given to the phrase "every pair of points x, y which are sufficiently near each other has the property $\mathrm{P}\{x, y\}$", since *a priori* we have no means of comparing the neighbourhoods of two different points. Now the notion of a pair of points near to each other arises frequently in classical analysis (for example, in propositions which involve uniform continuity). It is therefore important that we should be able to give a precise meaning to this notion in full generality, and we are thus led to define structures which are richer than topological structures, namely *uniform structures*. They are the subject of Chapter II.

With help of ɀɸ "precision can be given to the concept of sufficiently small error". In this direction, our idea was to suppose an "error" is but a pair $(a, a') \in E \times E$ of elements of E and to be thought of as *a 1-simplex of a simplicial set*, and that to give a precise meaning to the concept of sufficiently small error it is enough *to associate with E, by some means or other, for each finite Cartesian power of E, a family of subsets of "n-simplicies" $E \times \ldots \times E$ which are called neighbourhoods* — provided of course that these neighbourhoods satisfy certain conditions (namely, form an object $E_\bullet :$ ɀɸ, i.e. a contravariant functor from the category of finite linear orders to the category of filters.)

Following up "the first idea" [Bourbaki], let us first suppose that a set E carries a notion of "distance" between two elements which can be measured by a (positive) real number. Once the "distance" between any two elements of a set has been defined, it is clear how the "neighbourhoods" on $E \times E$ in ɀɸ should be defined: a subset of $E \times E$ will be a neighbourhood if for every element a it contains all pairs (a, a') of elements whose distance $\mathrm{dist}(a, a')$ is less than some preas- signed strictly positive number. Of course, we cannot expect to develop an interesting theory from this definition unless we impose certain conditions or axioms on the "distance" (for example, the inequalities relating the distances between the three vertices of a triangle which hold in Euclidean geometry should continue to hold for our generalized distance).

In this way we arrive at a generalization of topology. It is convenient to continue to use the language of topology: thus the 0-simplicies on which a "distance" has been defined are called *points*, and an object of the category ɀɸ itself is called a *space*.

Nevertheless, the ɀ𝝋-spaces so defined have a great many properties which can be stated without reference to the "distance" which gave rise to them. For example, every subset which contains a neighbourhood is again a neighbourhood, and the intersection of two neighbourhoods is a neighbourhood, and, more generally, the neighbourhoods form a functor $\Delta^{\mathrm{op}} \longrightarrow$ 𝝋. These properties and others have a multitude of consequences which can be deduced without any further recourse to the "distance" which originally enabled us to define neigh- bourhoods. We obtain statements in which there is no mention of magnitude or distance.

We are thus led at last to the general concept of a ɀ𝝋-space, which does not depend on any preliminary theory of the real numbers or topology. We shall say that a set E carries a ɀ𝝋-structure whenever we have associated with each finite Cartesian power of E, by some means or other, a family of subsets of $E \times .. \times E$ which are called neighbourhoods — provided of course that these neighbourhoods satisfy certain conditions (namely, form a simplicial object $\Delta^{\mathrm{op}} \longrightarrow$ 𝝋 in the category of filters; see §3.1.2 for an explanation how to reformulate the axioms of topology in terms of neighbourhoods as being a simplicial object). Of course, there are ɀ𝝋-spaces which are not associated to a set in this way.

The goal of this paper is to suggest to the reader that it may be worthwhile to view ɀ𝝋 as a replacement of the axioms of topology and to consider the question whether the system of axioms represented by ɀ𝝋 is broad enough for the present needs of topology/mathematics, without falling into excessive and pointless generality.

As [Bourbaki] have said, a topological structure on a set enables one to give an exact meaning to the phrase "whenever x is sufficiently near a, x has the property $P\{x\}$". But, apart from the situation in which a "distance" has been defined, it is not clear what meaning ought to be given to the phrase "every pair of points x,y which are suffi- ciently near each other has the property $P\{x, y\}$", since a priori we have no means of comparing the neighbourhoods of two different points. Now the notion of a pair of points near to each other arises fre- quently in classical analysis (for example, in propositions which involve uniform continuity). It is therefore important that we should be able to give a precise meaning to this notion in full generality, and we are thus led to define ɀ𝝋-structures which are richer than ones associated with topological structures, and in fact are associated with uniform structures which are the subject of Chapter II of [Bourbaki].

We do this as follows; we speculate that the fact that this is possible is an indication that the notion of an ɩφ-space is more flexible than the usual notion of a topological space.

Whenever a "distance" has been defined, to give a precise meaning to the notion of a pair of points near to each other, we associate with it an ɩφ-object such that its filter of 1-simplicies is defined as follows: a subset of $E \times E$ will be a neighbourhood if it contains all the pairs (a, a') of elements whose distance $\mathrm{dist}(a, a')$ is less than some preas- signed strictly positive number (note that in the previous construction of the ɩφ-structure corresponding to a topology, this number was allowed to depend on a, and this is why we had no means to compare the neighbourhoods of two different points). More generally, a subset of $E \times ... \times E$ will be a neighbourhood if it contains all the tuples $(a_1, ..., a_n)$ of elements such that the distance $\mathrm{dist}(a_i, a_j)$ for all $1 \leq i \leq j \leq n$ is less than some preassigned strictly positive number.

> **Once topological structures have been defined, it is easy to make precise the idea of *continuity*. Intuitively, a function is continuous at a point if its value varies as little as we please whenever the argument remains sufficiently near the point in question. Thus continuity will have an exact meaning whenever the space of arguments and the space of values of the function are topological spaces. The precise definition is given in Chapter I, § 2.**

Paraphrasing slightly these intuitive words of [Bourbaki,p.19], we say that intuitively, a function is continuous at a point if its value remains the same up to an error as small as we please whenever the argument remains the same up to a sufficiently small error. The precise meaning of this phrase in terms of ɩφ is straightforward: a map $f : X_\bullet \longrightarrow Y_\bullet$ in ɩφ is continuous iff for every $n \geq 0$, for every neighbourhood $\varepsilon \subset Y_n$ there is a neighbourhood $\delta \subset X_n$ such that $f(\delta) \subset \varepsilon$; equivalently, $f^{-1}(\varepsilon)$ is a neighbourhood in X_n.

Note that when we consider the ɩφ-structures defined above and take $n = 2$, we recover the standard definitions of continuity and uniform continuity [Bourbaki, I§2.1,II§2.1]. The similarity to the exposition of the definition of uniform continuity is particularly startling:

Definition 1. *A mapping f of a uniform space X into a uniform space X' is said to be uniformly continuous if, for each entourage V' of X', there is an entourage V of X such that the relation $(x, y) \in V$ implies $(f(x), f(y)) \in V'$.*

> In more expressive terms we may say that f is uniformly continuous if $f(x)$ and $f(y)$ are as close to each other as we please whenever x and y are close enough.

If we put $g = f \times f$, then Definition 1 means that *whenever V' is an entourage of X', $\overset{-1}{g}(V')$ is an entourage of X.*

3.1.1. The notion of limit via an endofunctor "shifting" dimension. — We now show how to rewrite the definition of a limit of a function as a Quillen lifting property involving an endofunctor of $[+1] : ɀ\boldsymbol{\varphi} \longrightarrow ɀ\boldsymbol{\varphi}$ "shifting" dimension.

Thus here we show that the notion of the "shift endofunctor $[+1] : \Delta^{op} \longrightarrow \Delta^{op}$ is related to that of a limit, and later we show that it appears also in other contexts in topology involving local properties, namely the definition of a locally trivial bundle, cf. §2.2.5 and §4.8.

As with continuity, the idea of a *limit* involves two sets, each endowed with suitable structures, and a mapping of one set into the other. For example, the limit of a sequence of real numbers a_n involves the set $\mathbf{N}$ of natural numbers, the set $\mathbf{R}$ of real numbers, and a mapping of the former set into the latter. A real number a is then said to be a limit of the sequence if, whatever neighbourhood V of a we take, this neighbourhood contains all the a_n except for a finite number of values of n; that is, if the set of natural numbers n for which a_n belongs to V is a subset of $\mathbf{N}$ whose complement is finite. Note that $\mathbf{R}$ is assumed to carry a topological structure, since we are speaking of neigbourhoods; as to the set $\mathbf{N}$, we have made a certain family of subsets play a particular part, namely those subsets whose complement is finite. This is a general fact: whenever we speak of limit, we are considering a mapping f of a set E into a topological space F, and we say that f has a point a of F as a limit if the set of elements x of E whose image $f(x)$ belongs to a neighbourhood V of a [this set is just the "inverse image" $\overset{-1}{f}(V)$] belongs, whatever the neighbourhood V, to a certain family $\mathfrak{F}$ of subsets of E, given beforehand. For the notion of limit to have the essential properties ordinarily attributed to it, the family $\mathfrak{F}$ must satisfy certain axioms, which are stated in Chapter I, § 6. Such a family $\mathfrak{F}$ of subsets of E is called a *filter* on E. The notion of a filter, which is thus inseparable from that of a limit, appears also in other contexts in topology; for example, the neighbourhoods of a point in a topological space form a filter.

Whenever we speak of limit, we are considering a mapping f of a set E into a topological space F, and we say that f has a point a of F as a limit if the set of elements x of E whose image $f(x)$ belongs to a neighbourhood V of a [this set is just the "inverse image" $f^{-1}(V)$] belongs, whatever the neighbourhood V, to a certain family $\mathfrak{F}$ of subsets of E, given beforehand.

In terms of ɀ𝔭, this is expressed as follows: the mapping $f_a : E \longrightarrow F \times F,\ x \mapsto (a, f(x))$ is continuous with respect to the filter on E defined by the family $\mathfrak{F}$, and the filter of neighbourhoods on $F \times F$ associated with the topology on F. However this is a mapping from 0-simplices to 1-simplices, and thus "shifts" dimension: this is not a problem, as the category Δ^{op} of finite linear orders admits an endofunctor $[+1] : \Delta^{\mathrm{op}} \longrightarrow \Delta^{\mathrm{op}}$ equipped with a natural transformation $[+1] \Longrightarrow \mathrm{id}$, and therefore ɀ𝔭 admits an endofunctor $[+1]$: ɀ𝔭 $\longrightarrow$ ɀ𝔭 shifting dimension equipped with a natural transformation $[+1] \Longrightarrow \mathrm{id}$.

Considerations above lead to the following reformulation of the notion of a limit in terms of ɀ𝔭:

– To give a mapping of sets $f : E \longrightarrow F$ is to give a map of simplicial sets

$$\overrightarrow{f} : \mathrm{Hom}_{\mathrm{Sets}}(-,E) \longrightarrow \mathrm{Hom}_{\mathrm{Sets}}(-,F)\,, \quad (x_1,..,x_n) \longmapsto (f(x_1),...,f(x_n))$$

– *The mapping f has a point a of F as a limit iff $\overrightarrow{f}$ factors as*

$$Hom_{\mathrm{Sets}}(-,E) \xrightarrow{\overrightarrow{f_a}} Hom_{\mathrm{Sets}}(-,F) \circ [+1] \longrightarrow \underline{Hom}_{\mathrm{sSets}}(-,F)$$

$$\overrightarrow{f_a} : (x_1,..,x_n) \longmapsto (a, f(x_1),...,f(x_n))$$

via a map continuous with respect to appropriate filters.

On $\mathrm{Hom}_{\mathrm{Sets}}(-,F)$ these are the filters associated with the topology on F. On $\mathrm{Hom}_{\mathrm{Sets}}(-,E)$ these are *the $\mathfrak{F}$-diagonal filters*, i.e. the finest filters such that the filter on 0-simplicies is $\mathfrak{F}$ and the degeneration map from the set of 0-simplicies to the set of n-simplicies is continuous for each $n > 0$; explicitly, a subset of $E^n = \mathrm{Hom}_{\mathrm{Sets}}(n,E)$ is a neighbourhood iff it contains $\{(x,x,..,x) : x \in \varepsilon\}$ for some $\varepsilon \in \mathfrak{F}$.

Let us check this is indeed a reformulation. As simplicial sets, the source of the morphism is connected whereas the target is not and its connected components are parametrised by $a \in F$. Hence, to pick such a decomposition is to pick an $a \in F$. Above we saw that the morphism $(\overrightarrow{f_a})_0 : E \longrightarrow F \times F$ of 0-simplicies is continuous iff f has a point a of F

as a limit. A slight extension of this argument shows this holds for $\overrightarrow{f_a}$ itself. See details at §2.1.3 and §4.9.

3.1.2. Axioms of topology as being simplicial. — A topology is a collection of (filters of) neighbourhoods of points compatible in some sense. We now show that it is "compatible" in the sense that it is "functorial", i.e. defines a functor from Δ^{op} to the category of filters.

This is almost explicit in the axioms $(\mathrm{V_I})$-$(\mathrm{V_{IV}})$ of [Bourbaki,I§1.2] of topology in terms of neighbourhoods if we rewrite them in terms of $\zeta\varphi$. We now quote:

$(\mathrm{V_I})$ ***Every subset of*** **X** ***which contains a set belonging to*** $\mathfrak{B}(x)$ ***itself belongs to*** $\mathfrak{B}(x)$.

$(\mathrm{V_{II}})$ ***Every finite intersection of sets of*** $\mathfrak{B}(x)$ ***belongs to*** $\mathfrak{B}(x)$.

$(\mathrm{V_{III}})$ ***The element*** x ***is in every set of*** $\mathfrak{B}(x)$.

$(\mathrm{V_{IV}})$ ***If*** **V** ***belongs to*** $\mathfrak{B}(x)$, ***then there is a set*** **W** ***belonging to*** $\mathfrak{B}(x)$ ***such that, for each*** $y \in \mathrm{W}$, **V** ***belongs to*** $\mathfrak{B}(y)$.

By Proposition 1, we may take W to be any open set which contains x and is contained in V.

PROPOSITION 2. ***If to each element*** x ***of a set*** **X** ***there corresponds a set*** $\mathfrak{B}(x)$ ***of subsets of*** **X** ***such that the properties*** $(\mathrm{V_I})$, $(\mathrm{V_{II}})$, $(\mathrm{V_{III}})$ ***and*** $(\mathrm{V_{IV}})$ ***are satisfied, then there is a unique topological structure on*** **X** ***such that, for each*** $x \in \mathrm{X}$, $\mathfrak{B}(x)$ ***is the set of neighbourhoods of*** x ***in this topology.***

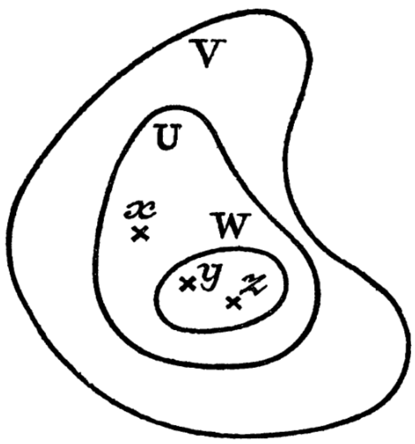

Figure 1.

Call a subset of $X \times X$ a neighbourhood iff it is of the form

$$\bigcup_{x\in X} \{x\} \times U_x \text{ where } U_x \in \mathfrak{B}(x)$$

Axiom (V_I) says that a subset containing a neighbourhood is itself a neighbourhood. Axiom (V_{II}) says that the neighbourhoods are closed under finite intersection. Hence, the first two axioms say that the neighbourhoods so defined form a filter on $E \times E$.

Axiom (V_{III}) states the continuity of the diagonal map $E \longrightarrow E \times E,\ x \longmapsto (x,x)$ from the set E equipped with the antidiscrete filter (i.e. the filter where E itself is the unique neighbourhood).

Axiom (V_{IV}) needs a little argument, as follows.

Equip $E \times E \times E$ with the coarsest filter such that the following coordinate projections $E \times E \times E \longrightarrow E \times E$ are continuous:

$$(x,y,z) \longmapsto (x,y) \text{ and } (x,y,z) \longmapsto (y,z)$$

Axiom (V_{IV}) says that the remaining coordinate projection $(x,y,z) \longmapsto (x,z)$ is continuous. To see this, consider the preimage of a neighbourhood containing $\{x\} \times V$. By continuity, there are neighbourhoods $\bigcup_{x'\in X} \{x'\} \times W_{x'}$ and $\bigcup_{y\in X} \{y\} \times V_y$ such that

$$\{x\} \times W_x \times X \bigcap X \times (\bigcup_{y\in X} \{y\} \times V_y) \subset \{x\} \times X \times V$$

Now take $W = W_x$ and see that $V_y \subset V$ for each $y \in W_x$.

Finally, note that the considerations above amount to the following

Proposition 2. ***If to each element x of a set X there corresponds a set $\mathfrak{B}(x)$ of subsets of X such that the properties (V_I), (V_{II}), (V_{III}) and (V_{IV}) are*** *satisfied, then there is a "unique 2-dimensional ¿Ƥ-structure on X" such that the set of neighbourhoods in $X \times X$ is*

$$\left\{ \bigcup_{x\in X} \{x\} \times U_x \ :\ U_x \in \mathfrak{B}(x) \right\}.$$

By this we mean the following: there is a unique object of ¿Ƥ such that

- ("¿Ƥ-structure on X") its underlying simplicial set is $\mathrm{Hom}_{\mathrm{Sets}}(-, X)$
- (V_{III}) the set $X_0 = X$ of 0-simplicies carries antidiscrete topology
- ((V_I)&(V_{II})) the filter on the set $X_1 = X \times X$ of 1-simplicies is

$$\left\{ \bigcup_{x\in X} \{x\} \times U_x \ :\ U_x \in \mathfrak{B}(x) \right\}$$

– $((\mathrm{V_{IV}}))$ $X_\bullet$ is 2-dimensional, i.e. the filter on $X_{n+1} = X^n$ is the coarsest filter such that the face maps

$$X_{n+1} = X^n \longrightarrow X_1 = X \times X, \quad (x_1, ..., x_n) \longmapsto (x_i, x_{i+1}),\ 0 < i < n$$

are continuous.

A similar reformulation can be given to the axioms of uniform structure [Bourbaki,II§1.1,Def.1], cf. Exercise 4.2.1.5.

3.1.3. A flexible notion of space. — The discussion above suggests this notion of space is somewhat more flexible than the usual notion of a topological space. Filters, uniform structures and topological spaces are $\wr\!\!\varphi$-spaces. and a limit of a function is an $\wr\!\!\varphi$-morphism.

This allows to talk in category-theoretic terms about equicontinuous sequences of functions and their limits, by considering

$$\mathrm{Hom}_{\mathrm{Sets}}(-,\mathbb{N}) \times \mathrm{Hom}_{\mathrm{Sets}}(-,X) \longrightarrow \mathrm{Hom}_{\mathrm{Sets}}(-,Y)$$

$$\mathrm{Hom}_{\mathrm{Sets}}(-,\mathbb{N}) \times \mathrm{Hom}_{\mathrm{Sets}}(-,X) \circ [+1] \longrightarrow \mathrm{Hom}_{\mathrm{Sets}}(-,Y) \circ [+1]$$

where the simplicial sets are equipped with various filters representing the topology or metric on X and Y, and the filter of cofinite subsets of $\mathbb{N}$. See §2.1.3 and §4.11.3 for a discussion.

We saw how the endofunctor $[+1] : \wr\!\!\varphi \longrightarrow \wr\!\!\varphi$ is used to talk about limits, a local notion. In a similar way it can be used to talk about families of functions defined locally. Let $X_\bullet$ and $Y_\bullet$ be $\wr\!\!\varphi$-objects corresponding to topological spaces X and Y. To give an $\wr\!\!\varphi$-morphism $X_\bullet \circ [+1] \longrightarrow Y_\bullet$ is to give a family of functions $f_x : X \longrightarrow Y$, $x \in X$ such that f_x is continuous in a neighbourhood of x (under some assumptions on X and Y). The definition of local triviality is formulated in terms of $\wr\!\!\varphi$ as follows: a $X \xrightarrow{p} B$ is a locally trivial bundle with fibre F iff in $\wr\!\!\varphi$ there is an isomorphism $B_\bullet \circ [+1] \times_{B_\bullet} X_\bullet \xrightarrow{(iso)} B_\bullet \circ [+1] \times F_\bullet$ over $B_\bullet \circ [+1]$, cf. §2.2.5 and §4.8 for explanation.

3.2. Algebraic topology. — Here we offer a couple of vague speculations about a possible intuition in $\wr\!\!\varphi$ originating in homotopy theory.

3.2.1. Simplicies as ε-discretised homotopies. — In a metric space, for $\epsilon > 0$ small enough, we may think of a sequence $x_0, .., x_T$, $\mathrm{dist}(x_t, x_{t+1}) < \epsilon, t = 0, .., T-1$ as an *ϵ-discretised homotopy* from x_0 to x_T, and the indices t, T as *time*. In terms of $\wr\!\!\varphi$, this sequence $\overrightarrow{x} = (x_0, ..., x_T)$ is an T-simplex the $\wr\!\!\varphi$-object $M_\bullet$ corresponding to the metric space, and the condition $\mathrm{dist}(x_t, x_{t+1}) < \epsilon, t = 0, .., T - 1$ says that the "consecutive" faces $\overrightarrow{x}[t < t+1] \in \varepsilon$ where $\varepsilon := \{(x,y) : \mathrm{dist}(x,y) < \epsilon\}$ is the neighbourhood associated with distance ϵ. This consideration suggests a generalisation of this intuition to an arbitrary object X : ɀ𝝋 : for a small enough neighbourhood $\varepsilon \subset X_\tau$ for $\tau < t' << T$, think of a simplex $s \in X_T$ as *ε-discretised homotopy* provided its "consecutive" faces $s[t_0 < ... < t_\tau] \in \varepsilon$ whenever $t_\tau \leq t_0 + t'$. Note that the essential asymmetry (direction of time) of this notion, or rather intuition, of homotopy, which is apparently a desirable property in the context of ∞-categories.

A well-known lemma says that two functions $f, g : A \longrightarrow M$ from an arbitrary topological space A to a (sufficiently nice) metric space M are homotopic iff for each $\epsilon > 0$ there is a ϵ-discretised homotopy $f = f_0, ..., f_T = g$ such that for any $x \in A$ $\sup_x \mathrm{dist}(f_t(x), f_{t+1}(x)) < \epsilon$. This allows us to think of homotopies of functions as simplicies in a certain function space with sup-metric.

3.2.2. Inner Hom and mapping spaces. — Think of the inner hom of the underlying simplicial sets of objects of ɀ𝝋 as a space of *discontinuous* functions. Such spaces of discontinuous functions are considered in probability theory as spaces of functions describing stochastic process, and there are standard metrics called *Levi, Levi-Prokhorov, and Skorokhod metric* on such spaces [Kolmogorov]. We need the following non-symmetric variant of Skorokhod metric defined on functions between metric spaces:

$$\mathrm{dist}(f,g) := \inf\{\epsilon > 0 : \forall x \exists y (\mathrm{dist}(x,y) < \epsilon \;\&\; \mathrm{dist}(f(x), g(y)) < \epsilon\}$$

This definition generalises to ɀ𝝋 as follows (see §2.3.2 details): For $N > 2n$, $\delta \subset X_N$ and $\varepsilon \subset Y_n$, a *$\varepsilon\delta$-Skorokhod neighbourhood in the Hom-set $Hom(X,Y)$* is the subset consisting of all the function $f : X \longrightarrow Y$ with the following property:

> there is a neighbourhood $\delta_0 \subset X_n$ such that each "δ_0-small" $x \in \delta_0$ has a "δ-small" "continuation" $x' \in X_N$, $x = x'[1..N]$ such that its "tail" maps into something "ε-small", i.e. $f(x'[N - n + 1..N]) \in \varepsilon$.

This defines the *Skorokhod filter on the Hom-set $Hom(X,Y)$* and thereby *Skorokhod neighbourhood structure on the inner Hom $\underline{Hom}_{\mathrm{sSets}}(X,Y)$* of the underlying simplicial sets of X and Y, and in fact a functor

$$\underline{\mathrm{Hom}}_{ɀ𝝋}(-,-) : ɀ𝝋^{\mathrm{op}} \times ɀ𝝋 \longrightarrow ɀ𝝋$$

which we call *Skorokhod mapping space*. The "usual" mapping space of *continuous* functions may be defined by taking the subspace of the Skorokhod mapping space consisting of simplicies whose 0-dimensional faces are $\wr\mathbf{P}$-morphisms.

The Skorokhod mapping spaces have the desirable property (cf. §2.3.2 and §4.6.2) that there is an evaluation map

$$\underline{\mathrm{Hom}}_{\wr\mathbf{P}}\Big(A, \underline{\mathrm{Hom}}_{\wr\mathbf{P}}(X,Y)\Big) \longrightarrow \underline{\mathrm{Hom}}_{\wr\mathbf{P}}(A\times X, Y)$$

and in fact there is a sort of left adjoint functor

$$\underline{\mathrm{Hom}}_{\wr\mathbf{P}}(A\ltimes X, Y) \xrightarrow{(iso)} \underline{\mathrm{Hom}}_{\wr\mathbf{P}}\Big(A, \underline{\mathrm{Hom}}_{\wr\mathbf{P}}(X,Y)\Big)$$

which admits a map $A\ltimes X \longrightarrow A\times X$.

In §2.3.1 and §2.3.3 we show a construction of [Besser], [Grayson, Remark 2.4.1-2], and [Drinfeld] can be interpreted in $\wr\mathbf{P}$ as saying that the geometric realisation may be thought as the Skorokhod space of *discontinuous* paths, and thus an endofunctor of $\wr\mathbf{P}$.

3.3. Ramsey theory and model theory. — Ramsey theory suggests $\wr\mathbf{P}$-spaces which do not come from topology. Given an sset X and a colouring c of simplices, call a simplex c-homogeneous iff all of its non-degenerate faces have the same c-colour. A collection of colourings allows one to define a notion of neighbourhood: a neighbourhood (of the diagonal) is a subset containing all the simplices homogeneous with respect to finitely many colours. This construction allows to generalise the notion of the Stone space of types in model theory if we consider formulae as colourings on the sset $n_\leq \longmapsto \mathrm{Hom}_{\mathrm{Sets}}(n,M)$ of tuples of elements of a model M : the neighbourhoods in the $\wr\mathbf{P}$-Stone space consists of sequences (with repetitions) indiscernible with respect to finitely many formulae. See §5.1-5.2 for details.

4. Definitions and constructions

4.1. The main category: the definition. — We now state formally the definition of the key categories of the paper.

4.1.1. The two categories of filters (sets with a notion of smallness.)— First we quote the definition of a filter by [Bourbaki, I§6.1, Def.I], again.

Definition I. ***A filter on a set X is a set $\mathfrak{F}$ of subsets of X which has the following properties:***

(F$_{\text{I}}$) ***Every subset of X which contains a set of $\mathfrak{F}$ belongs to $\mathfrak{F}$.***

(F$_{\text{II}}$) ***Every finite intersection of sets of $\mathfrak{F}$ belongs to $\mathfrak{F}$.***

(F$_{\text{III}}$) ***The empty set is not in $\mathfrak{F}$.***

Axiom (F$_{\text{II}}$) is equivalent to the conjunction of the following two axioms:

(F$_{\text{II}\,a}$) The intersection of two sets of $\mathfrak{F}$ belongs to $\mathfrak{F}$.

(F$_{\text{II}\,b}$) X belongs to $\mathfrak{F}$.

Axioms (F$_{\text{II}\,b}$) and (F$_{\text{III}}$) show that *there is no filter on the empty set.*

In order for a set of subsets which satisfies (F$_{\text{I}}$) also to satisfy (F$_{\text{II}\,b}$) it is necessary and sufficient that it is *not empty*. A set of subsets which satisfies (F$_{\text{I}}$) also satisfies (F$_{\text{III}}$) if and only if it is different from $\mathfrak{P}$(X).

Subsets in $\mathfrak{F}$ are called *neighbourhoods* or $\mathfrak{F}$-*big*. Unlike [Bourbaki], we do *not* require $(\mathrm{F}_{\mathrm{III}})$ and allow both $X = \varnothing$ and $\varnothing \in \mathfrak{F}$; necessarily $X \in \mathfrak{F}$..

Definition 4.1.1.1. — A *filter* on a set X is a collection of subsets of X called *neighbourhoods* satisfying $(\mathrm{F}_{\mathrm{I}})$ and $(\mathrm{F}_{\mathrm{II}})$ above.

A *morphism of filters* is a function of the underlying sets such that the preimage of a neighbourhood is necessarily a neighbourhood; we call such maps of filters *continuous*.

Let Φ denote the category of filters. Let $\mathfrak{F}$ denote the category of filters where we identify the maps equal almost everywhere: $\mathrm{Ob}\,\mathfrak{F} = \mathrm{Ob}\,\Phi$ and $\mathrm{Hom}_{\mathfrak{F}}(X,Y) = \mathrm{Hom}_{\Phi}(X,Y) / \approx_{\mathfrak{F}}$ where $f \approx_{\mathfrak{F}} g$ iff $\{x : f(x) = g(x)$ is a neighbourhood in X.

We sometimes refer to Φ as *the category of neighbourhoods*.

The *category Φ of filters* is equivalent[6] to the category of topological spaces with a base-point and morphisms being functions continuous *at* the base-point (not in a neighbourhood of the base-point), and—not a

[6] The equivalence is given by adjoining/removing the base-point: a filter F on a set X corresponds to the topological space with points $X \sqcup \{x_0\}$ where a subset is open iff it is of form $\{x_0\} \sqcup U$, $U \in F$ is an F-neighbourhood; a topological space X with a base-point x_0 corresponds to the filter on $X \setminus \{x_0\}$ induced by the neighbourhood filter of x_0. The requirement "no point other than the base-point goes into the base-point" means the map of sets $X \setminus \{x_0\} \longrightarrow Y \setminus \{y_0\}$ is well-defined, and continuity at the base-point means precisely that it is a morphism of filters.

natural requirement—no point other than the base-point goes into the base-point.

***Example 4.1.1.2* (Filters).** — – In a topological space, (not necessary open) neighbourhoods of a point form a filter.

- The filter consisting of subsets containing a given subset. For a set X, the filter of diagonals consists of subsets of $X \times .. \times X$ containing $\{(x, ..., x) : x \in X\}$.
- Given a measure μ on a set X, subsets of full measure form a filter; so do subsets of positive measure.
- An *ultrafilter* on a set X is a filter such that either A or $X \setminus A$ is a neighbourhood, i.e. is either open or closed. In other words, a not necessarily countably additive "measure" such that each subset is measurable and has measure either 0 or 1.
- The filter of cofinite subsets of a set: a subset is a neighbourhood iff its complement is finite.
- Sets of points of a metric space whose complement has finite diameter form a filter.
- Sets of pairs of points of a metric space which contain all pairs sufficiently close to each other, i.e. subsets of $M \times M$ which contain the following set for some $\epsilon > 0$:

$$\{(x, y) \in M \times M \, : \, \mathrm{dist}(x, y) < \epsilon\}$$

- Sets of pairs of points of a metric space which contain all pairs sufficiently far apart, i.e. subsets of $M \times M$ which contain the following set for some $d > 0$:

$$\{(x, y) \in M \times M \, : \, \mathrm{dist}(x, y) > d\}$$

- fixme: other examples..

4.1.2. Simplicial sets: notations and first definitions. — Recall a preorder $\leq \subseteq P \times P$, i.e. a reflexive transitive binary relation on a set P, defines a topology on P: a subset $U \subseteq P$ is open iff $x \in U$ whenever $x \leq y$ for some $y \in U$. Recall also that the preorder also defines a category whose objects are elements of P and where all diagrams commute: there is a necessarily unique morphism $x \longrightarrow y$ iff $x \leq y$. Monotone non-decreasing maps correspond to continuous maps and, resp., functors.

In fact, a preorder on a finite set can be viewed in three equivalent ways: as a reflexive transitive relation, a finite topological space, and a category where all diagrams commute. Maps (morphisms) of preorders are then viewed as monotone maps, continuous maps, and functors.

Let Δ be the category whose objects are finite linear orders $n_\leq := \{0, .., n-1\}$, $n \in \mathbb{N}$, and whose morphisms are non-decreasing maps; note that in Δ each isomorphism is an identity, in other words, there is a unique object in each isomorphism class. Let Δ^{op} be its opposite category. Let Δ_{big} denote the equivalent category of all finite linear orders and non-decreasing maps. A *simplicial object of a category* C is a functor $X_\bullet : \Delta^{\mathrm{op}} \longrightarrow C$. A *simplex* is an element of $X(n_\leq)$ for some $n > 0$. An n-simplex is an element of $X((n+1)_\leq)$.

An increasing sequence $1 \leq t_1 \leq ... \leq t_n \leq m$ determines a morphism $m_\leq \longrightarrow n_\leq$ in Δ^{op}. We denote the corresponding *faces and degenerations* of a simplex s by $s[t_1 \leq ... \leq t_n]$ or $s[t_1, ..., t_n]$.

In a category C, the set of all maps from an object X to Y is denoted by $\mathrm{Hom}_C(X, Y)$. The space of maps from X to Y, if defined, is denoted by $\underline{\mathrm{Hom}}_C(X, Y)$; typically it is an object of the category C itself.

Sometimes we borrow notation from type theory and write $X : C$ to indicate that X is an object of X.

4.1.3. The definition of the categories of simplicial sets with a notion of smallness.—

Definition 4.1.3.1. — Let $\mathsf{s}\boldsymbol{\varphi} = Func(\Delta^{\mathrm{op}}, \boldsymbol{\varphi})$ be the category of functors from Δ^{op}, the category opposite to the category Δ of finite linear orders, to the category $\boldsymbol{\varphi}$ of filters.

Let $\mathsf{s}\boldsymbol{\Phi}$ be the category $\mathsf{s}\boldsymbol{\varphi}$ localised as follows: $\mathrm{Ob}\, \mathsf{s}\boldsymbol{\Phi} = \mathrm{Ob}\, \mathsf{s}\boldsymbol{\varphi}$, $\mathrm{Hom}_{\mathsf{s}\boldsymbol{\Phi}}(X, Y) = \mathrm{Hom}_{\mathsf{s}\boldsymbol{\varphi}}(X, Y) / \approx$ where $f \approx_{\mathsf{s}\boldsymbol{\Phi}} g$ iff there is $N > 0$ such that for every $n > N$ there is a neighbourhood $\varepsilon \subset X_n$ such that for all $x \in \varepsilon$ it holds $f(x) = g(x)$.

We think of an object of $\mathsf{s}\boldsymbol{\varphi}$ as a simplicial set equipped with a notion of smallness, and that it provides us with a notion of *a space* which is more flexible than the notion of a topological space. We suggest no good name for these spaces and refer to such a space as either a simplicial neighbourhood, a neighbourhood structure, a simplicial filter; the reader preferring a short word may want to call it a situs.

We think of an object of $\mathsf{s}\boldsymbol{\Phi}$ as a space where we only care about *local* properties.

4.2. Topological and metric spaces as simplicial filters. — Here we show that ᶎφ contains both categories of topological and of metric spaces (with uniformly continuous maps) as full subcategories.

4.2.1. Topological and metric spaces as neighbourhood structures. — See §2.2.3 and §3.1 for examples and intuition.

***Definition 4.2.1.1* (Topological and metric spaces in ᶎφ)**

A metric $\text{dist} : X \times X \longrightarrow \mathbb{R}$ on a set X defines *a filter of ε-neighbourhoods of the diagonal* on $X^n = \text{Hom}_{\text{Sets}}(n_{\text{as Set}}, X)$: a subset $\varepsilon \subset X^n$ is a neighbourhood iff there is an $\epsilon > 0$ such that $(x_1, .., x_n) \in \varepsilon$ whenever $\text{dist}(x_i, x_j) < \epsilon$ for all $1 \le i, j \le n$.

A topology on a set X defines *a filter of topoic neighbourhoods of the diagonal* on $X^n = \text{Hom}_{\text{Sets}}(n_{\text{as Set}}, X)$ as follows:

1. for $n = 1$, X itself is the unique neighbourhood of $X = X^1$.
2. for $n = 2$, $U \subseteq X \times X$ is a neighbourhood iff for all $x \in X$ there is an open neighbourhood $U_x \ni x$ such that $\{x\} \times U_x \subseteq U$.
3. the filter on X^n is the coarsest filter such that maps $X^n \longrightarrow X \times X, (x_1, ..., x_n) \mapsto (x_i, x_{i+1})$ are maps of filters for each $0 < i < n$. Explicitly, a subset ε of X^n is *a neighbourhood* iff either of the following equivalent conditions holds:
   - there exist neighbourhoods $\varepsilon_i \subset X \times X, 0 < i < n$, such that $(x_1, ..., x_n) \in \varepsilon$ whenever for each $0 < i < n$, $(x_i, x_{i+1}) \in \varepsilon_i$.
   - for each $x \in X$ there a neighbourhood $\varepsilon_x \subseteq X^{n-1}$ such that $\{x\} \times \varepsilon_x \subseteq \varepsilon$

Denote the simplicial filters associated with a metric and a topology on X by $X_{ση}$ and $X_{ፐ}$, resp.

Exercise 4.2.1.2. — A verification shows that $-_{ፐ}$: Top $\longrightarrow$ ᶎφ and $-_{ση}$: (Metric Spaces, Uniformly Continuous Maps) $\longrightarrow$ ᶎφ define fully faithful embeddings of the category of topological spaces and the category of metric spaces and uniformly continuous maps.

Exercise 4.2.1.3. — Check these two embeddings have inverses.

- Check that the embedding of topological spaces admits an inverse $-_{ፐ^{-1}}$: ᶎφ $\longrightarrow$ Top defined as follows. See §2.2.4 for another brief exposition, and note the similarity to the definition of the topology associated with a uniform structure [Bourbaki,II§1.2,Prop.1,Def.3].

The set of points of $X_{\text{ፐ}^{-1}}$ is the set of 0-simplicies which are ε-small for each neighbourhood $\varepsilon \subset X_0$, i.e. $X_{\text{points}} := \bigcap_{\varepsilon \subset X_0 \text{ is a neighbourhood}} \varepsilon$. The topology is generated by the subsets that together with each point contain all ε-near points for some $\varepsilon \subset X_1$, i.e. $U_x \ni x$ is a neighbourhood of x iff there is a neighbourhood $\varepsilon \subset X_1$ such that $y \in U_x$ whenever $y \in X_{\text{points}}$ and $(x,y) \in \varepsilon$ or $(y,x) \in \varepsilon$.

- Check that for a topological space X, $(X_{\text{ፐ}})_{\text{ፐ}^{-1}} = X$, and that $([0.1]_{\leq})_{\text{ፐ}^{-1}} = [0,1]$ is the usual unit interval.
- Check whether the embedding of uniform spaces admits an inverse $-_{\sigma\eta^{-1}} : \zeta\text{ቕ} \longrightarrow$ UniformSpaces defined as follows.

  The set of points of the uniform space $X_{\sigma\eta^{-1}}$ corresponding to a simplicial neighbourhood X is the set X_0 of 0-simplicies. The uniform structure on X_0 is the coarsest such that the obvious map $X \longrightarrow X_{\sigma\eta^{-1}}$ is continuous.

Remark 4.2.1.4. — The neighbourhood structure of topoic subsets associated with a topology lacks symmetry of the filter associated with the neighbourhood structure of ε-neighbourhoods of the diagonal associated with a metric. This accords well with a remark of in the introduction of (Bourbaki, General Topology):

> a topological structure on a set enables one to give an exact meaning to the phrase "whenever x is sufficiently near a, x has the property $P\{x\}$". But, apart from the situation in which a "distance" has been defined, it is not clear what meaning ought to be given to the phrase "every pair of points x, y which are sufficiently near each other has the property $P\{x,y\}$", since a priori we have no means of comparing the neighbourhoods of two different points.

In fact, a metric gives rise to a functor $FiniteSets \longrightarrow \text{ቕ}$, and the notion of a uniform structure is equivalent to such a functor satisfying the "2-dimensionality" Condition of Def. 4.2.1.1(3).

Exercise 4.2.1.5. — Show that the definition of a uniform structure [Bourbaki,General Topology, II§1.1,Def.1] in fact describes a 2-dimensional (i.e. satisfying Def. 4.2.1.1(3)) symmetric simplicial filter, where symmetric means that it factors as $\Delta^{\text{op}} \longrightarrow FiniteSets^{\text{op}} \longrightarrow \text{ቕ}$.

4.2.2. Simplicial neighbourhoods associated with metric maps on the large scale. — We shall now show that ɀφ contains the opposite to the category of geodesic (on the large scale) metric spaces with surjective quasi-Lipschitz maps.

Call a map $f : X \longrightarrow Y$ of metric spaces *injective on the large scale* [Gromov,§0.2.D] iff there is a monotone unbounded real function $\lambda(d), d > 0$, such that either of the equivalent conditions hold:

- $\mathrm{dist}_X(x,y) \leq \lambda(\mathrm{dist}_Y(f(x),f(y)))$ for all $x,y \in X$
- $\mathrm{dist}_Y(f(x),f(y)) \geq \lambda^{-1}(\mathrm{dist}_X(x,y))$ for all $x,y \in X$

Associate with a metric space X the sset $\Delta^{\mathrm{op}} \longrightarrow Sets,\ n \longmapsto \mathrm{Hom}_{\mathrm{Sets}}(n,X)$ represented by the set of its points. Call a subset U of X^n a neighbourhood iff there is $d > 0$ and a set B of diameter $< d$ such that

- $(x_0,...,x_n) \in U$ whenever $x_0,...,x_n \notin B$ and
  $\mathrm{dist}(x_i,x_j) > d$ for all $0 \leq i < j \leq n$ such that $x_i \neq x_j$

Let $X_{\mathcal{L}}$: ɀφ be the simplicial neighbourhood obtained.

Exercise 4.2.2.1. — Check that $-_{\mathcal{L}}$ defines a contravariant fully faithful embedding of the category of geodesic (on the large scale) metric spaces with surjective quasi-Lipschitz maps, cf. [Gromov,§0.2.D,§0.2.A$'_2$].

- A morphism $X_{\mathcal{L}} \longrightarrow Y_{\mathcal{L}}$ in ɀφ is a map of metric spaces $X \longrightarrow Y$ injective on the large scale.
- If X and Y are geodesic and $f : X \longrightarrow Y$ is surjective, then f^{-1} is quasi-Lipschitz iff $f : X_{\mathcal{L}} \longrightarrow Y_{\mathcal{L}}$ is well-defined, i.e. iff f is injective on the large scale.
- Conclude that $-_{\mathcal{L}}$ defines a contravariant fully faithful embedding of the category of geodesic (on the large scale) metric spaces with surjective quasi-Lipschitz maps.
- Work out the geometric meaning of the lifting property defining connectedness (see Exercise 4.7.0.4 for notation and explanations):
  - $X \longrightarrow \{0=1\}_{\mathbb{T}} \rightthreetimes \{0,1\}_{\mathbb{T}} \longrightarrow \{0=1\}_{\mathbb{T}}$

Exercise(todo) 4.2.2.2. — In §2.3.1We define the notion of geometric realisation as an endofunctor of ɀφ. Define in ɀφ, and then work out the geometric meaning of, $X_{\mathcal{L}} \longrightarrow B(G_{\mathcal{L}})$: does $B(G_{\mathcal{L}})$ classify something which may be called a G-bundle on the large scale ?

4.3. The subdivision neighbourhood structure on a simplicial set. — We give precise meaning to "subdivide a simplex into simplexes small enough", as follows.

Definition 4.3.0.3. — Let X : sSet and let $\epsilon \in X_k$ be a simplex, $m \geq 0$. A subset $\varepsilon \subset X_n$ is (ϵ, m)*-neighbourhood*, resp. $(\epsilon, m)_>$*-neighbourhood* or $(\epsilon, m)_<$*-neighbourhood*, iff for each $N > 0$ and each simplex $\epsilon' : X_N$ such that ϵ is a face of ϵ' it holds:

- $\epsilon'[t_1 \leq ... \leq t_n] \in \varepsilon$ whenever $0 \leq t_1 \leq ... \leq t_n \leq t_1 + m \leq N$
- $\epsilon'[t_1 \leq ... \leq t_n] \in \varepsilon$ whenever $0 \leq t_1 \leq ... \leq t_n \leq m \leq N$
- $\epsilon'[t_1 \leq ... \leq t_n] \in \varepsilon$ whenever $N - m < t_1 \leq ... \leq t_n \leq N$

The *subdivision*, resp. >*-subdivision* or <*-subdivision*, filter on X_n is generated by the (ϵ, m)-neighbourhoods where $\epsilon : X_k$ varies through simplices of arbitrary dimension $k \geq 0$, $m > 0$.

Let X_{subd} denote the simplicial set equipped with the subdivision neighbourhood structure, and similarly for <-subdivision and >-subdivision neighbourhood structures.

Example 4.3.0.4. —
- For the simplicial set of Cartesian powers $\text{Hom}_{\text{Sets}}(-, M)$, $n \mapsto M^n$ of a set, the subdivision filter is trivial: M^n is the only neighbourhood of M^n.
- (see details in the next subsection) For a linear order $[0,1]_\leq$, in the simplicial set consider "co-represented" by I

$$n_\leq \longmapsto \text{Hom}_{\text{preorders}}(n_\leq, [0,1]_\leq),$$

the subdivision filter is generated by the sets $\{(t_1 \leq ... \leq t_n) : \text{dist}(t_i, t_{i+1}) < \epsilon\}$ where $\epsilon > 0$. Equivalently, this is the filter is generated by the subsets of simplices of diameter ϵ, for $\epsilon > 0$.

4.4. The real line interval $[0,1]$**.** — We now may define the interval object in ʑ𝝋 by equipping the sset corepresented by a linear order with the subdivision neighbourhood structure.

Definition 4.4.0.5. — Let *the interval object* $[0,1]_\leq$ *in* ʑ𝝋 be the sset

$$\Delta^{\text{op}} \longrightarrow \text{Sets}$$

$$n_\leq \longmapsto \text{Hom}_\leq(n_\leq, [0,1]_\leq)$$

equipped with the subdivision neighbourhood structure. Let $[0,1]_{\leq^-}$, $[0,1]_{\leq^+}$, and $[0,1]_{\leq^\pm}$ a denote the same sset equipped with the <-subdivision, >-subdivision, and the union of <-subdivision and >-subdivision neighbourhood structures, resp.

We also use similar notation for any preorder $(I, \leq)$.

Exercise 4.4.0.6. — The subdivision neighbourhood structure on $[0,1]_{\leq}$ is induced by the metric in the following sense: it is generated by the ϵ-neighbourhoods of the diagonal $\{0 \leq t_0 \leq ... \leq t_n \leq 1 : \mathrm{dist}(t_i, t_{i+1}) < \epsilon\}$, for $\epsilon > 0$.

Exercise 4.4.0.7. — Let X be a topological space, M a metric space.

- A continuous function $[0,1] \longrightarrow M$ is the same as a morphism $[0,1]_{\leq} \longrightarrow M_{\sigma q}$ in $\zeta\varphi$.
- A continuous function $[0,1] \longrightarrow X$ is the same as a morphism $[0,1]_{\leq^{\pm}} \longrightarrow X_{T}$ in $\zeta\varphi$.
- A upper semi-continuous function $[0,1] \longrightarrow X$ is the same as $[0,1]_{\leq^{-}} \longrightarrow X_{T}$ in $\zeta\varphi$.
- A lower semi-continuous function $[0,1] \longrightarrow X$ is the same as $[0,1]_{\leq^{+}} \longrightarrow X_{T}$ in $\zeta\varphi$.

4.5. A notion of homotopy based on the interval $[0,1]_{\leq}$. — In the standard way this notion of an interval leads to a notion of homotopy. Note that we later define a notion of homotopy based on a notion of the mapping space, which we feel is more appropriate.

***Remark 4.5.0.8* (Homotopy on $\zeta\varphi$).** — The definition above lets us define a notion of homotopy in $\zeta\varphi$ in the usual way: two maps $f, g : X \longrightarrow Y$ are *homotopic* iff there is a linear order I, elements i_f and i_g, a morphism $X \times I_{\leq} \xrightarrow{h} Y$ such that f factors as $X \times \{i_f\} \longrightarrow X \times I_{\leq} \xrightarrow{h} Y$ and g factors as $X \times \{i_g\} \longrightarrow X \times I_{\leq} \xrightarrow{h} Y$.

Exercise(todo) 4.5.0.9. — Compare this to the notion of homotopy defined in §4.6.3.

4.6. Mapping spaces, geometric realisation and a notion of homotopy. —

4.6.1. Discretised homotopies as Archimedean simplices. —

Definition 4.6.1.1. — A simplex s is *ε/n-fine* iff $s[t_1 \leq ... \leq t_k] \in \varepsilon$ whenever $t_1 \leq ... \leq t_k \leq t_1 + n$. A simplex s is *Archimedean* iff it can be split into finitely many of arbitrarily small parts, i.e. is a face of an

ε/n-fine simplex for every neighbourhood $\varepsilon \subset X_k$ and every $k, n > 0$. Call such an ε/n-fine simplex an ε/n-refinement of s.[7]

Call a set of simplices *bounded* iff its simplices are Archimedean and there is an upper bound on the dimension of their ε/n-refinements for each n and neighbourhood ε.

Exercise 4.6.1.2. — Check that for a metric space M, a pair of points $s = (x, y) \in M \times M$ form an Archimedean simplex in $M_{\sigma\eta}$ iff for each $\epsilon > 0$ there is an *ϵ-discretised path* $x = x_0, x_1, ..., x_l = y$, $\mathrm{dist}(x_t, x_{t+1}) < \epsilon$ for $0 \le t < l$, *from* $x = (x, y)[0]$ *to* $y = (x, y)[1]$.

Note we do not require that these ϵ-discretised path converge on an actual path; perhaps this hints the definition of an Archimedean simplex should be modified.

Check that in a metric space M with an inner metric a subset $B \subset M \times M$ is bounded iff there is a bound on the distance between points for $(x, y) \in B$, i.e. there is $d > 0$ such that $\mathrm{dist}(x, y) \le d$ whenever $(x, y) \in B$.

Exercise 4.6.1.3. — Archimedean simplices form a subobject (subfunctor) X_{Arch} of X, for $X : \zeta\varphi$. Check that a surjective map $X \longrightarrow Y$ induces a map $X_{\mathrm{Arch}} \longrightarrow Y_{\mathrm{Arch}}$.

Exercise 4.6.1.4. — A well-known lemma says that two functions $f, g : A \longrightarrow M$ from an arbitrary topological space A to a metric space M are homotopic iff there is a ϵ-discretised homotopy $f = f_0, ..., f_n = g$ such that for any $x \in A$ $\mathrm{dist}(f_t(x), f_{t+1}(x) < \epsilon$, under some assumptions on the metric space M; it is enough to assume that for every $\epsilon > 0$ there is $\delta > 0$ such that every ϵ-ball contains a contractible subset containing a δ-ball with the same centre. (Ref!)

Check that the lemma says that two functions $f, g : A \longrightarrow M$ are homotopic iff (f, g) is an Archimedean simplex of the mapping space $Func(A, M)$ with the sup-metric, or, equivalently, iff (f, g) is an Archimedean simplex of the mapping space $(Func(A, M)_{\sigma\eta})_{\mathrm{Arch}}$.

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[7] The geometric intuition suggests this definition should possibly be modified: ε/n-refinement of a simplex may "go off to infinity" as ϵ and n vary; in the case of a metric space, the ϵ-chains (i.e. ϵ-discretised homotopies) connecting two points go off to infinity rather than converge on an actual path). One may want to require something like that a simplex s is *Archimedean* iff for every neighbourhood $\varepsilon \subset X_k$ and every $k, n > 0$ it has an ε/n-refinement which in its ε-neighbourhood has δ/m-refinement with the same property, for every neighbourhood $\delta \subset X_l$ and every $l, m > 0$.

4.6.2. Skorokhod mapping spaces: the definition. — We now repeat somewhat more formally the definition of the Skorokhod filters given in §2.3.2.

Definition 4.6.2.1. — Let X, Y : ɀφ be objects of ɀφ. For $N > 2n$, $\delta \subset X_N$ and $\varepsilon \subset Y_n$, a *$\varepsilon\delta$-Skorokhod neighbourhood of the Hom-set $Hom_{\mathrm{sSets}}(X_{as\ sSet}, Y_{as\ sSet})$* of underlying simplicial sets of X and Y is the subset consisting of all the functions $f : X_{\text{as sSet}} \longrightarrow Y_{\text{as sSet}}$ with the following property:

> there is a neighbourhood $\delta_0 \subset X_n$ such that each "δ_0-small" $x \in \delta_0$ has a "δ-small" "continuation" $x' \in X_N$, $x = x'[1..N]$ such that its "tail" maps into something "ε-small", i.e. $f(x'[N-n+1..N]) \in \varepsilon$.
>
> As a formula, this is

$$\{f : X \longrightarrow Y \ : \ \exists \delta_0 \subset X_n \, \forall x \in \delta_0 \, \exists x' \in \delta (x = x'[1...n] \ \& \ f(x'[N{-}n{+}1, ..., N]) \in \varepsilon)\}$$

The *Skorokhod filter on $Hom_{\mathrm{sSets}}(X_{as\ sSet}, Y_{as\ sSet})$* is the filter generated by all the Skorokhod $\varepsilon\delta$-neighbourhoods for $N \geq 2n > 0$ (sic!), neighbourhoods $\delta \subset X_N$ and $\varepsilon \subset Y_n$.

As a formula, it is

$$\{f : X \longrightarrow Y \ : \ \exists \delta_0 \subset X_n \, \forall x \in \delta_0 \, \exists x' \in \delta (x = x'[1...n] \ \& \ f(x'[N{-}n{+}1, ..., N]) \in \varepsilon)\}$$

Let $\mathrm{Hom}_{\varphi}^{sSets}(X, Y)$ denote $\mathrm{Hom}_{\mathrm{sSets}}(X_{\text{as sSet}}, Y_{\text{as sSet}})$ equipped with the Skorokhod neighbourhood structure.

This allows to define mapping spaces in ɀφ by equipping the inner Hom of ssets with Skorokhod filters.

***Definition 4.6.2.2* (Mapping space).** — The *Skorokhod mapping space $\underline{Hom}^{sSets}_{ɀ\varphi}(X, Y)$* is the inner Hom $\underline{\mathrm{Hom}}_{\mathrm{sSets}}(X_{\text{as sSet}}, Y_{\text{as sSet}})$ of the underlying simplicial sets of X and Y equipped with the neighbourhood structure as follows. Equip the $(n-1)$-simplex $\Delta_{n-1} = \mathrm{Hom}_{\leq}(-, n_{\leq})$ with the antidiscrete filter, equip $X \times \Delta_{n-1} = X \times \mathrm{Hom}_{\leq}(-, n_{\leq})$ with the product filter, and, finally, equip the set of $(n{-}1)$-simplicies $\mathrm{Hom}_{\mathrm{sSets}}(X_{\text{as sSet}} \times \mathrm{Hom}_{\text{preorders}}(-, n_{\leq}), Y_{\text{as sSet}})$ with the resulting Skorokhod neighbourhood structure.

The *mapping space $\underline{Hom}_{ɀ\varphi}(X, Y)$* is the subspace of the Skorokhod mapping space consisting of "continuous functions", i.e. the simplicies "over" 0-simplicies in $\mathrm{Hom}_{ɀ\varphi}(X, Y)$. That is, it is formed by the simplicies $s \in \underline{\mathrm{Hom}}_{\mathrm{sSets}}(X_{\text{as sSet}}, Y_{\text{as sSet}})$ such that all its 0-dim faces $s[t] : X \longrightarrow Y$, $0 \leq t \leq \dim s$, are ɀφ-morphisms.

Remark 4.6.2.3. — It is possible to define a different neighbourhood structure. We give this definition to demonstrate that there are implicit choices made in formalisation, and care needs to be taken. Another reason is that we are not sure our definitions are most appropriate.

Call $U \subset \mathrm{Hom}_{\mathrm{sSets}}(X \times \mathrm{Hom}_{\leq}(-, n+1_{\leq}), Y)$ a neighbourhood iff for some $m \geq 0$, neighbourhoods $\delta \subset X_m$ and $\varepsilon \subset Y_m$, some (equivalently, all) sequences $\overrightarrow{t} := (0 \leq t_0 \leq \ldots \leq t_m \leq n)$, $\overrightarrow{s} := (0 \leq s_0 \leq \ldots \leq s_m \leq n)$ such that $t_0 \leq s_0$, $t_1 \leq s_1$,..., $t_m \leq s_m$, it holds

$f \in U$ whenever the following implication holds

$$f_n[\overrightarrow{t}]\big((\delta \times \theta)[\overrightarrow{t}]\big) \subset \varepsilon[\overrightarrow{t}] \implies f_n[\overrightarrow{s}]\big((\delta \times \theta)[\overrightarrow{s}]\big) \subset \varepsilon[\overrightarrow{s}]$$

For example, if $n = 0$ the only neighbourhood is $U = \mathrm{Hom}_{\mathrm{sSets}}(X \times \mathrm{Hom}_{\leq}(-, 1_{\leq}), Y)$ itself.

Exercise(todo) 4.6.2.4. — Let X be a locally compact space, and let M be a metric space such that each ball contains a contractible subset containing an open ball around the same point.

- Calculate the mapping space $\underline{\mathrm{Hom}}_{\mathbf{sP}}(X_{\mathrm{T}}, M_{\sigma\eta})$ and the Skorokhod mapping space $\underline{\mathrm{Hom}}^{sSets}_{\mathbf{sP}}(X_{\mathrm{T}}, M_{\sigma\eta})$.
- Check whether $f : X \longrightarrow M$ is homotopic to $g : X \longrightarrow M$ iff the simplex (f, g) is Archimedean in either $\underline{\mathrm{Hom}}_{\mathbf{sP}}(X_{\mathrm{T}}, M_{\sigma\eta})$ or in $\underline{\mathrm{Hom}}^{sSets}_{\mathbf{sP}}(X_{\mathrm{T}}, M_{\sigma\eta})$.

Exercise(todo) 4.6.2.5. — Check the following basic properties of the evaluation map for the Skorokhod mapping spaces or the mapping spaces.

- Check whether the isomorphism of the underlying ssets (natural in A, X, Y) is necessarily continuous:

$$\underline{ev}_* : \underline{\mathrm{Hom}}^{sSets}_{\mathbf{sP}}\Big(A, \underline{\mathrm{Hom}}^{sSets}_{\mathbf{sP}}(X, Y)\Big) \longrightarrow \underline{\mathrm{Hom}}^{sSets}_{\mathbf{sP}}(A \times X, Y)$$

- Check whether for the "point" object $\Delta_0 = \mathrm{Hom}_{\leq}(n_{\leq}, 1_{\leq})$ of $\mathbf{sP}$, both the Skorokhod mapping space and the mapping space from Δ_0 to an $Y : \mathbf{sP}$ is Y itself:

$$\underline{ev}_* : \underline{\mathrm{Hom}}^{sSets}_{\mathbf{sP}}(\mathrm{Hom}_{\leq}(n_{\leq}, 1_{\leq}), Y) \xrightarrow{(iso)} Y$$

Exercise 4.6.2.6. — Check whether $\underline{\mathrm{Hom}}^{sSets}_{\mathbf{sP}}(-, Y)$ has an "inner" analogue of the left adjoint in the following sense.

– Define a semi-direct product $A \ltimes X$ in ɿφ as follows. The underlying simplicial set is that of the direct product in sSets, or, equivalently, in ɿφ: $(A \ltimes X)_n := (A \times X)_n$. The filter is defined as follows: $\delta \subset A_n \times X_n$ is a neighbourhood iff there is a neighbourhood $\alpha \subset A_n$ such that for each "α-small" $a \in \alpha$ there is a neighbourhood $\delta_a \subset X_n$ such that $(a, x) \in \delta$ whenever $a \in \alpha$ and $x \in \delta_a$.
– Check whether there is an isomorphism in ɿφ natural in A, X, Y:

$$\underline{ev}_* : \underline{\mathrm{Hom}}^{sSets}_{ɿφ}\Big(A, \underline{\mathrm{Hom}}^{sSets}_{ɿφ}(X,Y)\Big) \xrightarrow{(iso)} \underline{\mathrm{Hom}}^{sSets}_{ɿφ}(A \ltimes X, Y)$$

– Ponder the similarity to the definition of the topoic filter in the definition of the ɿφ-neighbourhood structure associated with topological spaces.

4.6.3. A notion of homotopy based on the mapping space. — Now that we have a notion of a mapping space, we define a notion of homotopy using Archimedean simplices. Recall that a pair of continuous functions $f_0, f_1 : X \longrightarrow Y$ defines a map $X \times \Delta_1 \longrightarrow Y$, $(s, \theta) \mapsto (f_{\theta(0)}(s), ..., f_{\theta(n)}(s)) \in Y^{n+1} = Y_n$ and thus can be viewed as a 1-simplex in $\underline{\mathrm{Hom}}_{ɿφ}(X,Y) \subset \underline{\mathrm{Hom}}^{sSets}_{ɿφ}(X,Y)$.

***Definition 4.6.3.1* (Skorokhod homotopic).** — Say that *a map* $f : X \longrightarrow Y$ *is* Skorokhod homotopic *to* $g : X \longrightarrow Y$ iff the simplex (f, g) is Archimedean in $\underline{\mathrm{Hom}}^{sSets}_{ɿφ}(X,Y)$.

Say that *a map* $f : X \longrightarrow Y$ *is* homotopic *to* $g : X \longrightarrow Y$ iff the simplex (f, g) is Archimedean in $\underline{\mathrm{Hom}}_{ɿφ}(X,Y)$.

Exercise(todo) 4.6.3.2. — Are these notions symmetric? transitive? Study these notions of homotopy. For example:
– Let $X, Y : sSets$ be ssets equipped with antidiscrete filters: the only neighbourhood is the whole set. Check that this is equivalent to the definition of simplicial homotopy in [Goerss-Jardine, I§6].
– Check whether this gives the standard notion of homotopy of maps from topological spaces to metric spaces. Compare Exercise 4.6.1.4 and Exercise 4.6.2.4.

4.6.4. The geometric realisation via the approach of Grayson. — We now repeat more formally the construction of the geometric realisation due to [Besser], [Grayson, Remark 2.4.1-2], and [Drinfeld] sketched in §2.3.3.

[Besser, Def.3.3] and [Grayson, Remark 2.4.1-2] give a construction of the geometric realisation of a simplicial set based on the observation that the standard simplex $\Delta_n = \{(s_1, .., s_n) \in [0,1]^n : 0 \le s_1 \le ... \le s_n \le 1\}$ is the space of maps $[0,1]_\le \longrightarrow (n+1)_\le$ of preorders modulo some identifications, i.e.

$$\underline{\mathrm{Hom}}_{\mathrm{sSets}}\left(\mathrm{Hom}_\le(-,[0,1]_\le), \mathrm{Hom}_\le(-,(n+1)_\le)\right).$$

The notion of a mapping space in ɀφ suggests we should try to define the geometric realisation of a simplicial set X as the Skorokhod space $\underline{\mathrm{Hom}}^{sSets}_{\text{ɀφ}}\left(\underline{\mathrm{Hom}}_{preorders}(-,[0,1]_\le), X\right)$ of (discontinuous) paths in X equipped with an appropriate neighbourhood structure.

Definition 4.6.4.1. — The *Besser geometric realisation* of $X : \text{ɀφ}$ is the endofunctor

$$\text{ɀφ} \longrightarrow \text{ɀφ}, \quad X \longmapsto \underline{\mathrm{Hom}}^{sSets}_{\text{ɀφ}}([0,1]_\le, X)$$

The *Grayson subdivision* is the endofunctor

$$\text{ɀφ} \longrightarrow \text{ɀφ}, \quad X \longmapsto X \circ e$$

where $e : \Delta^{\mathrm{op}} \longrightarrow \Delta^{\mathrm{op}}$ is the endofunctor defined following [Grayson,§3.1, esp. Def.3.1.4, Def.3.1.8]):

$$n \longmapsto 2n,\ f : m \to n \longmapsto \{n{+}i \mapsto n{+}f(i),\ n{-}i \mapsto n{-}f(i),\ \text{for } i = 0, ..., n{-}1\}$$

Exercise(todo) 4.6.4.2. — Verify details of the argument in §2.3.3 and prove the following.

- Verify §2.3.3 gives a well-defined map of ssets
$$|X| \longrightarrow \mathrm{Hom}_{sSets}([0,1]_\le, X)$$
- Let X_{diag} denote the simplicial set X equipped with the finest neighbourhood structure such that the set X_0 of 0-simplicies is antidiscrete. Explicitly, a subset of X_n is a neighbourhood iff it contains the diagonal, i.e. the image of X_0 in X_n under the unique degeneracy map. Verify that $X_{\mathrm{diag}} : \text{ɀφ}$ is well-defined.
- Verify that for $X = \Delta_{n-1} = \mathrm{Hom}_\le(\cdot, n_\le)$, $n > 0$, the Hausdorffisation of the topologisation of the Skorokhod paths space is the standard

simplex:

$$\left(\underline{\mathrm{Hom}}_{ʅφ}^{sSets}\left([0,1]_{\leq},(\Delta_n)_{\mathrm{diag}}\right)_{\mathrm{T}^{-1}}\right)_{\mathrm{Hausdorff}} \xrightarrow{(iso)} |\Delta_n|$$

- Verify that $\underline{\mathrm{Hom}}_{\mathrm{sSets}}^{sSets}([0,1]_{\leq},-) : \mathrm{sSets} \longrightarrow \mathrm{sSets}$ preserves finite directed limits, and is also compatible with Skorokhod neighbourhood structure, i.e. that $\underline{\mathrm{Hom}}_{ʅφ}^{sSets}([0,1]_{\leq},-) : ʅφ \longrightarrow ʅφ$ also preserves finite directed limits.
- (todo) Conclude that for a finite simplicial set X there is an isomorphism of the geometric realisation and the Hausdorffisation of the topologisation of the Skorokhod paths space:

$$|X| \xrightarrow{(iso)} \left(\underline{\mathrm{Hom}}_{ʅφ}^{sSets}\left([0,1]_{\leq},X_{\mathrm{diag}}\right)_{\mathrm{T}^{-1}}\right)_{\mathrm{Hausdorff}}$$

- (todo) Give a precise meaning to the following argument. For every $n > 0$ the sequence of x_θ is determined, up to $\varepsilon = 1/n$, by x_θ where $\theta : n_{\leq} \longrightarrow [0,1], \theta = (0 < 1/n < ... < (n-1)/n)$. Hence, the topological geometric realisation of a simplicial set X is dense in the topologisation of its Skorokhod paths space.

Remark 4.6.4.3. — A.Smirnov suggested it maybe worthwhile to see whether the use of geometric realisation by [Suslin, On the K-theory of local fields] can be interpreted in terms of ʅφ.

4.7. The connected components functor π_0 as M2(l-lr)-replacement.— We observe that the connected components functor π_0 is analogues to the (co)fibrant replacement postulated by Axiom M2 of model categories where the (co)fibrant replacement is taken with respect to a morphism implicitly appearing in the definition of connectivity.

Recall that $\{0,1\} \longrightarrow \{0=1\}$ denotes the map of topological spaces gluing together the points of the discrete space with two points. As usual, we denote by $\{0,1\}_{\mathrm{T}} \longrightarrow \{0=1\}_{\mathrm{T}}$ the corresponding map in ʅφ.

Recall that $\{0,1\}_{\mathrm{T}}$ can be explicitly described as follows: $n \longrightarrow \{0,1\}^n$, and a subset of $\{0,1\}^n$ is a neighbourhood iff it contains the diagonal $\{(0,..,0),(1,...,1)\}$.

Exercise 4.7.0.4. — Check the following.

- A topological space X is connected iff $X \longrightarrow \{0=1\} \rightthreetimes \{0,1\} \longrightarrow \{0=1\}$.

– A simplicial set is connected iff $X \longrightarrow \{0=1\}_{\mathrm{T}} \rightthreetimes \{0,1\}_{\mathrm{T}} \longrightarrow \{0=1\}_{\mathrm{T}}$.

Denote by P^{l} and P^{r} the classes (properties) of morphisms defined with respect to the left, resp. right, lifting property:

$$P^{\mathrm{l}} := \{f \rightthreetimes g : g \in P\} \qquad P^{\mathrm{r}} := \{f \rightthreetimes g : f \in P\}$$

It is convenient to refer to P^{l} and P^{r} as the property of *left, resp. right, Quillen negation of property P*.

Definition 4.7.0.5. — In Top, let π_0 be the functor defined by the following $M2(l-lr)$ decomposition:

$$X \xrightarrow{(\{0,1\}\longrightarrow\{0=1\})^{\mathrm{l}}} \pi_0(X) \xrightarrow{(\{0,1\}\longrightarrow\{0=1\})^{\mathrm{lr}}} \{0=1\}$$

In ɀφ, let π_0 be the functor defined the following $M2(l-lr)$ decomposition:

$$X \xrightarrow{(\{0,1\}_{\mathrm{T}}\longrightarrow\{0=1\}_{\mathrm{T}})^{\mathrm{l}}} \pi_0(X) \xrightarrow{(\{0,1\}_{\mathrm{T}}\longrightarrow\{0=1\}_{\mathrm{T}})^{\mathrm{lr}}} \{0=1\}_{\mathrm{T}}$$

Exercise 4.7.0.6. — Check this definition is consistent with the usual definition of π_0 in sSets. Recall there is an embedding sSets $\longrightarrow$ ɀφ, $n \longmapsto (X_n)_{\mathrm{antidiscrete}}$ which sends a simplicial set X into itself equipped the antidiscrete filters, i.e. the filter such that the only neighbourhood is the whole set.

Check that the set of 1-simplices $\pi_0^{ɀφ}(X)_1$ is the set of connected components of $X_{\mathrm{antidiscrete}}$, and, in fact, $\pi_0^{ɀφ}(X) = \mathrm{diag}(\pi_0^{\mathrm{sSets}}(X)_{\mathrm{antidiscrete}})$.

Exercise 4.7.0.7. — Check this is consistent with the usual definition (notation) of π_0 on Top. For example, check the following.

Let X be a topological space such that the $\pi_0(X)$ is well-defined (behaved), e.g. X has finitely many connected components. Then in Top it holds $\pi_0^{ɀφ}(X_{\mathrm{T}}) = (\pi_0^{\mathrm{Top}}(X))_{\mathrm{T}}$, or, in other words,

$$X_{\mathrm{T}} \xrightarrow{(\{0,1\}_{\mathrm{T}}\longrightarrow\{0=1\}_{\mathrm{T}})^{\mathrm{l}}} \pi_0^{\mathrm{Top}}(X)_{\mathrm{T}} \xrightarrow{(\{0,1\}_{\mathrm{T}}\longrightarrow\{0=1\}_{\mathrm{T}})^{\mathrm{lr}}} \{0=1\}_{\mathrm{T}}$$

4.8. Locally trivial bundles. — Here we repeat somewhat more formally §2.2.5 about local trivial bundles.

It is said that being locally trivial means being locally a direct product. The precise meaning of this phrase in terms of ɀφ is straightforward: a map over a base B is locally trivial iff it becomes a direct product after pullback along $B[+1] \longrightarrow B$. We state this in the next § and then speculate whether this observation can be used to define a model structure on ʑφ.

4.8.1. Local triviality as being a product after pullback along $B[+1] \longrightarrow B$. —

Exercise 4.8.1.1. — A map $p : X \longrightarrow B$ of topological spaces is a locally trivial bundle with fibre F iff there is an ʑφ-isomorphism

$$\tau : B_{\intercal}[+1] \times_{B_{\intercal}} X_{\intercal} \xrightarrow{(iso)} B_{\intercal}[+1] \times F_{\intercal} \text{ over } B_{\intercal}[+1]$$

i.e. there is a commutative diagram as shown

$$\begin{array}{ccccc} B_{\intercal}[+1] \times F_{\intercal} & \xrightarrow{(iso)} & B_{\intercal}[+1] \times_{B_{\intercal}} X_{\intercal} & \longrightarrow & X_{\intercal} \\ \downarrow & & \downarrow & & \downarrow p \\ B_{\intercal}[+1] & \xrightarrow{\mathrm{id}} & B_{\intercal}[+1] & \longrightarrow & B_{\intercal} \end{array}$$

Verify the argument in §2.2.5 using the following steps.

- As simplicial sets, $B_{\intercal}[+1] \times_{B_{\intercal}} X_{\intercal} = \sqcup_{b\in B} X_{\intercal}$ and $B_{\intercal}[+1] \times F_{\intercal} = \sqcup_{b\in B} B_{\intercal} \times F_{\intercal}$ where $\sqcup$ denotes disjoint union.
- To give a map of these simplicial sets over $B_{\intercal}[+1]$ is to give for each $b \in B$ a map of sets $f_b : X \longrightarrow B \times F$ over B which extends to a map of the corresponding simplicial sets $X_{\intercal} \longrightarrow B_{\intercal} \times F_{\intercal}$.
- The maps $f_b : X \longrightarrow B \times F$, $b \in B$ represent a continuous ʑφ-isomorphism $B[+1] \times_B X \xrightarrow{(iso)} B[+1] \times F$ in ʑφ iff for every $b \in B$ there is a neighbourhood $U_b \ni B$ such that f_b defines a homeomorphism between $p^{-1}(U_b)$ and $U_b \times F$.
- Check whether the above holds for the ɱʑ-embedding of metric spaces.
- Check whether the above also holds for the localised category ʑφ.

Exercise(todo) 4.8.1.2. — Use the reformulation above to rewrite for ʑφ a definition of the long exact sequence of a (co)fibration in terms of the endofunctor $[+1] :$ ʑφ $\longrightarrow$ ʑφ and base change $B[+1] \longrightarrow B$.

4.8.2. Suggestions towards a model structure on the category of simplicial filters. —

Exercise(todo) 4.8.2.1. — Can this notion of local triviality be used to define a model structure on ፕ? For example, do the following.

– Calculate in ፕ

$$(f) := \{X \xrightarrow{p} B : \text{in ፕ } B[+1]\times_B X \longrightarrow B[+1] \text{ is of form } B[+1]\times F \longrightarrow B[+1]\}^{lr}$$

– Is it true that maps in (f) have the homotopy extension property, i.e. for any A $A \longrightarrow A \times [0,1]_\leq \rightthreetimes (f)$?
– Does this define the class of fibrations in the category of topological spaces, i.e. is it true under suitable assumptions that a map p of topological spaces is a fibration iff $p_{\mathrm{T}} \in (f)$?
– (todo) Does the morphism $[0,1]_\leq \circ [+1] \longrightarrow [0,1]_\leq$ have interesting universal properties or lifting properties ? Calulate $\{[0,1]_\leq \circ [+1] \longrightarrow [0,1]_\leq\}^{rl}$.
– (todo) Is there a model structure on ፕ where $(wc) := (f)^l$ and (f) are the classes of weak cofibrations and fibrations, resp.?
– (todo) More generally, define a model structure on ፕ or ፕ.

4.9. Taking limits of sequences and filters.— Here we show how to reformulate about taking limits, convergence of sequences, equicontinuous families of functions, Arzela-Ascoli theorems, compactness and completeness, with help of the endofunctor $[+1]$: ፕ $\longrightarrow$ ፕ "shifting dimension" and Quillen lifting properties.

4.9.1. Three embeddings $const, diag, cart$ of filters into simplicial filters. — Let F be a filter. There are two natural ways to equip $\mathrm{Hom}_{\mathrm{Sets}}(n, F)$ with a filter:

(a) the finest filter such that degeneracy (diagonal) map

$$\mathrm{Hom}_{\mathrm{Sets}}(1, F) = F \longrightarrow \mathrm{Hom}_{\mathrm{Sets}}(n, F) = F^n, x \longmapsto (x, x, ..., x)$$

is continuous.

(b) the coarsest filter such that all the face "coordinate projection" maps

$$\mathrm{Hom}_{\mathrm{Sets}}(n, F) = F^n \longrightarrow \mathrm{Hom}_{\mathrm{Sets}}(1, F) = F,\ (x_1, ..., x_n) \longmapsto x_i, 0 < i \leq n$$

are continuous.

Exercise 4.9.1.1. — Explicitly, these filters can be defined as:

(a') a subset of F^n is a *diag*-neighbourhood iff it contains $\{(x, x, .., x) : x \in U\}$ for some U an F-neighbourhood (b') a subset of F^n is a *cart*-neighbourhood iff it contains $U_1 \times U_2 \times .. \times U_n$, for some $U_1, .., U_n$ F-neighbourhoods

There is also the constant functor const : 𝝋 $\longrightarrow$ ɿ𝝋, $n \mapsto F$.

Exercise 4.9.1.2. — Check these define three fully faithful embeddings diag, cart, const : 𝝋 $\longrightarrow$ ɿ𝝋

4.10. Shift endofunctor $[+1] : \Delta \longrightarrow \Delta$. — The reformulation of the definition of limit in ɿ𝝋 uses the following "shift" endofunctor of Δ^{op} "forgetting the first coordinate". Let $[+1] : \Delta \longrightarrow \Delta$ denote the shift by 1 adding a new minimal element:

$$[+1] : (1 < ... < n) \longmapsto (-\infty < 1 < ... < n); \quad ([+1]f)(-\infty) := -\infty; \quad ([+1]f)(i) := i$$

The endofunctor is equipped with a natural transformation $[-1] : [+1] \Longrightarrow \mathrm{id} : \Delta^{\mathrm{op}} \longrightarrow \Delta^{\mathrm{op}}$, and its morphisms $X[+1] \xrightarrow{[-1]} X$ are particularly useful to us.

Now we recast in these terms a number of familiar notions in analysis: limit, Cauchy sequence, etc.

4.10.1. Cauchy sequences and their limits.— A sequence (a_n) of points of a metric space may be viewed as a map $\mathbb{N} \longrightarrow M$. Let $\mathbb{N}_{\mathrm{cof}}$ be $\mathbb{N}$ equipped with the filter of cofinite subsets: a subset U of $\mathbb{N}$ is *a neighbourhood* iff there is $N > 0$ such that $m \in U$ whenever $m > N$.

Exercise 4.10.1.1. — The sequence $(a_n) \in M$ is Cauchy iff it determines a continuous map

$$\mathrm{Hom}_{\mathrm{Sets}}(-, \mathbb{N}_{\mathrm{cof}})_{\mathrm{cart}} \to M_{\sigma\eta}, \quad (i_1, .., i_n) \longmapsto (a_{i_1}, ..., a_{i_n})$$

The sequence *converges* iff this map factors as

$$\mathrm{Hom}_{\mathrm{Sets}}(-, \mathbb{N}_{\mathrm{cof}})_{\mathrm{cart}} \longrightarrow M_{\sigma\eta}[+1] \xrightarrow{[-1]} M_{\sigma\eta},$$

and the map is necessarily of form $(i_1, .., i_n) \longmapsto (a_\infty, a_{i_1}, ..., a_{i_n})$ where a_∞ is the limit of the sequence.

Exercise 4.10.1.2. — A filter F on M is Cauchy iff $\mathrm{Hom}_{\mathrm{Sets}}(-, F)_{\mathrm{cart}} \longrightarrow M_{\sigma\eta}$ is well-defined in ɿ𝝋.

A Cauchy filter converges on M iff the morphism $\mathrm{Hom}_{\mathrm{Sets}}(-, F)_{\mathrm{cart}} \longrightarrow M_{\sigma\eta}$ factors as $F_{\mathrm{cart}} \longrightarrow M_{\sigma\eta}[+1] \longrightarrow M_{\sigma\eta}$.

A filter F converges on M iff the morphism $\mathrm{Hom}_{\mathrm{Sets}}(-, F)_{\mathrm{diag}} \longrightarrow M_{\sigma\eta}$ factors as $F_{\mathrm{diag}} \longrightarrow M_{\sigma\eta}[+1] \longrightarrow M_{\sigma\eta}$.

4.10.2. Limits on topological spaces.— The same construction works for topological spaces.

Let F be a filter on the set of points of a topological space T.

The inclusion (equality) of underlying subsets $F \subset T$ defines a morphism of sSets $\mathrm{Hom}_{\mathrm{Sets}}(n, F) \longrightarrow \mathrm{Hom}_{\mathrm{Sets}}(n, T)$.

Exercise 4.10.2.1. — A filter F converges on a topological space T iff the morphism $\mathrm{Hom}_{\mathrm{Sets}}(-, F)_{\mathrm{diag}} \longrightarrow T_{\mathfrak{T}}$ factors as $\mathrm{Hom}_{\mathrm{Sets}}(-, F)_{\mathrm{diag}} \longrightarrow T_{\mathfrak{T}}[+1] \longrightarrow T_{\mathfrak{T}}$.

4.11. Compactness and completeness.

4.11. Compactness and completeness.— We reformulate compactness and completeness in terms of iterated orthogonals/Quillen negation and morphisms representing typical examples of these notions. See the footnote in §2.1.4 for the definition of the Quillen negation/orthogonals.

4.11.1. Compactness and completeness as lifting properties/Quillen negation.—

Exercise 4.11.1.1. —

– A metric space X is complete iff each Cauchy filter converges, i.e. for each filter F it holds

$$\varnothing \longrightarrow \mathrm{Hom}_{\mathrm{Sets}}(-, F)_{\mathrm{cart}} \rightthreetimes X_{\sigma\eta}[+1] \longrightarrow X_{\sigma\eta}$$

– A topological space X is quasi-compact iff each ultrafilter converges, i.e. for each ultrafilter $\mathfrak{U}$ it holds

$$\varnothing \longrightarrow \mathrm{Hom}_{\mathrm{Sets}}(-, \mathfrak{U})_{\mathrm{diag}} \rightthreetimes X_{\mathfrak{T}}[+1] \longrightarrow X_{\mathfrak{T}}$$

– A metric space X is compact iff each ultrafilter converges, i.e. for each ultrafilter $\mathfrak{U}$ it holds

$$\varnothing \longrightarrow \mathrm{Hom}_{\mathrm{Sets}}(-, \mathfrak{U})_{\mathrm{diag}} \rightthreetimes X_{\sigma\eta}[+1] \longrightarrow X_{\sigma\eta}$$

– A metric space X is pre-compact iff each ultrafilter is Cauchy, i.e. for each ultrafilter $\mathfrak{U}$ it holds

$$\mathrm{Hom}_{\mathrm{Sets}}(-, \mathfrak{U})_{\mathrm{diag}} \longrightarrow \mathrm{Hom}_{\mathrm{Sets}}(-, \mathfrak{U})_{\mathrm{cart}} \rightthreetimes X_{\sigma\eta} \longrightarrow \bot$$

– A metric space X is complete iff each Cauchy ultrafilter converges, i.e. for each ultrafilter $\mathfrak{U}$ it holds

$$\varnothing \longrightarrow \mathrm{Hom}_{\mathrm{Sets}}(-, \mathfrak{U})_{\mathrm{cart}} \rightthreetimes X_{\sigma\eta}[+1] \longrightarrow X_{\sigma\eta}$$

– A map $X \longrightarrow Y$ of topological spaces is proper iff for each ultrafilter $\mathfrak{U}$ it holds

$$\varnothing \longrightarrow \mathrm{Hom}_{\mathrm{Sets}}(-,\mathfrak{U})_{\mathrm{diag}} \rightthreetimes X_{\mathfrak{T}}[+1] \longrightarrow X_{\mathfrak{T}} \vee_{X_{\mathfrak{T}}[+1]} Y_{\mathfrak{T}}[+1]$$

This lifting property is equivalent to [Bourbaki, I§10,2,Theorem 1d]:

Theorem I. ***Let*** $f: \mathrm{X} \to \mathrm{Y}$ ***be a continuous mapping. Then the following four statements are equivalent:***

a) f ***is proper.***

d) ***If*** $\mathfrak{U}$ ***is an ultrafilter on*** X ***and if*** $y \in \mathrm{Y}$ ***is a limit point of the ultrafilter base*** $f(\mathfrak{U})$, ***then there is a limit point*** x ***of*** $\mathfrak{U}$ ***such that*** $f(x) = y$.

4.11.2. Concise reformulations in terms of iterated orthogonals/Quillen negations.— The reformulations above lead to a concise expression in terms of iterated Quillen negations/orthogonals.

Consider the two element set $\{o, 1\}$ and let $\{o \leftrightarrow 1\}$ denote the filter with the unique neighbourhood on this set. Let $\{o < 1\}$ and $\{o > 1\}$ denote the filters on this set generated by $\{o\}$, resp. $\{1\}$.

For a filter F, let F_{cart} denote the simplicial set $\mathrm{Hom}_{\mathrm{Sets}}(-,F)_{\mathrm{cart}}$, and let $\sqcup$ denote the disjoint union (equivalently, coproduct).

Finally, let $\{o < 1\}_{\mathrm{cart}} \sqcup \{o > 1\}_{\mathrm{cart}} \longrightarrow \{o \leftrightarrow 1\}_{\mathrm{cart}}$ denote the obvious map.

Exercise 4.11.2.1. — – A topological space K is quasi-compact iff

$$K_{\mathfrak{T}}[+1] \longrightarrow K_{\mathfrak{T}} \in (\{o<1\}_{\mathrm{cart}} \sqcup \{o>1\}_{\mathrm{cart}} \longrightarrow \{o \leftrightarrow 1\}_{\mathrm{cart}})^{\mathrm{lr}}$$

– A map $X \longrightarrow Y$ of topological spaces is proper iff

$$X_{\mathfrak{T}}[+1] \longrightarrow X_{\mathfrak{T}} \vee_{X_{\mathfrak{T}}[+1]} Y_{\mathfrak{T}}[+1] \in (\{o<1\}_{\mathrm{cart}} \sqcup \{o>1\}_{\mathrm{cart}} \longrightarrow \{o \leftrightarrow 1\}_{\mathrm{cart}})^{\mathrm{lr}}$$

(Hint: First check that a filter F is an ultrafilter iff

$$\varnothing \longrightarrow F_{\mathrm{diag}} \rightthreetimes \{o<1\}_{\mathrm{cart}} \sqcup \{o>1\}_{\mathrm{cart}} \longrightarrow \{o \leftrightarrow 1\}_{\mathrm{cart}}$$

(the lifting property means that the preimage of either o or 1 is a neighbourhood). This is enough for the 'if' implication.)

Exercise 4.11.2.2. — Check whether the following holds for a reasonable class of metric spaces. A metric space is complete iff

$$M_{\sigma\eta}[+1] \longrightarrow M_{\sigma\eta} \in \{\mathbb{R}_{\sigma\eta}[+1] \longrightarrow \mathbb{R}_{\sigma\eta}\}^{\mathrm{lr}}$$

where $\mathbb{R}$ denotes the real line with the usual metric.

***Exercise 4.11.2.3* (Taimanov theorem).** — Show that a map f between T_4 topological spaces is proper iff in *Top* $f \in ((\{0\} \longrightarrow \{0 \to 1\})^{\mathrm{r}}_{<5})^{\mathrm{lr}}$. See [Gavrilovich-Pimenov,§2.2] for explanations.

Exercise(todo) 4.11.2.4. — Check whether the following reformulations of topological properties hold in the category of topological spaces. Can these expressions be interpreted in ʑ𝒫 ? What would they mean for topological spaces ?

For example, start by calculating the iterated orthogonals/negations below both in *Top* and ʑ𝒫 using the embedding of *Top* into ʑ𝒫. See [Gavrilovich-Pimenov,§5,3] for the explnatation of the notation and a list of other examples of topological properties expressed in terms of iterated Quillen orthogonals/negation.

- `{{}-->{o}}^r` is the class of surjections.
- `{{}-->{o}}^rr == {{x<->y->c}-->{x=y=c}}^l` is the class of subsets, i.e. maps of form $X \subset Y$ (where the topology on X is induced from Y)
- `{{}-->{o}}^rl` is the class of maps of form $X \longrightarrow X \sqcup D$ where D is discrete
- $\{\bullet\} \longrightarrow X \in$ `{{}-->{o}}^rll` iff X is connected and non-empty
- `{{}-->{o}}^l` $= \{A \longrightarrow B : A \neq \varnothing \text{ or } A = B = \varnothing\}$
- `{{}-->{o}}^lr` $= \{\varnothing \longrightarrow Y : \ Y \text{ arbitrary}\}$
- `{{}-->{o}}^lrr` is the class of maps which admit a section
- `{{c}-->{o->c}}^l` is the class of maps with dense image
- `{{x,y}-->{x=y}}^r == {{x<->y}-->{x=y}^l` is the class of injective maps
- `{{x<->y->c}-->{x<->y=c}}^l == {{c}-->{o->c}}^lr` is the class of closed subsets
- `{{x<->y<-c}-->{x<->y=c}}^l` is the class of open subsets
- $X \longrightarrow \{\bullet\} \in$ `{{a<-b->c<-d->e}-->{b=c=d}}^l` iff X is normal (T4)
- (Tietze lemma) $\mathbb{R} \longrightarrow \{\bullet\} \in$ `{{a<-b->c<-d->e}-->{b=c=d},{a<-b->c}-->{a=b=c}}^lr`
- in *Top* the following expression means something similar to the Urysohn lemma without taking care of the necessary(!) conditions like being first countable:

```
R-->{a<-b->c} (- {{a<-b->c<-d->e}-->{b=c=d}}^lr
```

Exercise(todo) 4.11.2.5. — Being proper can also be defined as "universally closed". Formulate an analogue of this definition in ʑ𝒫. The following steps may be of use.

– Reformulate the condition that a map of topological spaces is closed in terms of neighbourhoods. Namely, a map $f : X \longrightarrow Y$ is closed iff every system $U_x \ni x$ of neighbourhoods there exist a system $V_y \ni y$ of neighbourhoods such that $f^{-1}(V_y) \subset \bigcup_{f(x)=y} U_x$, i.e.
  - $f(x') \in V_y$ implies that $x' \in U_x$ for some x such that $f(x) = y$

– Reformulate the tube lemma in a similar manner.

– Rewrite the above in terms of the simplicial neighbourhoods $f_{\mathrm{T}} : X_{\mathrm{T}} \longrightarrow Y_{\mathrm{T}}$.

– Do the same for metric spaces.

– Ponder the syntactic similarity of the reformulations above to the characterisation of non-forking in terms of indiscernible sequences [Tent-Ziegler, 7.1.5] and to Definition 4.6.2.2 of neighbourhood structure of the mapping space.

Question 4.11.2.6. — Find a compact proper definition of compact spaces and proper maps. Note that you probably want the following to be examples of compact spaces: (i) function spaces in §4.11.5 coming from the Arzela-Ascoli theorems (ii) Stone spaces of indiscernible sequences in a model in §5.2.0.8.

4.11.3. Convergence of sequences of functions. — Here we reformulate various notions of uniform convergence of a family of functions as saying that a morphism in $\mathrm{ɀ}\varphi$ is well-defined.

Let $\{\mathbb{N}\}$ denote the trivial filter on $\mathbb{N}$ with a unique neighbourhood $\mathbb{N}$ itself, and $\mathbb{N}_{cofinite}$ denote the filter of cofinite subsets of $\mathbb{N}$.

A sequence $(f_i)_{i\in\mathbb{N}}$ of functions $f_i : X \longrightarrow M$ from a topological space X to a metric space M is *equicontinuous* if either of the following equivalent conditions holds:

– for every $x \in X$ and $\varepsilon > 0$, there exists a neighbourhood U of x such that $d_Y(f_i(x'), f_i(x)) \leq \varepsilon$ for all $i \in \mathbb{N}$ and $x' \in U$

– the map $X_{\mathrm{T}} \times \{\mathbb{N}\}_{\mathrm{const}} \longrightarrow M_{\sigma\eta}$, $(x,i) \longmapsto f_i(x)$ is well-defined

– the map $X_{\mathrm{T}} \times (\mathbb{N}_{cofinite})_{\mathrm{const}} \longrightarrow M_{\sigma\eta}$, $(x,i) \longmapsto f_i(x)$ is well-defined

– the map $X_{\mathrm{T}} \times \{\mathbb{N}\}_{\mathrm{diag}} \longrightarrow M_{\sigma\eta}$, $(x,i) \longmapsto f_i(x)$ is well-defined

– the map $X_{\mathrm{T}} \times (\mathbb{N}_{cofinite})_{\mathrm{diag}} \longrightarrow M_{\sigma\eta}$, $(x,i) \longmapsto f_i(x)$ is well-defined

If $X = (X, d_X)$ is also a metric space, replacing X_{T} by $X_{\sigma\eta}$ above gives us the notion of being *uniformly equicontinuous.* The family f_i is *uniformly equicontinuous* iff either of the following equivalent conditions holds:

- for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $d_Y(f_i(x'), f_i(x)) \le \varepsilon$ for all $i \in \mathbb{N}$ and $x', x \in X$ with $d_X(x, x') \le \delta$
- the map $X_{\sigma\eta} \times \{\mathbb{N}\}_{\text{const}} \longrightarrow M_{\sigma\eta}$, $(x, i) \longmapsto f_i(x)$ is well-defined
- the map $X_{\sigma\eta} \times (\mathbb{N}_{cofinite})_{\text{const}} \longrightarrow M_{\sigma\eta}$, $(x, i) \longmapsto f_i(x)$ is well-defined
- the map $X_{\sigma\eta} \times \{\mathbb{N}\}_{\text{diag}} \longrightarrow M_{\sigma\eta}$, $(x, i) \longmapsto f_i(x)$ is well-defined
- the map $X_{\sigma\eta} \times (\mathbb{N}_{cofinite})_{\text{diag}} \longrightarrow M_{\sigma\eta}$, $(x, i) \longmapsto f_i(x)$ is well-defined

Replacing diag by cart gives us the notion of *uniformly Cauchy*. The family f_i is *uniformly Cauchy* iff either of the following equivalent conditions holds:

- for every $\varepsilon > 0$ there exists a $\delta > 0$ and $N > 0$ such that $d_Y(f_i(x'), f_j(x)) \le \varepsilon$ for all $i, j > N$ and $x', x \in X$ with $d_X(x, x') \le \delta$.
- the map $X_{\sigma\eta} \times \{\mathbb{N}\}_{\text{cart}} \longrightarrow M_{\sigma\eta}$, $(x, i) \longmapsto f_i(x)$ is well-defined
- the map $X_{\sigma\eta} \times (\mathbb{N}_{cofinite})_{\text{cart}} \longrightarrow M_{\sigma\eta}$, $(x, i) \longmapsto f_i(x)$ is well-defined

An equicontinuous family f_i *converges* to a function f_∞ iff either of the following equivalent conditions holds:

- for every $\varepsilon > 0$ and $x \in X$ there exists a $\delta > 0$ and $N > 0$ such that $d_Y(f_\infty(x'), f_i(x')) \le \varepsilon$ for all $i > N$ and $x \in X$ with $d_X(x, x') \le \delta$.
- the following is a well-defined diagram:

$$\begin{array}{ccc} X_{\top}[+1] \times (\mathbb{N}_{cofinite})_{\text{diag}} & \dashrightarrow & M_{\sigma\eta}[+1] \\ \downarrow{\scriptstyle [-1]\times\text{id}} & & \downarrow{\scriptstyle [-1]\times\text{id}} \\ X_{\top} \times (\mathbb{N}_{cofinite})_{\text{diag}} & \xrightarrow{(f_1, f_2, \ldots)} & M_{\sigma\eta} \end{array}$$

  where the bottom morphism is $(x, i) \longmapsto f_i(x)$, $i > 0$, and the top morphism is $(x_0, i) \longmapsto f_\infty(x_0)$ $(x, i) \longmapsto f_i(x)$, $i > 0$.

  Moreover, the top morphism is necessarily of this form for some function $f_\infty : X \longrightarrow M$.

An uniformly equicontinuous family f_i *uniformly converges* to a function f_∞ iff either of the following equivalent conditions holds:

- for every $\varepsilon > 0$ there exists $N > 0$ such that $d_Y(f(x), f_i(x)) \le \varepsilon$ for all $i > N$ and $x \in X$.
- the following is a well-defined diagram:

$$\begin{array}{ccc} X_{\sigma\eta}[+1] \times (\mathbb{N}_{cofinite})_{\text{diag}} & \dashrightarrow & M_{\sigma\eta}[+1] \\ \downarrow{\scriptstyle [-1]\times\text{id}} & & \downarrow{\scriptstyle [-1]\times\text{id}} \\ X_{\sigma\eta} \times (\mathbb{N}_{cofinite})_{\text{diag}} & \xrightarrow{(f_1, f_2, \ldots)} & M_{\sigma\eta} \end{array}$$

where the bottom morphism is $(x,i) \longmapsto f_i(x)$, $i > 0$, and the top morphism is $(x_0,i) \longmapsto f_\infty(x_0) \quad (x,i) \longmapsto f_i(x)$, $i > 0$.

Moreover, the top morphism is necessarily of this form for some function $f_\infty : X \longrightarrow M$.

A uniformly equicontinuous family f_i has a subsequence which uniformly converges to a function f iff for some ultrafilter $\mathfrak{U}$ extending the filter $\mathbb{N}_{cofinite}$ of cofinite subsets either of the following equivalent conditions holds:

- there is a sequence $(i_j)_{j\in\mathbb{N}}$ such that for every $\varepsilon > 0$ there exists $N > 0$ such that $d_Y(f_\infty(x), f_i(x)) \leq \varepsilon$ for all $i > N$ and $x \in X$.
- the following is a well-defined diagram for some ultrafilter $\mathfrak{U}$ extending the filter $\mathbb{N}_{cofinite}$ of cofinite subsets:

$$\begin{array}{ccc} X\boldsymbol{\sigma\eta}[+1] \times \mathfrak{U}_{\text{diag}} & \dashrightarrow & M\boldsymbol{\sigma\eta}[+1] \\ \downarrow{\scriptstyle [-1]\times\text{id}} & & \downarrow{\scriptstyle [-1]\times\text{id}} \\ X\boldsymbol{\sigma\eta} \times (\mathbb{N}_{cofinite})_{\text{diag}} & \xrightarrow{(f_1,f_2,\ldots)} & M\boldsymbol{\sigma\eta} \end{array}$$

where the bottom morphism is $(x,i) \longmapsto f_i(x)$, $i > 0$, and the top morphism is $(x_0,i) \longmapsto f_\infty(x_0) \quad (x,i) \longmapsto f_i(x)$, $i > 0$.

Moreover, the top morphism is necessarily of this form for some function $f_\infty : X \longrightarrow M$.

Exercise(todo) 4.11.3.1. — M.Dubashinsky suggested it might be possible to attempt to reformulate Γ-convergence in these terms. "The natural setting of Γ-convergence are lower semicontinuous functions" [Braides, I,p.19], and this suggests that a Γ-convergent sequence of functions $f_i : X \longrightarrow \mathbb{R}$ something like a morphism $\mathbb{N}_{\leq} \times X \longrightarrow \mathbb{R}_{\leq}$ where $\mathbb{N}_{\leq} := \mathrm{Hom}_{\leq}(-, \mathbb{N}_{\leq})$ and $\mathbb{R}_{\leq} := \mathrm{Hom}_{\leq}(-, \mathbb{R}_{\leq})$ are ssets of non-decreasing (or non-increasing?) sequences equipped with appropriate filters, or perhaps a simplex of an appropriate Skorokhod mapping space.

4.11.4. Arzela-Ascoli theorems as diagram chasing. — We now see that the Arzela-Ascoli theorem can be reformulated in terms of diagram chasing. The following exercise is a summary of the reformulations above.

Exercise 4.11.4.1. — Check that the following diagrams represent the reformulation of the following Arzela-Ascoli theorem.

Theorem (Arzela-Ascoli). Let M be a complete metric space and X be a compact metric space, and let $f_i : X \longrightarrow Y$, $i \in \mathbb{N}$, be a sequence of functions. Then the following are equivalent:

(i) $(f_i)_{i\in\mathbb{N}}$ has a convergent subsequence.
(ii) $(f_i)_{i\in\mathbb{N}}$ is pointwise precompact and equicontinuous.
(iii) $(f_i)_{i\in\mathbb{N}}$ is pointwise precompact and uniformly equicontinuous.

As diagram chasing:

– (X is compact) it holds

$$X_{\boldsymbol{\sigma\eta}}[+1] \longrightarrow X_{\boldsymbol{\sigma\eta}} \in (\{o < 1\}_{\text{cart}} \sqcup \{o > 1\}_{\text{cart}} \longrightarrow \{o \leftrightarrow 1\}_{\text{cart}})^{\text{lr}},$$

or, equivalently, for each ultrafilter $\mathfrak{U}$ it holds

$$\varnothing \longrightarrow \mathfrak{U}_{\text{diag}} \rightthreetimes X_{\boldsymbol{\sigma\eta}}[+1] \longrightarrow X_{\boldsymbol{\sigma\eta}}$$

– (M is complete) for each ultrafilter $\mathfrak{U}$ it holds

$$\varnothing \longrightarrow \mathfrak{U}_{\text{cart}} \rightthreetimes M_{\boldsymbol{\sigma\eta}}[+1] \longrightarrow M_{\boldsymbol{\sigma\eta}}$$

or perhaps

$$M_{\boldsymbol{\sigma\eta}}[+1] \longrightarrow M_{\boldsymbol{\sigma\eta}} \in (\mathbb{R}_{\boldsymbol{\sigma\eta}}[+1] \longrightarrow \mathbb{R}_{\boldsymbol{\sigma\eta}})^{\text{lr}}$$

– ($(f_i)_{i\in\mathbb{N}}$ is pointwise precompact, i.e. for each point x there is a subsequence such that $(f_{i_j}(x))_j$ converges)
for each ultrafilter $\mathfrak{U}$ it holds

$$\begin{array}{ccc} X_{\text{diag}}[+1] \times \mathfrak{U}_{\text{diag}} & \dashrightarrow & M_{\boldsymbol{\sigma\eta.}}[+1] \\ \downarrow & & \downarrow \\ X_{\boldsymbol{T}} \times (\mathbb{N}_{cofinite})_{\text{diag}} & \xrightarrow{(f_i)} & M_{\boldsymbol{\sigma\eta.}} \end{array}$$

where $X_{\text{diag}} = (X_{\text{discrete}})_{\boldsymbol{\sigma\eta.}}$ denotes X equipped with the filter of diagonals, i.e. a subset is a neighbourhood iff it contains the diagonal.

– ($(f_i)_{i\in\mathbb{N}}$ being uniformly equicontinuous imply they converge uniformly)
for each ultrafilter $\mathfrak{U}$ it holds

$$\begin{array}{ccc} X_{\boldsymbol{\sigma\eta}}[+1] \times \mathfrak{U}_{\text{diag}} & \dashrightarrow & M_{\boldsymbol{\sigma\eta.}}[+1] \\ \downarrow & & \downarrow \\ X_{\boldsymbol{\sigma\eta.}} \times (\mathbb{N}_{cofinite})_{\text{diag}} & \xrightarrow{(f_i)} & M_{\boldsymbol{\sigma\eta.}} \end{array}$$

– ($(f_i)_{i\in\mathbb{N}}$ being equicontinuous imply being uniformly equicontinuous)

$$\begin{array}{ccc} X_{\mathrm{T}} \times (\mathbb{N}_{cofinite})_{\mathrm{diag}} & \longrightarrow & M_{\sigma\eta} \\ \downarrow & & \downarrow \\ X_{\sigma\eta} \times (\mathbb{N}_{cofinite})_{\mathrm{diag}} & \dashrightarrow & M_{\sigma\eta} \end{array}$$

4.11.5. Arzela-Ascoli theorems; compactness of function spaces. — Above we saw that to give a convergent sequence of functions is the same as to give a certain morphism in $\iota\varphi$, and that to take the limit of the sequence is to "lift" this morphism along shifts $X[+1] \xrightarrow{[-1]} X$ and $M[+1] \xrightarrow{[-1]} M$. This makes various Arzela-Ascoli theorems on pre-compactness of function spaces reminiscent of base change.

TODO 4.11.5.1. — Give a category theoretic approach to various Arzela-Ascoli theorems including the Prokhorov theorem on tightness of measures and Γ-convergence.

The following exercises may be of use.

Exercise(todo) 4.11.5.2. — Calculate the Skorokhod mapping spaces related to the morphisms above. Are these spaces pre-compact? complete? Is the subspace of continuous functions relatively compact within the mapping space under reasonable assumptions, i.e. is the subset

$$\mathrm{Hom}_{\iota\varphi}(X,Y) \subset \underline{\mathrm{Hom}}_{\iota\varphi}(X,Y)$$

relatively compact, and in what precise meaning ?

Calculate the following Skorokhod mapping spaces:

– equicontinuous:
  - $\underline{\mathrm{Hom}}^{sSets}_{\iota\varphi}(X_{\mathrm{T}} \times \{\mathbb{N}\}_{\mathrm{const}}, M_{\sigma\eta})$
  - $\underline{\mathrm{Hom}}^{sSets}_{\iota\varphi}(X_{\mathrm{T}} \times (\mathbb{N}_{cofinite})_{\mathrm{const}}, M_{\sigma\eta})$
  - $\underline{\mathrm{Hom}}^{sSets}_{\iota\varphi}(X_{\mathrm{T}} \times \{\mathbb{N}\}_{\mathrm{diag}}, M_{\sigma\eta})$
  - $\underline{\mathrm{Hom}}^{sSets}_{\iota\varphi}(X_{\mathrm{T}} \times (\mathbb{N}_{cofinite})_{\mathrm{diag}}, M_{\sigma\eta})$

– uniformly equicontinuous:
  - $\underline{\mathrm{Hom}}^{sSets}_{\iota\varphi}(X_{\sigma\eta} \times \{\mathbb{N}\}_{\mathrm{const}}, M_{\sigma\eta})$
  - $\underline{\mathrm{Hom}}^{sSets}_{\iota\varphi}(X_{\sigma\eta} \times (\mathbb{N}_{cofinite})_{\mathrm{const}}, M_{\sigma\eta})$
  - $\underline{\mathrm{Hom}}^{sSets}_{\iota\varphi}(X_{\sigma\eta} \times \{\mathbb{N}\}_{\mathrm{diag}}, M_{\sigma\eta})$

  - $\underline{\mathrm{Hom}}^{sSets}_{\text{ɫφ}}(X_{\text{ση}} \times (\mathbb{N}_{cofinite})_{\mathrm{diag}}, M_{\text{σʅ}})$
- uniformly Cauchy:
  - $\underline{\mathrm{Hom}}^{sSets}_{\text{ɫφ}}(X_{\text{ση}} \times \{\mathbb{N}\}_{\mathrm{cart}}, M_{\text{σʅ}})$
  - $\underline{\mathrm{Hom}}^{sSets}_{\text{ɫφ}}(X_{\text{ση}} \times (\mathbb{N}_{cofinite})_{\mathrm{cart}}, M_{\text{σʅ}})$

Exercise(todo) 4.11.5.3. — In measure theory Prokhorov's theorem relates tightness of measures to relative compactness (and hence weak convergence) in the space of probability measures. Reformulate the Prokhorov's theorem and give a uniform approach to both Prokhorov theorem and Arzela-Ascoli theorems.

Exercise(todo) 4.11.5.4. — Rewrite the Arzela-Ascoli theorems entirely in terms of iterated orthogonals/Quillen negations, taking M2-decompositions (i.e. (co)fibrant replacement), and, more generally, rules for manipulating labels on morphisms.

5. Model theory

5.1. Ramsey theory. — Let X : sSet be a simplicial set, and $c : X_n^{\mathrm{nd}} \longrightarrow C$ be a colouring of the set X_n^{nd} of non-degenerate n-simplices, i.e. an arbitrary function defined on the set of non-degenerate simplices of X of dimension n. Call a simplex $s : X_N$ c-homogeneous iff all its non-degenerate faces of dimension n have the same c-colour. Let $c(X)$ be the subsset of X consisting of c-homogeneous simplices in X.

***Exercise 5.1.0.5* (Ramsey theorem).** —
- Verify that $c(X)$: sSet is indeed a well-defined simplicial set; it is a disjoint union of subssets corresponding to different c-colours.
- (Ramsey theorem) Let the set of c-colours be finite. If X : sSet has non-degenerate simplices of arbitrarily high dimension, then so does $c(X)$. In more detail, if for unboundedly many N there is a simplex in X_N which is not a face of a simplex of lower dimension, then the same holds for $c(X)$.
- Take X : sSet to be $\mathrm{Hom}_{\mathrm{Sets}}(-, S)$ where S is an infinite set. Verify that the item above is the usual statement of Ramsey theorem: for each colouring of subsets of S of size n, there is an arbitrarily large subset of S such that all its subsets of size n have the same colour.

TODO 5.1.0.6. — Ponder the discussion of Ramsey theory-type theorems in [MLGrovov], [Gromov2014].

5.2. Indiscernible sequences in model theory. — Ramsey theory provides a basic tool in model theory known as the *indiscernible sequences.*

Definition 5.2.0.7 **(L-indiscernability neighbourhood structure on a model M^I)**

Let M be a model in a language L, and let $I_\leq$ be a linear order. For a n-ary formula φ of L, we say that a sequence $(a_i) \in M^I$ is *φ-homogeneous* iff

> $M \vDash \varphi(a_{i_1}, ..., a_{i_n}) \equiv \varphi(a_{j_1}, ..., a_{j_n})$ whenever $i_1 < i_2 < ... < i_n$, $j_1 < j_2 < ... < j_n$, and all the a_{i_k}'s are distinct, and all the a_{j_k}'s are distinct, i.e. $a_{i_k} \neq a_{i_l}$ and $a_{j_k} \neq a_{j_l}$ whenever $1 \leq k < l \leq n$

For a type π, a sequence is *π-homogeneous* iff it is φ-homogeneous for each formula in π. A *φ-neighbourhood of the diagonal in M^I* is the subset consisting of all the φ-homogeneous sequences. The *L-indiscernability filter on M^I* is generated by the φ-neighbourhoods of the diagonal, for φ a formula of L.

The requirement that all the a_{i_k}'s are distinct, and all the a_{j_k}'s are distinct, is needed to show that a non-decreasing map $f : J \longrightarrow I$ induces a continuous map $f^* : M^I \longrightarrow M^J$ of indiscernability filters.

Definition 5.2.0.8 **(The Stone space of a model.)**

Let M be a model in a language L, and let $A \subset M$ be a subset. Also assume that M is $\operatorname{card}(A)^+$-saturated. Call the *Stone space* $\mathcal{S}_1^M(A) = \mathcal{S}_1^M(A)$ *of a model M with parameters A* the sset $n_\leq \longmapsto \mathrm{Hom}_{\mathrm{Sets}}(n, M)$ where $M^n = \mathrm{Hom}_{\mathrm{Sets}}(n, M)$ is equipped with the $L(A)$-indiscernability neighbourhood structure. Similarly define $\mathcal{S}_n^M(A)$ as the Stone space of n-tuples.

The following exercise is based on the syntactic similarity of the characterisation of non-forking in terms of indiscernible sequences [Tent-Ziegler, Lemma 7.1.5], the reformulation Exercise 4.11.2.5 of 'being a closed map' in terms of neighbourhoods, and of Definition 4.6.2.2 of neighbourhood structure of the mapping space.

Exercise 5.2.0.9. — Check that the usual Stone space $S_1(A)$ is the Hausdorff quotient of the topologisation of ʑφ-Stone space $\mathcal{S}_1^M(A)$ whenever M is $\operatorname{card}(A)^+$-saturated.

TODO 5.2.0.10. — – Is $\mathcal{S}_n^M(A)$ quasi-compact? complete?

– Is the map $ረ_n^M(B) \longrightarrow ረ_n^M(A)$ proper, cf. Exercise 4.11.2.5 ? Is this related to the fact that every type has a non-forking extension?
– Interpret [Tent-Ziegler, Lemma 7.1.5] as a property of neighbourhoods of Stone spaces, e.g. that a projection of a certain subset is closed, cf. Exercise 4.11.2.5.
– Are there are non-trivial maps $ረ_n^M(A) \longrightarrow ረ_n^N(B)$ for different models M and N; what does it mean model-theoretically?
– More generally, ponder if ኔቀ-Stone spaces allow to define interesting homological or homotopical invariants of models.

TODO 5.2.0.11. — – Check whether a model M is stable iff $ረ_n^M(A)$ is symmetric for every $A \subset M$, i.e. $ረ_n^M(A) : \Delta^{\mathrm{op}} \longrightarrow ቀ$ factors as $ረ_n^M(A) : \Delta^{\mathrm{op}} \longrightarrow FiniteSets^{\mathrm{op}} \longrightarrow ቀ$
– Are there similar characterisations of e.g. simple or NIP theories in terms of their Stone spaces?

TODO 5.2.0.12. — Do these Stone spaces allow to express neatly the theory of forking?
– For example, do pushforwards mentioned in [Simon, Exercise 9.12 (distality of T^{eq})] are indeed pushforwards in ኔቀ?
– Reformulate the definition [Simon, Def. 9.28] of distality in terms of endomorphisms of ኔቀ-Stone spaces or similar objects of ኔቀ.

> Definition 9.28. Let I be any infinite sequence of tuples and A a base set of parameters. We say that I is distal over A if whenever J, a, $B \supseteq A$ satisfy:
> · J is indiscernible and realizes the EM-type of I;
> · $J = J_1 + (a) + J_2$, where both J_1 and J_2 are infinite with no endpoints;
> · the sequence $J_1 + J_2$ is indiscernible over B;
> then the sequence $J = J_1 + (a) + J_2$ is indiscernible over B.

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This continues [Gavrilovich-Pimenov]; see also acknowledgements there.

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