~~As we have already said,~~ a topological structure on a set enables one to give an exact meaning to the phrase "whenever x is sufficiently near a, x has the property $P(x)$". But, apart from the situation in which a "distance" has been defined, it is not clear what meaning ought to be given to the phrase "every pair of points x, y which are sufficiently near each other has the property $P(x, y)$", `or, more generally, to the phrase`

> `''every` n`-tuple of points` $x_1, x_2, ..., x_n$ `which are sufficiently near each other has the property` $P(x_1, ..., x_n)$`''`

since *a priori* we have no means of comparing the neighbourhoods of two different points. Now the notion of a pair of points near to each other arises frequently in classical analysis (for example, in propositions which involve uniform continuity,

> `and, in model theory, arises frequently the notion of a tuple of points satisfying a formula (for example, in propositions which involve indiscernible sequences and NTP-trees).`

It is therefore important that we should be able to give a precise meaning to this notion in full generality, and we are thus led to define structures which are richer than topological structures, namely uniform structures, `or, more generally`

$$\texttt{functors } FiniteLinearOrders^{\mathrm{op}} \longrightarrow \mathcal{F}ilters$$

> `from the category of finite linear orders to the category of filters.`

They are the subject of ~~Chapter II~~ `this talk`.

Much due to Martin Bays and Assaf Hasson.

If we start from the physical concept of approximation, it is natural to say that

> a subset A of a set E is a neighbourhood of an element a of A whenever we replace a by an element that "approximates" a, this new element will also belong to A, provided of course that the "error" involved is small enough;

or, in other words, if all the points of E which are "sufficiently near" a belong to A.

> This definition is meaningful whenever precision can be given to the concept of sufficiently small error or of an element sufficiently near another.

We consider

- sets of (what we think as) "errors"
- a neighbourhood is a subset containing "all errors small enough"

Metric space. A *neighbourhood* $\varepsilon = U \subset X \times X$ iff for some $\epsilon > 0$

$$\varepsilon = U \supset \{(x, y) : dist(x, y) < \epsilon\}$$

"$x \rightsquigarrow y$ up to error ϵ", hence a "give[s] a precise meaning to" "a pair of points near to each other"

Topological space. A subset $\varepsilon \subset X \times X$ is *a neighbourhood* iff for some open cover $U_x \ni x$ of X

$$\varepsilon \supset \bigcup \{x\} \times U_x = \{(x, y) : x \in X, y \in U_x\}$$

"$x \rightsquigarrow y$ up to error ϵ_x depending on x", hence "no means of comparing the neighbourhoods of two different points" *"consistency" structure:* elements of a k-φ-consistent tuple are "very near each other" or "small"

DEFINITION 7.2.1. 1. A formula $\varphi(x, y)$ has the *tree property* (with respect to k) if there is a tree of parameters $(a_s \mid \emptyset \neq s \in {}^{<\omega}\omega)$ such that:
a) For all $s \in {}^{<\omega}\omega$, $(\varphi(x, a_{si}) \mid i < \omega)$ is k-inconsistent.
b) For all $\sigma \in {}^{\omega}\omega$, $\{\varphi(x, a_s) \mid \emptyset \neq s \subseteq \sigma\}$ is consistent.
2. A theory T is *simple* if there is no formula $\varphi(x, y)$ with the tree property.

$\varepsilon = U \subset M^n$ *is a neighbourhood* iff

$$\varepsilon = U \supset \{(a_1, ..., a_n) : M \vDash \exists x \bigwedge_i \varphi(x, a_i)\}$$

Item b) is a precise meaning to the phrases

- "elements on the same branch are very near each other"
- "tuples on the same branch are small"

Indiscernability structure: in an indiscernible sequence, last few elements "approximate" a first few

$\varepsilon \subset M^n$ *is a neighbourhood* iff for some formula φ

- $(a_1, .., a_n) \in \varepsilon$ whenever any subsequence of distinct elements of $(a_1, ..., a_n)$ is φ-indiscernible (or can be continued to an infinite φ-indiscernible sequence,...)

On M — indiscrete: each 1-sequence is indiscernible.

On $M \times M$ — the **uniform** *Stone space of 1-types:* generated by

$$\{(x, y) : \varphi(x) \leftrightarrow \varphi(y)\}, \quad \varphi \text{ is a formula}$$

(a 2-sequence (a_1, a_2) is indiscernible iff $\mathrm{tp}(a_1) = \mathrm{tp}(a_2)$).

Examples.

An equivalence relation. Let $M := (|M|; \approx)$ be a set with an equivalence relation with infinitely many infinite equivalence classes. The indiscernability filter on $|M|^n$ is generated by

$$\{(a_1, ..., a_n) : a_1 \approx a_2 \approx ... \approx a_n \vee \bigwedge_{1 \leq i \leq j \leq n} a_i \not\approx a_j\}$$

Dense linear order. Let $M := (\mathbb{Q}; \leq)$ be a model of DLO. A neighbourhood is a subset containing

- all monotone tuples

$(\mathbb{Q}; +)$. The indiscernability filter on $|\mathbb{Q}|^n$ is generated by

$$\{(a_1, ..., a_n) : (\text{ distinct elements of } a_1, ..., a_n \text{ are lin.indep.})$$
$$\vee(\text{ at most two distinct elements })\} \quad // \text{ not for } \infty\text{-indi}$$

A neighbourhood is a subset containing tuples such that either

- distinct elements are $\mathbb{Q}$-linearly independent
- there are at most two distinct elements (not for ∞-indi)

$(\mathbb{C}; +, *)$. The indiscernability filter on $\mathbb{C}^n$ is generated by

$$\{(a_1, ..., a_n) : a_1 = a_2 = ... = a_n \vee (a_1, ..., a_n \text{ are alg.indep.})\}$$
$$\cup\{(a_1, ..., a_n) : \text{ solutions of an equation with Galois group the sym}$$

1. Continuous maps of filters

In a filter, call big subsets *neighbourhoods*. A map of filters is *continuous* iff

- the preimage of a big set is big
- the preimage of a neighbourhood is a neighbourhood.

Think of

- a filter as *a space with a notion of smallness*:

This enables the usual topological intuition/phrasing:

- a property holds *for all sufficiently small x* iff it holds "locally" on a neighbourhood.

Example (uniformly continuous). *Recall for a metric space X the filter on $X \times X$ is generated by*

$$\{(x,y) : dist(x,y) < \epsilon\}, \quad \epsilon > 0$$

Therefore: a map $f : X \longrightarrow Y$ induces a continuous map $X \times X \longrightarrow Y \times Y$ iff for each neighbourhood

$$\varepsilon := \{(y_1,y_2) \in Y \times Y : dist(y_1,y_2) < \epsilon\}$$

there is a neighbourhood

$$\delta := \{(x_1,x_2) : dist(x_1,x_2) < \delta\}$$

such that

$$f(\delta) \subset \varepsilon \qquad \textit{(as if in an old analysis textbook!)}$$

Think of the pair (x_1,x_2) as *a (sufficiently) small error*, or of x_2 as an *approximation* of x_1.

A sequence $(a_i)_{i\in\omega} \in Y$ is Cauchy iff the induced maps

- $(\omega^{\text{cofinite}} \times \omega^{\text{cofinite}} \longrightarrow Y \times Y$
- $(\{(i,j) : i < j, i,j \in \omega\}, \text{obvious filter}) \longrightarrow Y \times Y$

are continuous. Indeed, for each neighbourhood

$$\varepsilon := \{(y_1,y_2) \in Y \times Y : dist(y_1,y_2) < \epsilon\}$$

there is a neighbourhood

$$\delta := \{(i,j) : i,j > N\} \text{ or resp. } \delta := \{(i,j) : j \geq i > N\}$$

such that

$$a_\bullet(\delta) \subset \varepsilon \text{ i.e. } (a_i,a_j) \in \delta \qquad \text{(as if in an old analysis textbook!)}$$

A sequence $(a_i)_{i\in\omega} \in M$ is eventually (order) indiscernible iff the induced map

$$\{(i_1,..,i_n) : i_1 \leq ... \leq i_n, i_1,..,i_n \in \omega\}_{\text{“tail” filter}} \longrightarrow M^n$$

is continuous.

Indeed: for each formula φ and a neighbourhood

$$\varepsilon := (\varphi\text{-indiscernible sequences with repetitions})$$

there is $N \in \omega$ and a neighbourhood

$$\delta := \{(i_1,..,i_n) : N \leq i_1 \leq ... \leq i_n, i_1,..,i_n \in \omega\}$$

such that

$$a_\bullet(\delta) \subset \varepsilon$$

$(a_{i_1},...,a_{i_n})$ is φ-indiscernible with repetitions for $N \leq i_1 \leq ... \leq i_n$

A sequence $(a_i)_{i\in\omega} \in M$ *is eventually* **totally** *indiscernible* iff the induced map

$$(\omega^{\text{cofinite}})^n \longrightarrow M^n$$

is continuous.

Indeed: for each formula φ and a neighbourhood

$$\varepsilon := (\varphi\text{-indiscernible sequences with repetitions})$$

there is $N \in \omega$ and a neighbourhood

$$\delta := \{(i_1, .., i_n) : i_1, ...i_n > N, i_1, .., i_n \in \omega\}$$

such that

$$a_\bullet(\delta) \subset \varepsilon$$

$(a_{i_1}, ..., a_{i_n})$ is φ-indiscernible with repetitions for $i_1, ...i_n > N$

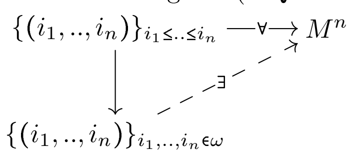
Each eventually indiscernible sequence is eventually totally indiscernible in terms of arrows: for each injective map $\omega \longrightarrow M$, if induced maps

$$\{(i_1, .., i_n) : i_1 \le ... \le i_n, i_1, .., i_n \in \omega\}^{\text{“tail” filter}} \longrightarrow M^n$$

are continuous for all n, they factor as

$$\{(i_1, .., i_n) : i_1 \le ... \le i_n, i_1, .., i_n \in \omega\}^{\text{“tail” filter}} \longrightarrow (\omega^{\text{cofinite}})^n \longrightarrow M^n$$

Same as a diagram (a Quillen lifting property/negation):

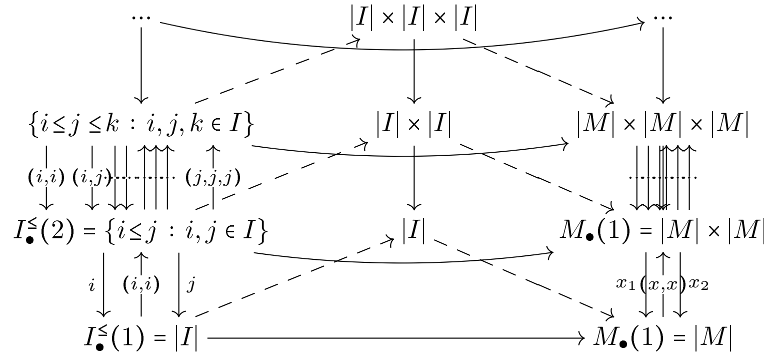


each eventually indiscernible sequence is eventually totally indiscernible

Now “collect” these data as functors: .

$\omega_\bullet^{\le tails}, \omega_\bullet^{tails}, M_\bullet : FiniteLinearOrders^{\text{op}} \longrightarrow \Phi ilters$

take an arbitrary linear order I instead of ω. The sequence $I_\bullet^{\le} \longrightarrow |I|_\bullet \longrightarrow M_\bullet$ “expands” to a large commutative diagram:



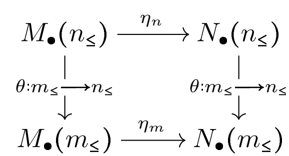
2. Definitions

Definition. Φ *is the category of filters and continuous maps.*

Definition. *The category* $s\Phi$ *of simplicial filters consists of functors*

$$M_\bullet : FiniteLinearOrders^{op} \longrightarrow \Phi ilters$$

A morphism $M_\bullet \longrightarrow N_\bullet$ *is a natural transformation: for each finite linear orders* $n_\le$, $m_\le$, *and a non-decreasing map* $\theta : m_\le \longrightarrow n_\le$ *the obvious diagram commutes:*



3. More intuition/philosophy

It is convenient to continue to use the language of geometry: thus the elements of a set on which ~~a "distance"~~ the notion of neighbourhood has been defined are called points, and the ~~set~~ functor itself is called a space.

We are thus led at last to the general concept of a generalised topological space, namely functors

$$FiniteLinearOrders^{\mathrm{op}} \longrightarrow \Phi ilters,$$

which does not depend on any preliminary theory of the ~~real numbers~~ syntax and language.

Benefits? We obtain statements in which there is no mention of ~~magnitude or distance~~ syntax or language.

- reformulations of stability and simplicity as a Quillen lifting property/negation
- Shelah representation of stable theories

Can formulate new questions, for example, as $Top \subset s\Phi$ is a full subcategory, may generalise the notion of

- compactness and locally trivial bundles

from topological spaces to $s\Phi$-spaces and ask

- is $M_\bullet$ compact iff M is stable ?
- what it mean for a map $M_\bullet \longrightarrow N_\bullet$ be locally trivial ?
- Quillen negation/lifting property trickery

Homotopy theory? Develop homotopy theory for generalised spaces and apply it to $M_\bullet$...

- spaces of indiscernible sequences $Maps(I^{\leq tails}, M_\bullet)$

4. Stability in $s\Phi$

"each indiscernible sequence is an indiscernible set"

Corepresented simplicial set. View a set $|I|$ as a functor $|I|_\bullet : FiniteLinearOrders^{\mathrm{op}} \longrightarrow \Phi ilters$:

$$|I|_\bullet(n_\leq) := |I|^n = Hom_{Sets}(n, |I|)$$

$$|I|^n \longrightarrow |I|^m,\ (x_1, ..., x_n) \longmapsto (x_{i_1}, ..., x_{i_m}), i_1 \leq .... \leq i_m$$

Equip $|I|^n$ with the indiscrete filter.

In more explicit notation:

$$\text{sets}: \quad |I|, \quad |I\times I|, \quad |I\times I\times I|, \quad ....$$

$$\text{filters}: \quad \{|I|\}, \{|I\times I|\},\ \{|I\times I\times I|\}....$$

$$\text{morphisms}: \quad |I| \xrightarrow{x\mapsto(x,x)} |I|\times|I| \underset{(x,y)\mapsto y}{\overset{(x,y)\mapsto x}{\longrightarrow}} |I|......$$

The generalised Stone space of a model. For each n, equip

$$|M_\bullet(n_\leq)| := M^n, \quad n > 0$$

$$M, \quad M\times M, \quad M\times M\times M, \quad ....$$

with the indiscernability filter:

$\varepsilon \subset M^n$ *is a neighbourhood* iff for some formula φ

- $(a_1, .., a_n) \in \varepsilon$ whenever any subsequence of distinct elements of $(a_1, ..., a_n)$ is φ-indiscernible (or can be continued to an infinite φ-indiscernible sequence,...)

Intuitions:

- in an indiscernible sequence, last few elements "approximate" a first few

An indiscernible set. A map $a : |I| \longrightarrow |M|$ induces a natural transformation $|I|_\bullet \longrightarrow M_\bullet$ iff the sequence $(a_i)_{i\in I}$ is *totally* indiscernible.

Indeed it says that $I^n \subset a^{-1}(\varepsilon)$ for each neighbourhood $\varepsilon \subset M^n$ as above.

An indiscernible sequence. Let $I^\leq$ be an order. Make $I^\leq_\bullet$ "remember" the ordering:

$$I^\leq_\bullet(n_\leq) \coloneqq Hom_{preorders}(n_\leq, I^\leq) = \{(x_1, ..., x_n) : x_1 \leq ... \leq x_n\}$$

$$I^\leq(n_\leq) \longrightarrow I^\leq(m_\leq),\ (x_1, ..., x_n) \longmapsto (x_{i_1}, ..., x_{i_m}), i_1 \leq .... \leq i_m$$

Equip $|I|^n$ with the indiscrete filter. In more explicit notation:

$$\text{sets}: \quad I \quad \{(a,b) \in I^2 : a \leq b\} \quad \{(a,b,c) \in I^3 : a \leq c\}$$

$$\text{filters}: \quad I \quad \{\{(a,b) \in I^2 : a \leq b\}\} \quad \{\{(a,b,c) \in I^3 : a \leq c\}\}$$

$$\text{morphisms}: \quad I \xrightarrow{x\mapsto(x,x)} \{(x,y) \in I \times I : x \leq y\} \underset{(x,y)\mapsto y}{\overset{(x,y)\mapsto x}{\longrightarrow}} I......$$

Now $I^\leq(n_\leq)$ consists only of *ordered* sequences, hence continuity — $I^\leq \subset a^{-1}(\varepsilon)$ for each neighbourhood ε — means an indiscernible *sequence.*

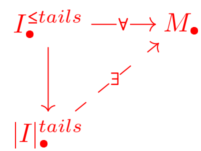
Each indiscernible sequence is an indiscernible set. Category theoretically in ɀφ: each injective map $I^\leq \longrightarrow M_\bullet$ factors as

$$I^\leq_\bullet \longrightarrow |I|_\bullet \longrightarrow M_\bullet$$

What about non-injective maps? Consider on the same underlying sets the *tail* filters $I^{\leq tails}_\bullet$ and $|I|^{tails}_\bullet$:

- a "tail" neighbourhood contains all tuples with large enough elements

Stability as diagram chasing in a category



In a model M each indiscernible sequence of *elements* is an indiscernible set iff each map $I^{\leq tails} \longrightarrow M_\bullet$ factors as

$$I^{\leq tails}_\bullet \longrightarrow |I|^{tails}_\bullet \longrightarrow M_\bullet$$

Conclusions.

- "We are thus led at last to the general concept of ~~a topological space~~ `stability`, which does not depend on any preliminary theory of the ~~real numbers~~ `syntax and language`."
- Nothing happened — very tautological
- just introduced weird bookkeeping notation
- in another universe/textbook, we'd define stability while playing with simplest examples of arrows in ɀφ and the Quillen lifting property
- Is it of any use ? Don't know !
- An analogy (how deep?) between analysis and model theory:

$$\omega^{\leq tails}_\bullet \longrightarrow M_\bullet$$

  is
  - eventually indiscernible sequence, for a model M
  - a Cauchy sequence, for a metric space M

• *"consistency" structure.* Recall: elements of a k-φ-consistent tuple are "very near each other"

Definition 7.2.1. 1. A formula $\varphi(x,y)$ has the *tree property* (with respect to k) if there is a tree of parameters $(a_s \mid \emptyset \neq s \in {}^{<\omega}\omega)$ such that:
 a) For all $s \in {}^{<\omega}\omega$, $(\varphi(x,a_{si}) \mid i < \omega)$ is k-inconsistent.
 b) For all $\sigma \in {}^{\omega}\omega$, $\{\varphi(x,a_s) \mid \emptyset \neq s \subseteq \sigma\}$ is consistent.
2. A theory T is *simple* if there is no formula $\varphi(x,y)$ with the tree property.

Transcribing the definition of simplicity. Now in a verbose manner we step-by-step follow the "android" approach of [Hasson-Gavrilovich] to transcribe to $s\Phi$ the definition of the tree property in [Tent-Ziegler].

Consistent instances of $\varphi(x,-)$. Equip $|M|^n$ with the filter generated by the set

$$\{(a_1,...,a_n) \in |M|^n : M \vDash \exists x \bigwedge_{1\leq i\leq n} \varphi(x,a_i)\}$$

Motivation 1: The definition 7.2.1 talks about the formulas $\exists x \bigwedge_{1\leq i\leq n} \varphi(x,a_i)$ implicitly, or rather about tuples satisfying these formulas.

This turns the simplicial set hom $-|M|$ corepresented by the set of elements of M into a simplicial filter which we denote by $M_\bullet^{\exists x \wedge\varphi(x,-)}$. Accordingly,

- call a tuple $(a_1,...,a_n) \in |M|^n$ *small* or $\exists x\varphi(x,-)$*-small* iff the set $\{\varphi(x,a_1),..,\varphi(x,a_n)\}$ is consistent, i.e. $M \vDash \exists x \bigwedge_{1\leq i\leq n} \varphi(x,a_i)$.

Motivation 2: The definition speaks of consistency of formulas of form $\varphi(x,a_s)$. Which is what we use to express this as the property of continuity of a morphism in $s\Phi$.

The tree object. Define $({}^{<\omega}\omega)_\bullet$:

- $({}^{<\omega}\omega)_\bullet(n_\leq) := \{(a_1 \leq ... \leq a_n), (a_1 \leq ... \leq a_n) : a_1,..,a_n \in M^n$
- equipped with indiscrete filters
- $({}^{<\omega}\omega)_\bullet(n_\leq) \longrightarrow M^n$

$a_\bullet : ({}^{<\omega}\omega)_\bullet(n_\leq) \longrightarrow M^n$ is continuous iff

- the tree $(a_s \mid \varnothing \neq s \in {}^{<\omega}\omega)$ satisfies b) above

Item b) For all $\sigma \in {}^{\omega}\omega$ $\{\varphi(x,a_s) \mid \varnothing \neq s \subseteq \sigma\}$ *is consistent.* just says that the morphism

$$({}^{<\omega}\omega)_\bullet^\leq(n) \longrightarrow M_\bullet^{\exists x\wedge\varphi(x,-)}(n)$$

$$({}^{<\omega}\omega)_\bullet^\leq \longrightarrow M_\bullet^{\exists x\wedge\varphi(x,-)}$$

defined by the parameters $(a_s)_s$ is continuous when $({}^{<\omega}\omega)_\bullet^\leq$ is equipped with indiscrete filters.

Item a) For all $s \in {}^{<\omega}\omega$, $(\varphi(x,a_{si}) \mid i < \omega)$ *is* k*-inconsistent.* says that we cannot extend *non-trivially*

$$({}^{<\omega}\omega)_\bullet^\leq \longrightarrow M_\bullet^{\exists x\wedge\varphi(x,-)}$$

continuously to a larger domain with indiscrete filter:

- any "to-be-inconsistent" k-tuple $((si_j) \mid 1 \leq j \leq k)$ "from" $(\varphi(x,a_{si}) \mid i < \omega)$ lies outside the preimage of the $M_\bullet^{\exists x\wedge\varphi(x,-}$ big subset of inconsistent tuples.

More formally: Hence,

- add to $({}^{<\omega}\omega)^{\leq}_{\bullet}$ tuples $(si_j)\,|\,1 \leq j \leq k)$ representing potentially "φ–inconsistent" tuples in M.
- equip $({}^{<\omega}\omega_{\text{fans}})_{\bullet}(k)$ with any filter such that
  - each tuple of distinct elements "inconsistent by Item a)" lies outside of *some* neighbourhood

By definition then

- TP for φ implies that we cannot extend

$$({}^{<\omega}\omega)^{\leq}_{\bullet} \longrightarrow M_{\bullet}^{\exists x \wedge \varphi(x,-)}$$

continuously to this larger domain with such filters

The converse. Need to enlarge the filters for converse to hold. Can do the same for *isomorphic copies* of ${}^{<\omega}\omega$ inside of itself. Hence, define:

- a subset ε is not big iff for some isomorphic copy $\sigma \subset {}^{<\omega}\omega$

$$\varepsilon \setminus \sigma^{\leq}_{\bullet}(k)$$

  consists only of tuples with $< k$ distinct elements
- equiv., a neighbourhood is a set containing at least one tuple required to be inconsistent by Item a) wrt σ for each isomorphic copy $\sigma \subset {}^{<\omega}\omega$ of ${}^{<\omega}\omega$

Then tautologically: the map does not extend consistently iff the preimage of the set of consistent tuples is not big, i.e. for some isomorphic copy $\sigma \subset {}^{<\omega}\omega$ contains only tuples allowed to be consistent. Thus, that copy σ is a counterexample to TP.

In notation: Item a) considers consistency of tuples of formulas

$$(\varphi(x, a_{si})|i < \omega)$$

and, implicitly,

- the linear orders $si \leq sj$ iff $i \leq j$, $s \in {}^{<\omega}\omega$.

Hence, we consider a simplicial set containing these tuples:

$$({}^{<\omega}\omega_{\text{fans}})^{\leq}_{\bullet}(n_{\leq}) := \{(si_1, ..., si_n) : s \in {}^{<\omega}\omega, 1 \leq i_1 \leq ... \leq i_n < \omega\}, n < \omega$$

where ${}^{<\omega}\omega_{\text{fans}}$ is the *fan* partial order defined by

$$a_{si} \leq a_{s'i'} \text{ iff } s = s' \text{ and } i \leq i'$$

Note that

$$({}^{<\omega}\omega)^{\leq}_{\bullet}(1_{\leq}) = ({}^{<\omega}\omega_{\text{fans}})^{\leq}_{\bullet}(1_{\leq}) = {}^{<\omega}\omega$$

Then we modify the definition of $({}^{<\omega}\omega_{\text{fans}})^{\leq}_{\bullet}(k_{\leq})$ so that it talks about arbitrary descendants rather than the si's:

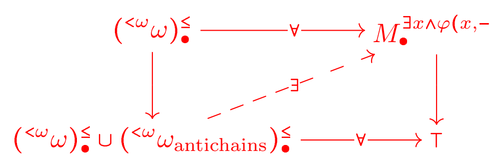
$$|({}^{<\omega}\omega_{\text{antichains}})^{\leq}_{\bullet}(n_{\leq})| := \{(s_1, ..., s_n) : s_i \leq_{lex} s_j \forall 1 \leq i \leq j \leq n, \text{ and } s$$

where $\leq_{lex}$ is the lexicographic order on ${}^{<\omega}\omega$.

A subset $\varepsilon \subseteq ({}^{<\omega}\omega_{\text{antichains}})^{\leq}_{\bullet}(n_{\leq})$ is *big* iff for each isomorphic copy $\sigma \subseteq {}^{<\omega}\omega$ of ${}^{<\omega}\omega$ the set $\varepsilon \cap (\sigma_{\text{antichains}})_{\bullet}(n_{\leq})$ contains a non-degenerate simplex, i.e. a tuple with all elements distinct.

Simplicity. Formula φ has no tree property in a model M iff each continuous map $({}^{<\omega}\omega)^{\leq}_{\bullet} \longrightarrow M_{\bullet}^{\exists x \wedge \varphi(x,-)}$ factors as

$$({}^{<\omega}\omega)^{\leq}_{\bullet} \longrightarrow ({}^{<\omega}\omega)^{\leq}_{\bullet} \cup ({}^{<\omega}\omega_{\text{antichains}})^{\leq}_{\bullet} \longrightarrow M_{\bullet}^{\exists x \wedge \varphi(x,-)}$$



6. Shelah representation of stable models via equivalence relations

Here we deal with another external property, *representability*. This notion was a try to formalize the intuition that "the class of models of a stable first order theory is not much more complicated than the class of models $M = (A, \ldots, E_t, \ldots)_{s \in I}$ where E_t^M is an equivalence relation on A refining E_s^M for $s < t$; and I is a linear order of cardinality $\leq |T|$. It was first defined in Cohen-Shelah [SC16], where it was shown that one may characterize stability and $\aleph_0$-stability by means of representability.

The results are phrased below, and the full definition appears in Definition 1.2, but first consider a simplified version. We say that a a model M is $\mathfrak{k}$-representable for a class $\mathfrak{k}$ when there exists a structure $\mathbf{I} \in \mathfrak{k}$ with the same universe as M such that for any n and two sequences of length n from M, if they realize the same quantifier free type in $\mathbf{I}$ then they realize the same (first order) type in M. Of course, T is $\mathfrak{k}$-representable if every model of T is $\mathfrak{k}$-representable. We prove, e.g. that T is superstable iff it is $\mathfrak{k}_\kappa^{\text{unary}}$-representable for some κ where $\mathfrak{k}_\kappa^{\text{unary}}$ is the class of structures with exactly κ unary functions (and nothing else).

Let $(J; \{\approx_i : i < \kappa\})$ be a structure in the language with equivalence relations, and nothing else.

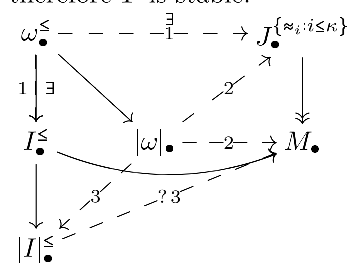
Let $J_\bullet^{\{\approx_i : i<\kappa\}}$ be the model associated with *quantifier-free formulas*: the filter on J^n is generated by, for $i < \kappa$

$$\{(a_1, ..., a_n) : a_1 \approx_i a_2 \approx_i ... \approx_i a_n \vee \bigwedge_{1 \leq i \leq j \leq n} a_i \not\approx_i a_j\}$$

Proposition. *A theory T is stable if there is κ such that for each model M of T there is a structure $\mathbf{J}$ on the same domain, $|\mathbf{J}| = |M|$, with at most κ equivalence relations $\approx_\alpha, \alpha < \kappa$ (and nothing else), such that either*

- *there is a $\mathfrak{z}\boldsymbol{\varphi}$-surjection $\mathbf{J}_\bullet^{\{\approx_\alpha : \alpha < \kappa\}} \longrightarrow M_\bullet$*
- *equiv., each quantifier-free indiscernible sequence in $\mathbf{J}$ is necessarily indiscernible in M (hence, if infinite, order indiscernible)*

Proof(easy). Let M be a large enough model of T and let $\mathbf{J}_\bullet^{\{\approx_\alpha : \alpha < \kappa\}} \longrightarrow M_\bullet$ be an $\mathfrak{z}\boldsymbol{\varphi}$-surjection. A long enough sequence indiscernible in a model M of T has an infinite subsequence quantifier-free indiscernible in $\mathbf{J}$, as the number of quantifier-free types in $\mathbf{J}$ is bounded. In $\mathbf{J}$, a quantifier-free indiscernible sequence is necessarily quantifier-free order-indiscernible, and therefore order-indiscernible in M, because $\mathfrak{z}\boldsymbol{\varphi}$-morphisms preserve indiscernible sequences. Hence, in M every long enough indiscernible sequence has an infinite order-indiscernible subsequence, and hence is order-indiscernible itself. Hence, any large enough and saturated enough model of T is stable, and therefore T is stable.



□

Question. What about "iff"? Is it superstability?
Note the argument only uses *symmetry* of

$$J_\bullet^{\{\approx_i : i \leq \kappa\}} : FiniteLinearOrders^{\text{op}} \longrightarrow \boldsymbol{\varphi}ilters$$

i.e. that it factors as

$$J_\bullet^{\{\approx_i : i \leq \kappa\}} : FiniteLinearOrders^{\text{op}} \longrightarrow FiniteSets^{\text{op}} \longrightarrow \boldsymbol{\varphi}ilters$$

and that filters are generated by $< \kappa$ neighbourhoods.

"IFF" is more technical and is implied by Shelah representation.

Shelah representation talks about *types*, hence we can only construct a ⁊𝝋-surjection

$$\mathbf{J}_\bullet^{\mathrm{qftp}(\{\approx_\alpha:\alpha<\kappa\})} \longrightarrow M_\bullet$$

where $\mathrm{qftp}(\{\approx_\alpha : \alpha<\kappa\})$ is the set of quantifier-free types in $\mathbf{J}$.

Proposition. *A theory T is stable iff there is κ such that for each model M of T there is a structure $\mathbf{J}$ on the same domain, $|\mathbf{J}| = |M|$, with at most κ equivalence relations $\approx_\alpha, \alpha<\kappa$ (and nothing else), such that*

- *there is a ⁊𝝋-surjection $\mathbf{J}_\bullet^{qftp(\{\approx_\alpha:\alpha<\kappa\})} \longrightarrow M_\bullet$*

Reducing to equivalence relations. Infinite quantifier-free indiscernible sequences in a structure in a language consisting only of equality and unary functions $(f,g,..)$, are the same as in the structure with equivalence relations

$$f(x)=g(x) \qquad\qquad f(x)=f(y)$$

Lemma. *In a theory in a language consisting only of equality and unary functions, which we assume closed under composition, the quantifier-free type of an indiscernible sequence of $n\geq 3$ elements is isolated, among types of indiscernible sequences, by a formula of the form*

$$\bigwedge_{\substack{1\leq i\leq n\\(f,g)\in F_1}} f(x_i)=g(x_i)\ \&\ \bigwedge_{\substack{1\leq i\leq n\\(f,g)\in F_2}} f(x_i)\neq g(x_i)$$

$$\&\ \bigwedge_{\substack{i<j\\ f\in F_3}} f(x_i)=f(x_j)\ \&\ \bigwedge_{\substack{i<j\\ f\in F_4}} f(x_i)\neq f(x_j)$$

for some sets F_1,F_2 of pairs of unary functions, and some sets F_3,F_4 of unary functions.

Proof(easy). Indeed, let $f(x_1)=g(x_2)$ be in the quantifier-free type of an indiscernible sequence (a_1,a_2,a_3). Then so are $f(x_1)=g(x_3)$, $f(x_2)=g(x_3)$, and therefore $f(x_1)=f(x_2)=g(x_2)=g(x_3)$, which is equivalent to the conjunction of $f(x_1)=f(x_2)$, $f(x_2)=g(x_2)$, and $g(x_2)=g(x_3)$ of the required form, and implies the formula $f(x_1)=g(x_2)$ we started with. □

Proof ($\Longrightarrow$). Let M be a model of T, and let $\mathbf{J}'$ be a structure in a language consisting only of equality and unary functions representing M as in Theorem 3.1(7).

Let $\mathbf{J}$ be the model constructed from $\mathbf{J}'$ by the Lemma.

By definition of representation [CoSh:919, Def. 2.1], a quantifier-free indiscernible sequence in $\mathbf{J}'$ is necessarily indiscernible in M, hence the same is true for $\mathbf{J}$ by Lemma.

Hence, the identity map $|\mathbf{J}|\longrightarrow|M|$ induces an ⁊𝝋-morphism

$$\mathbf{J}^{\{\approx_i:i<\kappa\}} \longrightarrow M_\bullet$$

which is surjective. □

Category-theoretic interpretation. We only use that $J_\bullet^{\{\approx_i:i<\kappa\}}$ is symmetric: the functor

$$J_\bullet^{\{\approx_i:i<\kappa\}} : FiniteLinearOrders^{\mathrm{op}} \longrightarrow \boldsymbol{\varphi}ilters$$

factors as

$$J_\bullet^{\{\approx_i:i<\kappa\}} : FiniteLinearOrders^{\mathrm{op}} \longrightarrow FinSets^{\mathrm{op}} \longrightarrow \boldsymbol{\varphi}ilters$$

and this factorisation implies that

> each (quantifier-free) indiscernible sequence is (quantifier-free) order indiscernible

Lemma. *A theory T is stable iff there is a cardinal κ such that for each model $M \models T$ there is a surjective $\wr\mathcal{F}$-morphism*

$$J_\bullet \longrightarrow M_\bullet$$

from a ("2-dimensional") "symmetric" simplicial filter

$$J_\bullet : FiniteLinearOrders \longrightarrow FiniteSets^{op} \longrightarrow \mathcal{F}ilters$$

with at most κ neighbourhoods, or, equivalently, such that its filter structure is pulled back from at most κ morphisms to filters of form

$$J_\bullet : FiniteLinearOrders \longrightarrow FiniteSets^{op} \longrightarrow \mathcal{F}$$

where for each $n > 0$ $|J_\bullet(n_\leq)|$ is a finite set.

Proof. Implied by Lemmas above. □

Remark. Uniform structures are "1-dimensional" "symmetric" simplicial filters such that $J_\bullet(1)$ is indiscrete.

It will be interesting to compare this to [Boney, Erdos-Rado Classes, Thm 6.8].

7. Larger context: Quillen negation / orthogonality / lifting property

What we have ?

- A way of transcribing model theory into weird bookkeeping notation
- transcribing is tautological, brings no new ideas
- by itself, entirely useless, just a way of bookkeeping

Benefits ?

- places some definitions into a new topological context
- new point of view, possibly intuition
- diagram chasing trickery, especially
  - Quillen negation /lifting property
- eventually, maybe homotopy theory

Homotopy theory (maybe). $\wr\varphi$ contains full categories with rich homotopy theory:

- $Top \longrightarrow \wr\varphi \longrightarrow Top$
- $sSets \longrightarrow \wr\varphi \longrightarrow sSets$ (in several ways)
- Geometric realisation

$$|\cdot| : sSets \longleftrightarrow Top : Sing$$

gives rise to adjoint(?) endofunctors

$$sSets \longrightarrow \wr\varphi \longleftrightarrow \wr\varphi \longrightarrow Top$$

Maybe applies to $M_\bullet$ or $M_\bullet^{\exists x \wedge \varphi(x,-)}$?

Locally trivial bundles. Local triviality can be defined by a diagram in $\wr\varphi$.

A morphism $p : X \longrightarrow B$ of topological spaces is

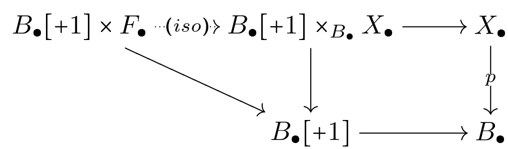
- locally trivial with fibre F iff in $\wr\varphi$

the map $p_\bullet : X_\bullet \longrightarrow B_\bullet$ of their associated objects

- becomes "trivial" after pullback along the obvious map $B_\bullet \circ [+1] \longrightarrow B_\bullet$ "forgetting the first coordinate":

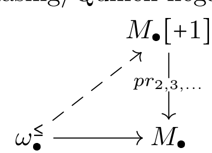
Namely, there is an isomorphism in $\wr\varphi$ over $B_\bullet$

$$B_\bullet \circ [+1] \times_{B_\bullet} X_\bullet \xleftrightarrow{(iso)} B_\bullet \circ [+1] \times F_\bullet \quad \text{over } B_\bullet$$



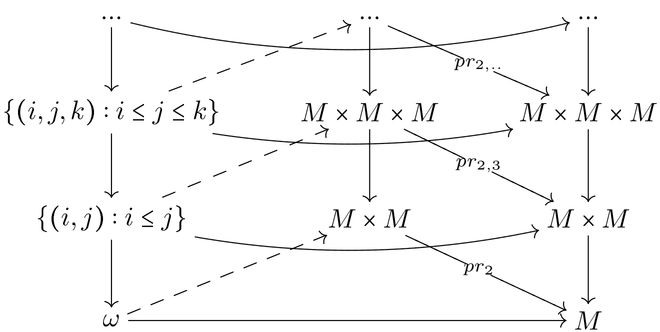
That is, a map $X \xrightarrow{p} B$ is *locally trivial* iff there is an $\wr\varphi$-isomorphism $B_\bullet[+1] \times F_\bullet \xrightarrow{(iso)} B_\bullet[+1] \times_{B_\bullet} X_\bullet$ over $B_\bullet[+1]$.

Topology and analysis (maybe). $\wr\varphi$ contains both topological and metric spaces. Also filters, hence can talk about limits (uniformly convergent sequences, Arzela-Ascoli etc) as diagram chasing/Quillen negation



For a metric space M, A *Cauchy* sequence is a map $\omega_\bullet^{\leq} \longrightarrow M_\bullet$. M is complete iff the diagram on the left holds.

(see next page)



Recall: $\varepsilon \subset M^n$ *is a neighbourhood* iff for some $\epsilon > 0$

$$\varepsilon \supset \{(x_1, ..., x_n) : dist(x_i, x_j) < \epsilon \text{ for } 0 \leq i \leq j \leq n\}$$

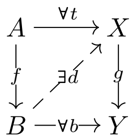
A $\varepsilon \subset \{(i,j) : i \leq j\}$ *is a neighbourhood* iff for some $N > 0$

$$\varepsilon \supset \{(i,j) : i, j > N\}$$

For topological spaces, sequences of functions is similiar.

8. Quillen negation / orthogonality / lifting property

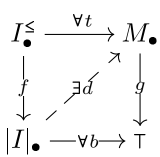
The Quillen negation/orthogonality/lifting property is often used define properties of morphisms starting from an explicitly given class of morphisms, in particular from a list of (counter)examples.



Definition. $A \xrightarrow{f} B \,\rightthreetimes\, X \xrightarrow{g} Y$
f has the left lifting property wrt g
f is orthogonal to g

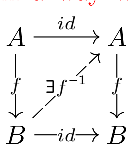
Notation. $X_\bullet, Y_\bullet$ denote objects of $\wp$, the subscript $\bullet$ indicating it is a functor.

For a model M, let $M_\bullet$ denote *the generalised Stone space of* 1*-types of M* as constructed above; and let $\omega_\bullet^{\leq tails}$ and $|\omega|_\bullet^{tail}$ denote the objects of $\wp$ corresponding the linear order $(\omega, \leq)$ and the set ω equipped with the cofinite filter (filter of tails).

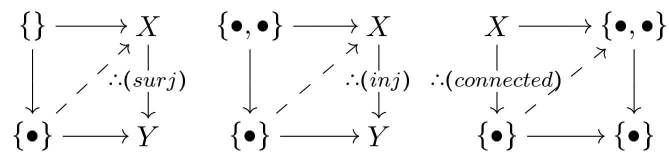


"Unary" Stability
$I_\bullet^{\leq} \xrightarrow{f} |I|_\bullet \,\rightthreetimes\, M_\bullet \xrightarrow{g} \top$ in $\wp$
each order indiscernible sequence of *elements* (not tuples) of M
is totally indiscernible

Taking the orthogonal of a class C is a simple way to define a class of morphisms excluding non-isomorphisms from C, in a way which is useful in a diagram chasing computation:



This diagram defines isomorphism
(in any category)
$A \xrightarrow{f} B \,\rightthreetimes\, A \xrightarrow{f} B$
iff f is an isomorphism



These diagrams "parse" as follows:

$\{\} \longrightarrow \{\bullet\} \rightthreetimes X \xrightarrow{\therefore(surj)} Y$ iff each point $\bullet \in X$ lifts to point in Y

$\{\bullet,\bullet\} \longrightarrow \{\bullet\} \rightthreetimes X \xrightarrow{\therefore(inj)} Y$ iff the images of points $\bullet_{\text{left}}, \bullet_{\text{right}}$ in X coincide (the square commutes), then the points itself coincide $\bullet_{\text{left}} = \bullet_{\text{right}} \in X$ (the upper triangle commutes).

$X \xrightarrow{\therefore(connected)} \{\bullet\} \rightthreetimes \{\bullet,\bullet\} \longrightarrow \{\bullet\}$ iff X being a union of two closed open subsets (preimages of $\{\bullet,\bullet\}$) implies one of them is empty (the diagonal arrows picks one of these sets), meaning that X is connected

Define left/right Quillen negation/orthogonal:

$$C^l := \{f : f \rightthreetimes g \forall g \in C\}, \qquad C^r := \{g : f \rightthreetimes g \forall f \in C\}$$

- A useful intuition is to think that the property of left-lifting against a class C is a kind of negation of the property of being in C, and that right-lifting is also a kind of negation.

9. Examples of Quillen negations

Examples in model theory. We have shown that: Stability and simplicity are defined by (where I is an infinite linear order)

- M is stable iff $(M^{eq})_\bullet \longrightarrow \top \in \{I_\bullet^{\leq} \longrightarrow |I|_\bullet^{\leq}\}^1$
- M has NTP for formula $\varphi(-,-)$ iff

$$M_\bullet^{\exists x \wedge \varphi(x,-)} \longrightarrow \top \in \{{}^{<\omega}\omega_\bullet^{\leq} \longrightarrow ({}^{<\omega}\omega)_\bullet^{\leq} \cup ({}^{<\omega}\omega_{\text{antichains}})_\bullet^{\leq}\}^1$$

The Quillen negation ideology of "often used define properties of morphisms starting from an explicitly given class of morphisms" suggests we interpret Shelah's 'try to formalise the intuition that "the class of models of a stable first order theory is not much more complicated than the class of models $M = (A, \dots, E_t, \dots)_{s \in I}$ where E_t^M is an equivalence relation on A refining E_s^M for $s < t$; and I is a linear order of cardinality $\le |T|$ "' as the following conjecture (but says nothing of which specific class of morphisms associated with equivalence relations to consider):

Conjecture. A saturated enough model N is stable iff in ɀφ

$$(N^{\mathrm{eq}})_\bullet \longrightarrow \top \;\in\; \{(A, \dots, E_t, \dots)_{s \in I})_\bullet \longrightarrow \top \,:\, A, E_t \text{ as above } \}^{\mathrm{lr}}$$

Question. What it means for a saturated enough model N that $(N^{\mathrm{eq}})_\bullet \longrightarrow \top$ is in either of the classes calculated in ɀφ

$$\{(\mathbb{Q}; \le)_\bullet \longrightarrow \top\}^{\mathrm{lr}}$$

$$\{(\text{an equivalence relation})_\bullet \longrightarrow \top\}^{\mathrm{lr}} \quad \text{(implies } N \text{ is stable)}$$

$$\{(\mathbb{C}; +, *)_\bullet \longrightarrow \top\}^{\mathrm{lr}} \quad \text{(stability? implies } N \text{ is stable)}$$

fibrant-cofibrant decompositions. Often for a property P each arrow decomposes both as

$$\bullet \xrightarrow{(P)^{\mathrm{l}}} \cdot \xrightarrow{(P)^{\mathrm{lr}}} \star$$

$$\bullet \xrightarrow{(P)^{\mathrm{rl}}} \cdot \xrightarrow{(P)^{\mathrm{r}}} \star$$

Take P as above ...

What properties are defined by analogues of M2 fibrant-cofibrant decompositions? "Stable approximation" ?

Examples of notions defined by iterated Quillen negations in the category of FiniteGroups.

```
nilpotent group: H-->HxH \in {0-->*}^lr
soluble group:   0-->H \in {0-->A: A abelian}^lr
p-group:   H-->0 \in {Z/pZ-->0}^rr
```

"Computer" syntax for basic notions of topology.

```
compact:    { {o}-->{o->c}}^r_{<5}^lr              //
surjective: {{}-->{o}}^r == { {}-->{o} }^rrl     //  sin
connected: {{}-->{o}}^rll )((. {{x,y}-->{x=y}}^l ..(.=.
discrete:  {{}-->{o}}^rl  )(.
subset:    {{}-->{o}}^rr=={{x<->y->c}-->{x=y=c}}^l )). =
```

```
proper maps:  {{o}-->{o->c}}^r_{<5}^lr   ' )<5() '\.
dense image:  {{c}-->{o->c}}^l   .('\.
injection:    {{x,y}-->{x=y}}^r == {{x<->y}-->{x=y}^l   ..). == '~'('
closed subset: {{x<->y->c}-->{x<->y=c}}^l == {{c}-->{o->c}}^lr '~'\.('~'=. == .()`\.
open subset:   {{x<->y<-c}-->{x<->y=c}}^l  ._./'(._.='
normal (T4): {{a<-b->c<-d->e}-->{b=c=d}}^l    /V\(/\
Tietze lemma: R-->{o} (- {{a<-b->c<-d->e}-->{b=c=d},{a<-b->c}-->{a=b=c}}^lr
Uryhson lemma: R-->{a<-b->c} (- {{a<-b->c<-d->e}-->{b=c=d}}^lr (not quite!)
retract: {{*-->{o}}^l    *(.
neighbourhood retract: {Y->-oo}-->{X->-oo} \in  {{*-->{o}}^l *(.  (almost!)
```