Math. Log. Quart. **62,** No. 3, 140–154 (2016) / **DOI** 10.1002/malq.200920105

Stable theories and representation over sets

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Received 16 October 2009, revised 24 June 2013, accepted 23 October 2013
Published online 6 May 2016

In this paper we give characterizations of the stable and $\aleph_0$-stable theories, in terms of an external property called representation. In the sense of the representation property, the mentioned classes of first-order theories can be regarded as "not very complicated".

1 Introduction

1.1 Motivation and main results

Our motivation to investigate the properties under consideration in this paper comes from the following thesis:

> It is very interesting to find dividing lines and it is a fruitful approach in investigating quite general classes of models. A "natural" dividing property "should" have equivalent internal, syntactical, and external properties (cf. [5] for more).

Of course, we expect the natural dividing lines will have many equivalent definitions by internal and external properties.

The class of stable complete first order theories T is well known (cf. [4]); it has many equivalent definitions by "internal, syntactical" properties, such as the order property. As for external properties, one may say "for every $\lambda \geq |T|$ for some model M of T we have $\mathbf{S}(M)$ has cardinality $> \lambda$" is such a property (characterizing instability). Anyhow, the property "not having many κ-resplendent models" (or equivalently, having at most one in each cardinality) is certainly such an external property (cf. [8]).

Here we deal with another external property, *representability*. The results are phrased below, and the full definition appears in Definition 2.1, but first consider a simplified version. We say that a a model M is $\mathfrak{k}$-representable for a class $\mathfrak{k}$ when there exists a structure $\mathbf{I} \in \mathfrak{k}$ with the same universe as M such that for any n and two sequences of length n from M, if they realize the same quantifier free type in $\mathbf{I}$ then they realize the same (first order) type in M. Of course, T is $\mathfrak{k}$-representable if every model of T is $\mathfrak{k}$-representable. We prove, e.g., that T is stable iff it is $\mathfrak{k}_\kappa^{\text{unary}}$-representable for some κ where $\mathfrak{k}_\kappa^{\text{unary}}$ is the class of structures with exactly κ unary functions (and nothing else).

The main results presented in this paper are:

- **Characterization of stable theories (Theorem 3.1)**: For a complete first-order theory T, the following conditions are equivalent:

  1. T is stable.
  2. T is representable in $\mathrm{Ex}^1_{|T|^+,|T|}(\mathfrak{k}^{\mathrm{eq}})$ (cf. Definitions 2.1 & 2.10).
  3. For some cardinals $\mu_1, \kappa_1, \mu_2, \kappa_2$, it holds that T is representable in $\mathrm{Ex}^1_{\mu_1,\kappa_1}(\mathrm{Ex}^2_{\mu_2,\kappa_2}(\mathfrak{k}^{\mathrm{eq}}))$ (cf. Definition 2.11).

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4. T is representable in $\mathrm{Ex}^0_{\mu,\kappa}(\mathfrak{k}^{\mathrm{eq}})$ for some cardinals μ, κ (cf. Definition 2.8).
5. T is representable in $\mathrm{Ex}^0_{0,|T|}(\mathfrak{k}^{\mathrm{eq}})$.

- **Characterization of $\aleph_0$-stable theories (Theorem 4.1):** For a complete first-order theory T, the following conditions are equivalent:

    1. T is $\aleph_0$-stable.
    2. T is representable in $\mathrm{Ex}^2_{\aleph_0,\aleph_0}(\mathfrak{k}^{\mathrm{eq}})$.
    3. T is representable in $\mathrm{Ex}^1_{\aleph_0,2}(\mathfrak{k}^{\mathrm{eq}})$.
    4. T is representable in $\mathrm{Ex}^{0,\mathrm{lf}}_{\aleph_0,2}(\mathfrak{k}^{\mathrm{eq}})$ (cf. Definition 2.9)

In the attempt to extend the framework of representation it seemed natural, initially, to conjecture that if we consider representation over linear orders rather than over sets, we could find an analogous characterizations for dependent theories. However, such characterizations would imply strong theorems on existence of indiscernible sequences. Lately (cf. [2]), some dependent theories were discovered for which it is provably "quite hard to find indiscernible subsequences", implying that this conjecture would fail in its original formulation.

Nevertheless, this raises various further questions:

1. Can we characterize by representability the property "T is strongly dependent", and similarly for the various relatives (cf. [6])?
2. For a natural number n, what is the class of theories representable by the class $\mathfrak{k}^n_\kappa$ of structures with just κ n-place functions (or relations)?

The characterization of the superstable case will be treated in [7].

1.2 Set-theoretic Preliminaries

We use the greek letters μ, κ, λ for cardinal numbers. The letter T will be used to denote a first-order theory; $\mathfrak{C}_T$ is a monster model for it. The set-theoretic facts that are used in this article are stated here for the sake of completeness. We also prove the special version of the Δ-system lemma that is used later.

Theorem 1.1 (Fodor) *Let λ a regular uncountable cardinal, $f : \lambda \to \lambda$ such that $f(\alpha) < \alpha$ for all $0 < \alpha < \lambda$, Then $\{\alpha < \lambda : f(\alpha) = \beta\}$ is a stationary set of λ for some $\beta < \lambda$ (cf. [1]).*

Corollary 1.2 *If $f : \lambda \to \mu$, $\lambda > \mu$ regular, then $f^{-1}(\{\alpha\})$ is a stationary set of λ for some $\alpha < \mu$.*

Theorem 1.3 *Let λ be a regular cardinal, $|W| = \lambda$, $|S_t| < \mu$ (for $t \in W$) such that $\chi < \lambda \to \chi^{<\mu} < \lambda$. Then:*

1. *(The Δ-system lemma) There exist $W' \subseteq W$, $|W'| = \lambda$ and S such that $s \neq t \to S_t \cap S_s = S$ holds for all $s, t \in W'$.*
2. *If $\bar{z}_t = \langle z^\alpha_t : \alpha < \alpha(t)\rangle$ enumerates S_t for $t \in W$, then we can add:*
    (a) $t \in W' \Rightarrow \alpha(t) = \alpha_0$ holds for some α_0.
    (b) For some $U \subseteq \alpha_0$ it holds that $s \neq t \in W' \Rightarrow \bar{z}_t \restriction U = \bar{z}_s \restriction U$, $U = \{\alpha < \alpha_0 : z^\alpha_t = z^\alpha_s\}$.
    (c) For some equivalence relation E on α_0 it holds that $t \in W' \Rightarrow z^\alpha_t = z^\beta_t \leftrightarrow (\alpha, \beta) \in E$.

Proof. For (1), cf. [1,4]. Let us prove (2): The map $t \to \alpha(t)$ fulfills the assumptions of Theorem 1.1 ($\alpha(t) < \mu < \lambda$), therefore (a) holds for some $W_0 \subseteq W$. By part 1 there exist $S \subseteq \{z^\alpha_t : \alpha < \alpha_0, t \in W_0\}$, $W_1 \subseteq W_0$ of cardinality λ, such that $S = S_t \cap S_s$ for all $t \neq s$. Define a map $W_1 \ni t \mapsto U_t$ where: $U_t = \{\alpha < \alpha_0 : z^\alpha_t \in S\}$, Since the range has cardinality $2^{|\alpha_0|} \leq 2^{<\mu} < \lambda$ this map also fulfills the assumptions of Theorem 1.1, and we get that for some $W_2 \subseteq W_1$ of cardinality λ and U it holds that $t \in W_2 \to U_t = U$. The range of the map $t \to \bar{z}_t \restriction U$ is ${}^U S$ whose cardinality is $\leq |\alpha_0|^{|\alpha_0|} < \lambda$, and by another use of Theorem 1.1 we get $W_3 \subseteq W_2$ of cardinality λ such that (b) holds. The map $t \to E_t$ where $E_t = \left\{(\alpha, \beta) : z^\alpha_t = z^\beta_t,\ \alpha, \beta < \alpha_0\right\}$ has cardinality at most $|\alpha_0|^{|\alpha_0|}$ and again by Theorem 1.1 the result holds for some E and $W' \subseteq W_3$ of cardinality λ, now W' is as required. □

1.3 Model-theoretic and Stability-theoretic preliminaries

This subsection is organized in three parts: general, stable and $\aleph_0$-stable theories.

General

For the rest of this paper, T is assumed to be a complete first-order theory. By $\mathbf{EC}(T)$, we denote the elementary class of T, i.e., the class of all models satisfying T. By $\mathbf{EC}_\lambda(T)$, we denote the class of models of cardinality λ satisfying T. We will use the name "structure" for any triple consisting of a set (the structure's universe or domain), a vocabulary (i.e., function symbols and relation symbols with prescribed arities), and an interpretation relation. Structures will be denoted by $\mathbf{I}$, $\mathbf{J}$ and their domains by I, J, respectively. The word "model" will only be used for the elements of $\mathbf{EC}(T)$. Models will be denoted by M, N and their domains by $|M|$, $|N|$, respectively. The fraktur letter $\mathfrak{k}$ will denote a class of structures in a fixed vocabulary $\tau_{\mathfrak{k}}$. For a model M and set $A \subseteq |M|$, $\mathbf{S}^m(A, M)$ denotes all the complete m-types in M over A. If $M = \mathfrak{C}_T$, we may omit it.

Although any model is a structure, we shall use the words "structure" and "model" carefully since in this paper we use *structures* to analyse *models* of the theory T. Usually we will deal with quantifier-free properties of the structures, but with general first-order properties of the models.

Definition 1.4 For a set $A \subseteq \mathfrak{C}_T$, we let $\mathbf{FE}(A)$ denote the set of formulas $\varphi(x, y, \overline{a})$ such that $T \models$ "$\varphi(x, y, \overline{a})$ is an equivalence relation with finitely many classes".

The formula $\varphi(\overline{x}, \overline{a})$ *divides* over a set A iff there exists a sequence $\langle \overline{a}_n : n < \omega\rangle$ such that $\mathrm{tp}(\overline{a}_n, A) = \mathrm{tp}(\overline{a}, A)$ for all $n < \omega$, but there exists an $m < \omega$ such that $\models \neg\exists\overline{x} \bigwedge_{n\in w} \varphi(\overline{x}, \overline{a}_n)$ holds for all $w \in [\omega]^m$. The type $p(\overline{x})$ *forks* over A if there exist formulas $\varphi_i(\overline{x}, \overline{a}_i)$ $(i < n)$, such that for all $i < n, \varphi_i$ divides over A and $p(\overline{x}) \vdash \bigvee_{i<n} \varphi_i(\overline{x}, \overline{a}_i)$. We say that a formula $\varphi(x, \overline{c})$ (with parameters from $\mathfrak{C}$) is almost over $A \subseteq \mathfrak{C}$ iff for some $E(x, y) \in \mathbf{FE}(A)$ and some $d_i \in \mathfrak{C}$ $(i < n)$ it holds that $T \models E(x, d_i) \leftrightarrow \varphi(x, \overline{c})$. A formula is over $A \subseteq \mathfrak{C}$ iff it is equivalent in T to a formula with parameters taken only from A.

Theorem 1.5 (Forking is preserved under elementary maps) *If* $\mathrm{p}(\overline{x})$ *forks over A, and f is an elementary map in M,* $\mathrm{dom}(f) \supseteq \mathrm{dom}(p) \cup A$, *then $f(p)$ forks over $f(A)$ (cf.* [4, III.1.5]).

Theorem 1.6 *(cf.* [4, III.2.2(2)]) *There are (up to logical equivalence mod T) at most $|T| + |A|$ formulas almost over A.*

Stable theories

A theory T is called κ-stable iff $|\mathbf{S}^m(A, M)| \leq \kappa$ for every $M \models T$, set $A \subseteq M$, $|A| \leq \kappa$ and $m < \omega$. The theory T is called stable if T is κ-stable for some κ.

Theorem 1.7 (The order property) *A theory T is unstable if and only if there exist a formula $\varphi(\overline{x}, \overline{y})$ and a sequence $\langle \overline{a}_n : n < \omega\rangle$ such that $\models \varphi(\overline{a}_i, \overline{a}_j)^{\mathrm{if}(i<j)}$ holds for all $i, j < \omega$ (cf.* [4, II.2.13]).

For the rest of this subsection, T is assumed to be stable.

Theorem 1.8 (Type definability in stable theories) *For every formula $\varphi(\overline{x}, \overline{y})$ there exists another formula $\psi_\varphi(\overline{y}, \overline{z})$ satisfying for every $A \subseteq \mathfrak{C}$, $|A| \geq 2$ and $\overline{b} \in \mathfrak{C}$ that there exists $\overline{c} \in A$ such that for every $\overline{a} \in A$*

$$\models \varphi(\overline{b}, \overline{a}) \ \Leftrightarrow \models \psi_\varphi(\overline{a}, \overline{c}).$$

(Put differently, for any $\overline{b}$ the (φ, A)-type of $\overline{b}$ is (ψ_φ, A)-definable; cf. [4, II.2.2].)

Theorem 1.9 *For stable T and distinct types $p, q \in \mathbf{S}(B)$ non-forking over $A \subseteq B$, there exists $E \in \mathbf{FE}(A)$ such that*

$$p(x) \cup q(y) \vdash \neg E(x, y)$$

*(cf. [*4*, III.2.9(2)]).*

We let $\kappa(T)$ be the first cardinal such that for any model $M \models T$, any increasing sequence of sets $\langle A_i \subseteq |M| : i \leq \kappa \rangle$ and any type $p(\bar{x}) \in \mathbf{S}^m(A_\kappa)$ there is some $i < \kappa$ such that $p \restriction A_{i+1}$ does not fork over A_i (cf. also [4, III, Definition 3.1]).

Theorem 1.10 *Assuming that T is stable, then $\kappa(T) \leq |T|^+$ and for every $B \subseteq |M|$ and every type $p(\bar{x}) \in \mathbf{S}^m(B)$, there exists $A \subseteq B$ with $|A| < \kappa(T)$ such that p does not fork over A (cf.* [4, III, Corollaries 3.2 & 3.3]*).*

Lemma 1.11 *Suppose $A \subset M \models T$, and $a \in M$. There exists $B \subseteq A$, $|B| \leq |T|$ such that for every $\overline{c} \in A$ and $\varphi(\overline{x}, \overline{c})$ almost over B there exists $\vartheta(\overline{x}, \overline{d})$ over B such that*

$$M \models \forall \overline{x}(\vartheta(\overline{x}, \overline{d}) \leftrightarrow \varphi(\overline{x}, \overline{c})),$$

and $\mathrm{tp}(a, A, M)$ *does not fork over B.*

Proof. First, define an increasing sequence B_n by induction on n. Let $|B_0| < \kappa(T) \leq |T|^+$, $B_0 \subseteq A$ such that $\mathrm{tp}(a, A)$ does not fork over B_0. Now assume B_n was defined and let

$$S_n := \left\{\varphi(\overline{x}, \overline{c}) \in \mathcal{L}_T : \overline{c} \subseteq A, \varphi(\overline{x}, \overline{c}) \text{ is almost over } B_n\right\}$$

By Theorem 1.6 there exist at most $|T| + |B_n| = |T|$ non equivalent formulas almost over B_n. Therefore, without loss of generality, $|S_n| \leq |T|$ and define B_{n+1} as follows:

$$B_{n+1} := B_n \cup \left\{\overline{c} : \varphi(\overline{x}, \overline{c}) \in S_n\right\}$$

That the required properties of $B := \bigcup_{n<\omega} B_n$ hold is easily verified.

$\aleph_0$-Stable theories

For the rest of this subsection, T is assumed to be $\aleph_0$-stable.

Theorem 1.12 *Let $p \in \mathbf{S}(A)$. For a given $B \supseteq A$ there are only finitely many non-forking extensions of p in $\mathbf{S}(B)$ (cf.* [4, III]*).*

Corollary 1.13 *For $p \in \mathbf{S}(A)$ there exists a finite $B \subseteq A$ such that p is the unique non-forking extension of $p \restriction B$ to $\mathbf{S}(A)$.*

Proof. By Theorem 1.12 there are finitely many non-forking extensions of $p \restriction B$ in $\mathbf{S}(A)$, therefore there exists a finite $B_0 \subseteq A$ such that $q_0 \restriction B_0 \neq q_1 \restriction B_0$ holds for every distinct $q_0, q_1 \in \mathbf{S}(A)$ non-forking extensions of p. Also p does not fork over some finite $B_1 \subseteq A$. Now, the conclusion easily follows for $B = B_0 \cup B_1$. □

Theorem 1.14 *A formula $\varphi(\overline{x}, \overline{c})$ is equivalent to a formula over B if and only if $\varphi(\overline{x}, f(\overline{c})) \equiv \varphi(\overline{x}, \overline{c})$ holds for every $f \in \mathrm{Aut}(\mathfrak{C}/B)$ (cf.* [4, III.2.3(2)]*).*

2 Structure classes and representation

We start with a number of conventions:

The vocabulary is a set of individual constants, (partial) function symbols and finitary and relation symbols (=*predicates*) with fixed arity; e.g., for a function symbol F, $\mathrm{arity}_\tau(F)$ is the number of places of the symbol F. Individual constants may be considered as 0-place function symbols; here, function symbols are interpreted as partial functions.

A structure $\mathbf{I} = \langle \tau, I, \models \rangle$ is a triple of vocabulary, universe (domain) and the interpretation relation for the vocabulary: let $|\mathbf{I}| = I$, $\|\mathbf{I}\|$ the cardinality of I and $\tau_{\mathbf{I}} = \tau$; $\mathbf{I}$ is called a τ-*structure*. By $\mathfrak{k}$, we denote a class of structures in a given vocabulary $\tau_{\mathfrak{k}}$.

By $\mathcal{L}^\tau_{\mathrm{qf}}$, we denote the set of quantifier-free formulas with terms from $\tau_{\mathfrak{k}}$. That is, finite Boolean combinations of atomic formulas, where atomic formulas (for τ) have the form $P(\sigma_0, \ldots, \sigma_{n-1})$ or $\sigma_0 = \sigma_1$ for some n-ary predicate $P \in \tau$, $\sigma_0, \ldots$ are terms in the closure of the variable by function (and partial function) symbols.

If Δ is a set of formulas in the vocabulary τ, $\mathbf{I}$ a τ-structure, $\bar{a} = \langle a_i : i < \alpha \rangle \in {}^\alpha|\mathbf{I}|$, then

$$\mathrm{tp}_\Delta(\bar{a}, B, \mathbf{I}) = \left\{\varphi(\bar{x}, \bar{b}) : \varphi(\bar{x}, \bar{y}) \in \Delta,\ \mathbf{I} \models \varphi(\bar{a}, \bar{b}),\ \bar{b} \in {}^{\mathrm{lh}(\bar{y})} B\right\}$$

2.1 Defining representations

We now reach the central definitions:

Definition 2.1 Consider a structure $\mathbf{I}$ and a set of formulas $\Delta \subseteq \mathcal{L}_{\mathbf{I}}$ (they are either as in the conventions at the beginning of this section or just first order).

For a structure $\mathbf{J}$, a function $f : |\mathbf{I}| \to |\mathbf{J}|$ is called a Δ-*representation* of $\mathbf{I}$ in $\mathbf{J}$ iff

$$\mathrm{tp}_{\mathrm{qf}}(f(\overline{a}), \varnothing, \mathbf{J}) = \mathrm{tp}_{\mathrm{qf}}(f(\overline{b}), \varnothing, \mathbf{J}) \;\Rightarrow\; \mathrm{tp}_{\Delta}(\overline{a}, \varnothing, \mathbf{I}) = \mathrm{tp}_{\Delta}(\overline{b}, \varnothing, \mathbf{I})$$

for any two sequences $\overline{a}, \overline{b} \in {}^{<\omega} I$ of equal length.

We say that $\mathbf{I}$ is Δ-*represented in a class of models* $\mathfrak{k}$ if there exists a $\mathbf{J} \in \mathfrak{k}$ such that $\mathbf{I}$ is Δ-represented in $\mathbf{J}$. For two classes of structures $\mathfrak{k}_0, \mathfrak{k}$ we say that $\mathfrak{k}_0$ is Δ-*represented in* $\mathfrak{k}$ if every $\mathbf{I} \in \mathfrak{k}_0$ is Δ-represented in $\mathfrak{k}$. We say that a first-order theory T is Δ-*represented in* $\mathfrak{k}$ if $\mathbf{EC}(T)$ is Δ-represented in $\mathfrak{k}$. We may omit Δ from the notation for the set of first order formulas or use qf for the set of quantifier-free formulas.

For a sequence $\overline{s}$ we use the shorthand $\overline{a}_{\overline{s}} := a_{s_0} {}^\frown \dots {}^\frown a_{s_{\mathrm{lh}(\overline{s})-1}}$.

For a structure $\mathbf{I}$, $\Delta \subseteq \mathcal{L}_T$ and sequences $\overline{a}_t (t \in I)$ from $\mathfrak{C}_T$ we say that $\mathbf{a} = \langle \overline{a}_t : t \in I \rangle$ is a Δ-*indiscernible sequence over* A in M if $\mathrm{tp}_{\Delta}(\overline{a}_{\overline{t}}, A, M) = \mathrm{tp}_{\Delta}(\overline{a}_{\overline{s}}, A, M)$ holds for all $\overline{s}, \overline{t} \subseteq I$ with the same quantifier-free type in $\mathbf{I}$. (This definition also appears in [9] and [3, II].) For a class of structures $\mathfrak{k}$, $M \models T$ and subset $A \subseteq |M|$ we denote by $\mathrm{Ind}_{\Delta}(\mathfrak{k}, A, M)$ the class of structures $\mathbf{a} = \langle \overline{a}_t : t \in I \rangle$ ($I \in \mathfrak{k}, \overline{a}_t \subset |M|$) which are Δ-indiscernible in M over $A \subseteq |M|$. We omit the respective symbol from the above notation in the specific cases $\Delta = \mathcal{L}(\tau_M)$, $M = \mathfrak{C}$ and $A = \varnothing$.

Finally, by $\mathfrak{k}^{\mathrm{eq}}$ we denote the class of structures of the vocabulary $\{=\}$.

2.2 The free algebras $\mathcal{M}_{\mu,\kappa}$

Definition 2.2 For a given structure $\mathbf{I}$, we define the structure $\mathcal{M}_{\mu,\kappa}(\mathbf{I})$ as the structure whose vocabulary is $\tau_{\mathbf{I}} \cup \langle F_{\alpha,\beta} : \alpha < \mu, \beta < \kappa \rangle$, with a β-ary function symbol $F_{\alpha,\beta}$ for all $\alpha < \mu, \beta < \kappa$. The vocabulary of $\mathbf{I}$ includes a unary relation symbol I for the structure's universe, and we will assume $F_{\alpha,\beta} \notin \tau_{\mathbf{I}}$.

We now define a structure $\mathcal{M}_{\mu,\kappa,\zeta}(\mathbf{I})$ by recursion:

1. $\mathcal{M}_{\mu,\kappa,0}(\mathbf{I}) := |\mathbf{I}|$;
2. for limit ζ, $\mathcal{M}_{\mu,\kappa,\zeta}(\mathbf{I}) = \bigcup_{\xi<\zeta} \mathcal{M}_{\mu,\kappa,\xi}(\mathbf{I})$; and
3. for $\zeta = \gamma + 1$,

$$\mathcal{M}_{\mu,\kappa,\zeta}(\mathbf{I}) = \mathcal{M}_{\mu,\kappa,\gamma}(\mathbf{I}) \cup \left\{ F_{\alpha,\beta}(\overline{b}) : \overline{b} \in {}^{\beta}\mathcal{M}_{\mu,\kappa,\gamma}(\mathbf{I}),\ \alpha < \mu,\ \beta < \kappa \right\}$$

where $F_{\alpha,\beta}(\overline{b})$ is treated as a formal object. Then the universe for the structure $\mathbf{I}$ is

$$\mathcal{M}_{\mu,\kappa}(\mathbf{I}) = \bigcup_{\gamma \in \mathrm{Ord}} \mathcal{M}_{\mu,\kappa,\gamma}(\mathbf{I}).$$

By Remark 2.3 below, $\mathcal{M}_{\mu,\kappa}(\mathbf{I})$ is a set and not a proper class. The symbols in $\tau_{\mathbf{I}}$ have the same interpretation as in $\mathbf{I}$. In particular, α-ary functions may be interpreted as $(\alpha+1)$-ary relations. The β-ary function $F_{\alpha,\beta}(\overline{x})$ is interpreted as the mapping $\overline{a} \mapsto F_{\alpha,\beta}(\overline{a})$ for all $\overline{a} \in {}^{\beta}|\mathcal{M}_{\mu,\kappa}(I)|$, where $F_{\alpha,\beta}(\overline{a})$ on the right side of the mapping is the formal object. If $\mu = \kappa = \aleph_0$ we may omit them from the notation.

We denote

$$\mathrm{reg}(\kappa) = \begin{cases} \kappa & \kappa = \mathrm{cf}\kappa, \\ \kappa^+ & \text{otherwise,} \end{cases}$$

for every cardinal κ.

Remark 2.3 Since $\mathrm{reg}(\kappa) \geq \kappa$ is regular, for all $\beta < \kappa$ and sequences of terms $\sigma_i(\overline{c}_i) \in \mathcal{M}_{\mu,\mathrm{reg}(\kappa)}(i < \beta)$ there exists $\gamma < \mathrm{reg}(\kappa)$ such that $\sigma_i(\overline{c}_i) \in \mathcal{M}_{\gamma}$ for all $i < \beta$. Therefore $F_{\alpha,\beta}(\langle \sigma_i(\overline{c}_i) : i < \beta \rangle) \in \mathcal{M}_{\gamma+1} \subseteq \mathcal{M}_{\mu,\mathrm{reg}(\kappa)}$, hence $\mathcal{M}_{\mu,\kappa}(S) = \mathcal{M}_{\mu,\kappa,\mathrm{reg}(\kappa)}(S)$ and particularly $\mathcal{M}_{\mu,\kappa}(S)$ is a set (though defined as a class).

We observe that $\|\mathcal{M}_{\mu,\kappa}(S)\| \le (\mu + |S|)^{<\mathrm{reg}(\kappa)}$. We can prove by induction on $\gamma \le \mathrm{reg}(\kappa)$ that $|\mathcal{M}_\gamma| \le (\mu + |S|)^{<\mathrm{reg}(\kappa)}$.

For a sequence $\overline{a} \subseteq \mathcal{M}_{\mu,\kappa}(S)$ we define its closure under subterms as the set $\mathrm{cl}(\overline{a})$ defined by induction on the construction of the term and the sequence length as $\mathrm{cl}(\overline{a}) := \overline{a}$ for $\overline{a} \subseteq S$. If $\mathrm{lh}(\overline{a}) = 1$ and $a_0 = F_{\alpha,\beta}(\overline{b})$ then $\mathrm{cl}(\overline{a}) := \{a_0\} \cup \bigcup \{\mathrm{cl}(b_i) : i < \beta\}$. Otherwise, $\mathrm{cl}(\overline{a}) := \bigcup \{\mathrm{cl}(a_i) : i < \mathrm{lh}(\overline{a})\}$.

Observation 2.4 *For any set S and sequence $\overline{a} \subset (\mathcal{M}_{\mu,\kappa}(S))$ of length $< \mathrm{reg}(\kappa)$, the closure of $\overline{a}$ under subterms has cardinality less than* $\mathrm{reg}(\kappa)$.

Remark 2.5 If λ is regular, $\lambda \ge \kappa$ then for every set S and sequence $\overline{a} \in {}^{<\lambda}(\mathcal{M}_{\mu,\kappa}(S))$ there exist a subset $S' \subseteq S$ of cardinality $\chi < \lambda$ and a term $\overline{\sigma}(\overline{b}) \in \mathcal{M}_{\mu,\kappa}(S')$ such that $\overline{a} = \overline{\sigma}(\overline{b})$.

Definition 2.6 Denote $\vartheta_{\mu,\kappa} := (\mathrm{reg}(\kappa) + \mu)^{<\mathrm{reg}(\kappa)}$. (In particular, $\vartheta_{\mu,\kappa} = \|\mathcal{M}_{\mu,\kappa}(\mathbf{I})\|$ for $\|\mathbf{I}\| = \mathrm{reg}(\kappa)$.)

Definition 2.7 Consider a free algebra $\mathcal{M}(S)$. We shall say that a set $\mathcal{A}$ of $\tau_{\mathcal{M}}$-terms is a *minimal system of terms for $\mathcal{M}$* if for every term $\sigma(\overline{v}) \in \mathcal{M}(S)$ there exists a single $\sigma'(\overline{x}) \in \mathcal{A}$ such that for some $\overline{u} \in S$ without repetitions it holds that $\sigma(\overline{v}) = \sigma'(\overline{u})$.

It follows from the Axiom of Choice that every free algebra has a minimal system of terms.

2.3 Extensions of classes of structures

For a class of structures $\mathfrak{k}$, we define several classes of structures that are based on $\mathfrak{k}$.

Definition 2.8 We let $\mathrm{Ex}^0_{\mu,\kappa}(\mathfrak{k})$ be the class of structures $\mathbf{I}^+$ which, for some $\mathbf{I} \in \mathfrak{k}$ satisfy $|\mathbf{I}^+| = |\mathbf{I}|$; $\tau_{\mathbf{I}^+} = \tau_{\mathbf{I}} \cup \{P_\alpha : \alpha < \mu\} \cup \{F_\beta : \beta < \kappa\}$ for new unary relation symbols P_α and new unary function symbols F_β; if $\mu > 0$ then $\langle P^{\mathbf{I}^+}_\alpha : \alpha < \mu\rangle$ is a partition of $|\mathbf{I}|$; and $\langle F^{\mathbf{I}^+}_\beta : \beta < \kappa\rangle$ are partial unary functions.

Definition 2.9 We let $\mathrm{Ex}^{0,\mathrm{lf}}_{\mu,\kappa}(\mathfrak{k})$ be the class of structures in $\mathrm{Ex}^0_{\mu,\kappa}(\mathfrak{k})$ for which the closure of every element under the new functions is finite. (Here, "lf" stands for "locally finite".)

Definition 2.10 We let $\mathrm{Ex}^1_{\mu,\kappa}(\mathfrak{k})$ be the class of structures in $\mathrm{Ex}^0_{\mu,\kappa}(\mathfrak{k})$ for which $F_\beta(P_\alpha) \subseteq P_{<\alpha} := \bigcup_{\gamma<\alpha} P_\gamma$ holds for every $\alpha < \mu$, $\beta < \kappa$.

Definition 2.11 We let $\mathrm{Ex}^2_{\mu,\kappa}(\mathfrak{k})$ be the class of structures of the form $\mathbf{I}^+ = \mathcal{M}_{\mu,\kappa}(\mathbf{I})$, for some $\mathbf{I} \in \mathfrak{k}$ (cf. Definition 2.2).

In the following, we use the convention that $\mathrm{Ex}_{\mu,\kappa}$ will be one of the above classes.

2.4 Some properties of representation and extension classes

Let us note several properties of representations:

Observation 2.12 *Let $\mathbf{I}_1, \mathbf{I}_2, \mathbf{I}_3$ be structures. If $f : I_1 \to \mathbf{I}_2$ is a Δ-representation of $\mathbf{I}_1$ in $\mathbf{I}_2$ and $g : \mathbf{I}_2 \to \mathbf{I}_3$ is an $\mathcal{L}^{\mathrm{qf}}_{\mathbf{I}_2}$-representation of $\mathbf{I}_2$ in $\mathbf{I}_3$, then $g \circ f$ is a Δ-representation of $\mathbf{I}_1$ in $\mathbf{I}_3$.*

Observation 2.13 *We have that $\mathrm{Ex}^{0,\mathrm{lf}}_{\mu,\kappa}(\mathfrak{k})$ and $\mathrm{Ex}^1_{\mu,\kappa}(\mathfrak{k})$ are included in $\mathrm{Ex}^0_{\mu,\kappa}(\mathfrak{k})$.*

Observation 2.14 *For κ finite, we have that $\mathrm{Ex}^{0,\mathrm{lf}}_{\mu,\kappa}(\mathfrak{k}) \supseteq \mathrm{Ex}^1_{\mu,\kappa}(\mathfrak{k})$.*

Proof. Let $\mathbf{I} \in \mathrm{Ex}^1_{\mu,\kappa}(\mathfrak{k})$ and fix $a \in |\mathbf{I}|$. Consider the tree formed by finite sequences $\bar{\eta} \in {}^{<\omega}[\kappa]$ for which $(F_{\eta_{\ell-1}} \circ \ldots \circ F_{\eta_0})(a)$ is well-defined (i.e., $(F_{\eta_s} \circ \ldots \circ F_{\eta_0})(a) \in \mathrm{dom}(F_{\eta_{s+1}})$ for every $s < \ell - 1$, where $\ell = \mathrm{lh}(\bar{\eta})$). Now, since it holds that $F_\beta(P_\alpha) \subseteq P_{<\alpha}$ for any $\beta < \kappa$, $\alpha < \mu$, and by the well-ordering of the ordinals each branch of this tree is finite. Kőnig's Lemma implies that the tree is finite, and since there is a map from the tree onto the closure of a under the functions, we see that $\mathbf{I}$ is actually locally finite. □

For any cardinals μ and κ, $\mathfrak{k}$ is qf-representable in all the extension classes of $\mathfrak{k}$ defined above. The classes $\mathrm{Ex}_{\mu_1+\mu_2,\kappa_1+\kappa_2}(\mathfrak{k})$ and $\mathrm{Ex}_{\mu_2,\kappa_2}(\mathrm{Ex}_{\mu_1,\kappa_1}(\mathfrak{k}))$ are qf-representable in each other (for $\mathrm{Ex} = \mathrm{Ex}^0, \mathrm{Ex}^{0,\mathrm{lf}}, \mathrm{Ex}^1$, or Ex^2).

Observation 2.15 *If $\mu_2 \leq \mu_1$, $\kappa_2 \leq \kappa_1$ then $\mathrm{Ex}_{\mu_2,\kappa_2}(\mathfrak{k})$ is qf-representable in $\mathrm{Ex}_{\mu_1,\kappa_1}(\mathfrak{k})$ (for $\mathrm{Ex} = \mathrm{Ex}^0, \mathrm{Ex}^{0,\mathrm{lf}}, \mathrm{Ex}^1$, or Ex^2).*

Observation 2.16 *For any $\mu_1, \mu_2, \kappa_2, \kappa_1 \geq \aleph_0$, the class $\mathrm{Ex}^2_{\mu_2,\kappa_2}(\mathrm{Ex}^1_{\mu_1,\kappa_1}(\mathfrak{k}))$ is qf-representable in $\mathrm{Ex}^1_{\mu_1,\kappa_1}(\mathrm{Ex}^2_{\mu_2,\kappa_2}(\mathfrak{k}))$.*

P r o o f . Note that $\mathrm{Ex}^1_{\mu_1,\kappa_1}(\cdot)$ expands each structure of the class $\mathfrak{k}$ by preserving the structure and enriching it with a partition and partial functions between classes of the partition (particularly such that the order of the classes is preserved). On the other hand, $\mathrm{Ex}^2_{\mu_2,\kappa_2}(\cdot)$ extends the structure by adding formal terms. Now, consider structures $\mathbf{I} \in \mathfrak{k}$, $\mathbf{I}^+ \in \mathrm{Ex}^1_{\mu_1,\kappa_1}(\mathfrak{k})$, $\mathbf{I}^{++} \in \mathrm{Ex}^2_{\mu_2,\kappa_2}(\mathrm{Ex}^1_{\mu_1,\kappa_1}(\mathfrak{k}))$, such that $\mathbf{I}^{++}$ is formed from $\mathbf{I}^+$ which is formed from $\mathbf{I}$ in the obvious way, using the extension classes we defined. In order to show that $\mathbf{I}^{++}$ is qf-representable in the latter class, we extend $\mathbf{I}^+$ to a new structure $\mathbf{I}^*$ by defining the partition P'_α as follows: we will take the partition $\{P_\alpha\}$ (of $|\mathbf{I}^+|$ only) and extend it to a partition of $|\mathbf{I}^{++}|$. Define (as sets) $P'_n = P_{n-1}$ for all $2 \leq n < \omega$, $P'_\alpha = P_\alpha$ for all $\omega \leq \alpha$, and $P'_0 = |\mathbf{I}^{++}| \backslash |\mathbf{I}^+|$. Note that we did not extend the domain of the structure, nor have we changed any of the partial or full functions. It should be clear that the identity map $\mathbf{I}^{++} \to \mathbf{I}^*$ is a qf-representation as required. □

Observation 2.17 *The class $\mathrm{Ex}^1_{2^\kappa,\kappa}(\mathfrak{k}^{\mathrm{eq}})$ is qf-representable in $\mathrm{Ex}^1_{0,\kappa}(\mathfrak{k}^{\mathrm{eq}})$.*

P r o o f . Let $\langle I, \langle P_\alpha : \alpha < 2^\kappa\rangle, \langle F_\beta : \beta < \kappa\rangle\rangle$ be the vocabulary of $\mathbf{I}^+$. Without loss of generality $|P_0^{\mathbf{I}^+}| \geq 2$ (every model such that $|P_0^{\mathbf{I}^+}| = 1$ can be represented in such a model). We select two distinct $t_0, t_1 \in P_0^{\mathbf{I}^+}$ and let $h : 2^\kappa \to \mathcal{P}(\kappa)$ a bijection. Consider the structure $\mathbf{I}' = \langle I, \langle F_\beta : \beta < \kappa\rangle, \langle G_\beta : \beta < \kappa\rangle\rangle$ whose universe is $|\mathbf{I}^+|$, $F_\beta^{\mathbf{I}'} = F_\beta^{\mathbf{I}^+}$ and also define for all $\gamma < \kappa$, $x \in P_\gamma^{\mathbf{I}^+}$,

$$G_\beta^{\mathbf{I}'}(x) = \begin{cases} t_0 & \gamma \in h(\beta), \\ t_1 & \gamma \notin h(\beta). \end{cases}$$

Here, I means that $P_0 \in \tau_{\mathbf{I}'}$ and $P_0^{\mathbf{I}'} = I$. It is easy to verify that the identity is a $\mathcal{L}^{\mathrm{qf}}_{\mathbf{I}^+}$-representation of $\mathbf{I}^+$ in $\mathbf{I}'$. □

Observation 2.18 *The class $\mathrm{Ex}^2_{\mu,\aleph_0}(\mathfrak{k}^{\mathrm{eq}})$ is qf-representable in $\mathrm{Ex}^1_{\mu,2}(\mathfrak{k}^{\mathrm{eq}})$ for $\mu \geq \aleph_0$ (in fact, it is representable by a structure with only two unary functions).*

P r o o f . Consider a structure from $\mathrm{Ex}^2_{\mu,\aleph_0}(\mathfrak{k}^{\mathrm{eq}})$. By the definition, we may assume that the structure is $\mathcal{M}_{\mu,\aleph_0}(S)$ for some $S \in \mathfrak{k}^{\mathrm{eq}}$. Let $\langle \sigma_\alpha(\overline{x}_\alpha) : \alpha < \mu\rangle$ be a minimal system of terms of $\mathcal{M}_{\mu,\aleph_0}(S)$ (cf. Definition 2.7, ω is the upper bound on function symbol arities, μ is the number of functions). Without loss of generality, $\mathrm{lh}(\overline{x}_0) = 1$, $\sigma_0(x_0) = x_0$. We now construct a structure $\mathbf{I}^+$ whose vocabulary is $\langle I, f_{\mathrm{last}}, f_{\mathrm{head}}, \langle P_\beta : \beta < \mu\rangle\rangle$ and whose universe is

$$I^{\mathbf{I}^+} = \left\{\langle \alpha, i, s_0, \ldots, s_i\rangle : \alpha < \mu,\ i < \mathrm{lh}(\overline{x}_\alpha),\ s_0, \ldots, s_i \in S\right\}$$

Let $\langle\langle \alpha_\beta, i_\beta\rangle : \beta < \mu\rangle$ enumerate the pairs $\left\{\langle \alpha, i\rangle : \alpha < \mu,\ i < \mathrm{lh}(\overline{x}_\alpha)\right\}$ in increasing lexicographic order. Let $P_\beta^{\mathbf{I}^+}$ be the set of sequences in $|\mathbf{I}^+|$ whose head is $\langle \alpha_\beta, i_\beta\rangle$, and let

$$f_{\mathrm{last}}^{\mathbf{I}^+}(\langle \alpha, i, s_0, \ldots, s_i\rangle) := \langle 0, 0, s_i\rangle$$

$$f_{\mathrm{head}}^{\mathbf{I}^+}(\langle \alpha, i, s_0, \ldots, s_i\rangle) := \langle \alpha, i-1, s_0, \ldots, s_{i-1}\rangle \quad (i > 0)$$

we define a map $h : \mathcal{M}_{\mu,\aleph_0}(S) \to |\mathbf{I}^+|$ by $h(\sigma_\alpha(\overline{v})) = \langle \alpha, \mathrm{lh}(\overline{x}_\alpha) - 1, v_0, \ldots, v_{\mathrm{lh}(\overline{x}_\alpha)-1}\rangle$. That h is a qf-representation of $\mathcal{M}_{\mu,\aleph_0}(S)$ in $\mathbf{I}^+$ is easy to verify. □

We say that a function f with domain and range contained in a structure $\mathbf{I}$ is a *partial automorphism* when for every sequence $\overline{a} \in |\mathbf{I}|$ of members of $\mathrm{dom}(f)$, it holds that $\mathrm{tp}_{\mathrm{qf}}(\overline{a}, \varnothing, \mathbf{I}) = \mathrm{tp}_{\mathrm{qf}}(f(\overline{a}), \varnothing, \mathbf{I})$.

3 Stable theories

The central result for this section is the following theorem:

Theorem 3.1 *Let T be a complete first-order theory. Then the following conditions are equivalent:*

1. *T is stable.*
2. *T is representable in* $\mathrm{Ex}^1_{0,|T|}(\mathfrak{k}^{\mathrm{eq}})$ *(cf. Definitions 2.1 & 2.10).*
3. *T is representable in* $\mathrm{Ex}^1_{|T|^+,|T|}(\mathfrak{k}^{\mathrm{eq}})$.
4. *T is representable in* $\mathrm{Ex}^1_{2^{|T|},|T|}(\mathfrak{k}^{\mathrm{eq}})$.
5. *For some cardinals $\mu_1, \kappa_1, \mu_2, \kappa_2$, it holds that T is representable in* $\mathrm{Ex}^1_{\mu_1,\kappa_1}(\mathrm{Ex}^2_{\mu_2,\kappa_2}(\mathfrak{k}^{\mathrm{eq}}))$ *(cf. Definition 2.11).*
6. *T is representable in* $\mathrm{Ex}^0_{\mu,\kappa}(\mathfrak{k}^{\mathrm{eq}})$ *for some cardinals μ, κ (cf. Definition 2.8).*
7. *T is representable in* $\mathrm{Ex}^0_{0,|T|}(\mathfrak{k}^{\mathrm{eq}})$.

We can add to the above list of equivalences the following:

5′. For some $\mu_1, \kappa_1, \mu_2, \kappa_2$, and $\lambda > (\mu_1 + |T|)^{<\kappa_1} + (\mu_2 + |T|)^{<\kappa_2}$, it holds that $\mathbf{EC}_\lambda(T)$ is representable in $\mathrm{Ex}^1_{\mu_1,\kappa_1}(\mathrm{Ex}^2_{\mu_2,\kappa_2}(\mathfrak{k}^{\mathrm{eq}}))$

This is a better condition, as it is a condition only on one $\mathbf{EC}_\lambda(T)$; similar modifications can be done for 2., 3., 4., 6., or 7.

Proof. Theorem 3.11 below proves 1.⇒3. Observation 2.15 implies 2.⇒3.⇒4. immediately. Similarly, 4.⇒5. and 7.⇒6. are immediate by properties of representations. 5.⇒1. follows from Theorem 3.3. 4.⇒2. follows from Observations 2.12 & 2.17. So far we have equivalence of conditions 1. to 5.. Now, 2.⇒7. is immediate since $\mathrm{Ex}^0_{0,|T|}(\mathfrak{k}^{\mathrm{eq}}) \supseteq \mathrm{Ex}^1_{0,|T|}(\mathfrak{k}^{\mathrm{eq}})$. We leave 6.⇒1. without a complete proof, since it is very similar to 5.⇒1.

3.1 Stability of representable theories

We shall first prove the first direction of the main theorem. Namely, that a theory which is representable in $\mathrm{Ex}^2_{\mu_2,\kappa_2}(\mathrm{Ex}^1_{\mu_1,\kappa_1}(\mathfrak{k}^{\mathrm{eq}}))$ is stable. The method relies on the combinatorial properties of models of stable theories, particularly that all order indiscernibles are indiscernible sets.

Theorem 3.2 *Assume that*

1. *T is representable in* $\mathrm{Ex}^2_{\mu_2,\kappa_2}(\mathrm{Ex}^1_{\mu_1,\kappa_1}(\mathfrak{k}^{\mathrm{eq}}))$, *for fixed cardinals $\mu_1, \mu_2, \kappa_1, \kappa_2$;*
2. *$\mu \geq \mathrm{reg}(\kappa_2) + \mu_1^+$;*
3. *$\lambda > \mu + \vartheta_{\mu_2,\kappa_2} + \kappa_1$ is a regular cardinal (cf. Definition 2.6); and*
4. *$\lambda > \chi^{<\mu}$ for all cardinals $\chi < \lambda$.*

Then, for every sequence $\bar{b} = \langle \bar{b}_\alpha : \alpha < \lambda \rangle \subseteq {}^{<\mu}[\mathfrak{C}_T]$ of length $< \mu$ there exists $S \in [\lambda]^\lambda$ such that $\langle \bar{b}_\alpha : \alpha \in S \rangle$ is an indiscernible set.

Proof. Let $M \models T$ such that $\bar{b}_\alpha \in {}^{<\mu}|M|$ for all $\alpha < \lambda$ and assume that $f : M \to \mathbf{I}^+ := (\mathcal{M}_{\mu_2,\kappa_2}(I), P_\alpha, F_\beta)_{\alpha<\mu_1,\beta<\kappa_1}$ is a representation, $I = \bigcup_{\alpha<\mu_1} P_\alpha$ where $\langle P_\alpha : \alpha < \mu_1 \rangle$ is a partition of I, $F_\beta : P_\alpha \to \bigcup_{\gamma<\alpha} P_\gamma$ (So, $\mathrm{dom}(F_\beta) = I$) and let $\bar{a}_\alpha = f(\bar{b}_\alpha)$ for all $\alpha < \lambda$.

Without loss of generality, we can add the following assumptions

(a) Each $\bar{a}_\alpha$ is closed under subterms in $\mathcal{M}_{\mu_2,\kappa_2}(I)$. (Since $\mu \geq \kappa_2$ is regular, we can apply Observation 2.4 for (μ, κ_2).)
(b) The set $\{F_\beta : \beta < \mu_1\}$ is closed under composition. (Including the empty composition which is the identity; recall that those are unary functions.)
(c) Each $\bar{a}_\alpha$ is closed under the partial functions F_β. (To find the closure of $\bar{a}_\alpha$ under the functions we need to add at most μ_1 elements, so the closure of $\bar{a}_\alpha$ is $< \mu$.)
(d) $\mathrm{lh}(\bar{a}_\alpha) = \xi = |\xi|$ for all $\alpha < \lambda$ (Since $\lambda = \bigcup_{\xi<\mu} \{\alpha < \lambda : \xi = \mathrm{lh}(\bar{a}_\alpha)\}$ and $\lambda > \mu$ is regular, and by reordering.)

The rest of the proof is by taking subsequences of the original sequence, while preserving the length λ, as follows (in brackets we note the common property of the desired subsequence):
First subsequence (sequences constructed by the same terms): By Remark 2.5, for each $i < \xi, \alpha < \lambda$ there exist terms $\sigma_{\alpha,i}(\overline{x}_{\alpha,i})$ in the language of $\mathcal{M}_{\mu_2,\kappa_2}$ and sequences $\overline{t}_{\alpha,i} \in {}^{<\mathrm{reg}(\kappa_2)}I$ such that $a_{\alpha,i} = \sigma_{\alpha,i}(\overline{t}_{\alpha,i})$, and also $|\bigcup\left\{\overline{x}_{\alpha,i:\alpha<\lambda,i<\xi}\right\}| \le \mathrm{reg}(\kappa_2)$. Since $\lambda > [\vartheta_{\mu_2,\kappa_2}]^{\xi}$ is regular, there exist $\langle\sigma_i(\overline{x}_i) : i < \xi\rangle$, $S_0 \in [\lambda]^{\lambda}$ such that $\langle\sigma_{\alpha,i}(\overline{x}_{\alpha,i}) : i < \xi\rangle = \langle\sigma_i(\overline{x}_i) : i < \xi\rangle$ for all $\alpha \in S_0$.
Second subsequence (the quantifier free type of $\overline{a}_\alpha$ relative to the P_α): since $(\kappa_1)^{\xi} < \lambda$, there exists a $S_1 \in [S_0]^{\lambda}$ such that the function $\alpha \mapsto \left\{(i, \beta) \in \xi \times \kappa_1 : a^i_\alpha \in P_\beta\right\}$ is constant on S_1 (denote this constant as the relation R_1).
Third subsequence (the quantifier free type of $\overline{a}_\alpha$ relative to the F_α): since $\xi^{\mu_1+\xi} \le \xi^{<\mu} < \lambda$, there exists a $S_2 \in [S_1]^{\lambda}$ such that the function $\alpha \mapsto \left\{(\beta, \zeta_0, \zeta_1) : \zeta_0, \zeta_1 < \xi,\ \beta < \mu_1,\ F_\beta(a^{\zeta_0}_\alpha) = a^{\zeta_1}_\alpha\right\}$ is constant on S_2 (denote this constant as the relation R_2).

Final subsequence: By the Δ-system lemma (Theorem 1.3(2)), there exist $S_3 \in [S_2]^{\lambda}$, $U \subseteq \xi$, $E \subseteq \xi \times \xi$ such that

(a) $\overline{a}_\alpha\restriction U = \overline{a}_\beta\restriction U$ for all $\alpha, \beta \in S_3$;
(b) E is an equivalence relation such that for all $\alpha \in S_3$: $a^i_\alpha = a^j_\alpha \leftrightarrow (i, j) \in E$;
(c) $a^i_\alpha = a^j_\beta \to i, j \in U$ for all $\alpha \neq \beta \in S_3$.

We now show that for any finite $\overline{u}, \overline{v} \subseteq S_3$ of length ℓ without repetition, it holds that $\overline{a}_{\overline{v}}$ and $\overline{a}_{\overline{u}}$ have the same quantifier-free type in $\mathbf{I}^+$.

Let $\varphi(\overline{x}_{\ell\times\xi})$ an atomic formula. By symmetry, it suffices to show that $\varphi(\overline{a}_{\overline{u}}) \to \varphi(\overline{a}_{\overline{v}})$.

Case 1: $\varphi(\overline{x}_{\ell\times\xi}) =$ "$\sigma_1(\overline{x}_{\ell\times\xi}) = \sigma_2(\overline{x}_{\ell\times\xi})$". The proof is carried by induction on the complexity of the term σ_1.

For $\sigma_1(\overline{x}_{\ell\times\xi}) = F_{\alpha,\beta}(\overline{\sigma}^*_1(\overline{x}_{\ell\times\xi}))$ it follows from properties of the free algebra that for some sequence of terms $\overline{\sigma}^*_2(\overline{x}_{\ell\times\xi})$ it holds that $\sigma_2(\overline{x}_{\ell\times\xi}) = F_{\alpha,\beta}(\overline{\sigma}^*_2(\overline{x}_{\ell\times\xi}))$ and also $\sigma^*_{1,i}(\overline{a}_{\overline{u}}) = \sigma^*_{2,i}(\overline{a}_{\overline{u}})$ for all $i < \alpha$. The induction hypothesis implies that $\sigma^*_{1,i}(\overline{a}_{\overline{v}}) = \sigma^*_{2,i}(\overline{a}_{\overline{v}})$ and thus $\sigma_1(\overline{a}_{\overline{v}}) = \sigma_2(\overline{a}_{\overline{v}})$ as required.

For $\sigma_1(\overline{x}_{\ell\times\xi}) = F_{\alpha^*_1}(\sigma^*_1(\overline{x}_{\ell\times\xi}))$, the validity of $\varphi(\overline{a}_{\overline{u}})$ implies that $\sigma_2(\overline{a}_{\overline{u}}) = \sigma_1(\overline{a}_{\overline{u}}) \in I$. It is easy to verify (by induction on the complexity of the term) that the terms $\sigma_s(s = 1, 2)$ contains only symbols from $\overline{x}_{\ell\times\xi}$, F_α (since $\mathrm{dom}(F_\alpha) \subseteq I$). Now, for a finite sequence of ordinals $\overline{\alpha}$, denote $F_{\overline{\alpha}} := F_{\alpha_0} \circ \ldots \circ F_{\alpha_{\mathrm{lh}(\overline{\alpha})}}$, ($F_{\langle\rangle}$ is the identity). It is easy to verify that the term $\sigma_s(\overline{x}_{\ell\times\xi})$ takes the form $F_{\overline{\alpha}_s}(x_{i_s,\zeta_s})$ for some sequence $\overline{\alpha}$. And the formula φ can be rewritten as $F_{\overline{\alpha}_1}(x_{i_1,\zeta_1}) = F_{\overline{\alpha}_2}(x_{i_2,\zeta_2})$.

Since the family $\langle F_\alpha : \alpha < \mu_1\rangle$ is closed under composition (cf. above), there exists a $\beta_s < \mu_1$ such that $F_{\overline{\alpha}_s} = F_{\beta_s}$. The sequences $\overline{a}_{u_{i_s}}$ are closed under $\langle F_\alpha : \alpha < \mu_1\rangle$, hence for some $\zeta^*_s < \xi$ it holds that $F_{\beta_s}(a_{u_{i_s},\zeta_s}) = a_{u_{i_s},\zeta^*_s}$ and $a_{u_{i_1},\zeta^*_1} = a_{u_{i_2},\zeta^*_2}$. The former implies $\langle\beta_s, \zeta_s, \zeta^*_s\rangle \in R_2$ and the latter implies that $\zeta^*_1, \zeta^*_2 \in U$ and $\langle\zeta^*_1, \zeta^*_2\rangle \in E$. Now, since $\overline{a}_{v_{i_1}}\restriction U = \overline{a}_{v_{i_2}}\restriction U$ it follows that $F_{\beta_s}(a_{v_{i_s},\zeta_s}) = a_{v_{i_s},\zeta^*_s}$ and $a_{v_{i_1},\zeta^*_1} = a_{v_{i_2},\zeta^*_2}$, so easily $\models \varphi(\overline{a}_{\overline{v}})$.

Case 2: $\varphi(\overline{x}_{\ell\times\xi}) = P_\alpha(\sigma(\overline{x}_{\ell\times\xi}))$. That $\models \varphi(\overline{a}_{\overline{v}})$ implies that $\sigma(\overline{x}_{\ell\times\xi}) = F_{\overline{\alpha}}(x_{i,\zeta})$ for some $i < \ell, \zeta < \xi$. Now by the closure of the functions under composition, the formula is equivalent to $P_\alpha(F_\beta(x_{i,\zeta}))$. And for some ζ^* we get that $F_\beta(a_{u_i,\zeta}) = a_{u_i,\zeta^*}$ and $P_\alpha(a_{u_i,\zeta^*})$ implying $\langle\beta, \zeta, \zeta^*\rangle \in R_2$ and $\langle\alpha, \zeta^*\rangle \in R_1$, respectively. Similar arguments give $\models \varphi(\overline{a}_{\overline{u}})$.

Theorem 3.3 *If T is representable in* $\mathrm{Ex}^2_{\mu_2,\kappa_2}(\mathrm{Ex}^1_{\mu_1,\kappa_1}(\mathfrak{k}^{\mathrm{eq}}))$*, then T is stable.*

Proof. Assume towards contradiction that T is unstable. By the order property (Theorem 1.7), and compactness, we can construct a sequence $\langle\overline{a}_i : i < \lambda\rangle$, where

$$\lambda = \beth_2(\mu + \vartheta_{\mu_2,\kappa_2} + \kappa_1)^+,\ \mu = \mathrm{reg}(\kappa_2) + \mu_1^+$$

such that $\models \varphi(\overline{a}_i, \overline{a}_j)^{\mathrm{if}(i<j)}$ holds for all $i, j < \lambda$.

Now by the assumptions let $f : M \to \mathbf{I}^+$ be a representation of M in $\mathrm{Ex}^2_{\mu_2,\kappa_2}(\mathrm{Ex}^1_{\mu_1,\kappa_1}(\mathfrak{k}^{\mathrm{eq}}))$. It is easily verified that the conditions in Theorem 3.2 hold. Hence, there exists $S \in [\lambda]^{\lambda}$ such that $\{\overline{a}_i : i \in S\}$ is an indiscernible set and particularly $\models \varphi(\overline{a}_i, \overline{a}_j) \leftrightarrow \varphi(\overline{a}_j, \overline{a}_i)$ holds for all $i, j \in S$, contradicting the assumption. □

3.2 Stability implies representability

Here we turn to proving the other direction of the main result. We recall several facts about stable theories (from [4, II & III]).

Theorem 3.4 *If $p(\overline{x})$ forks over A and $B \subseteq A$, $q \vdash p$ then q forks over B* (monotonicity of forking). *Moreover, for a stable T, $\mathrm{tp}(\overline{a}, A \cup \overline{b})$ does not fork over A iff $\mathrm{tp}(\overline{b}, A \cup \overline{a})$ does not fork over A* (symmetry)*; and for sets $A \subseteq B \subseteq C$ such that $\mathrm{tp}(\overline{a}, C)$ does not fork over B, and $\mathrm{tp}(\overline{a}, B)$ does not fork over A it holds that $\mathrm{tp}(\overline{a}, C)$ does not fork over A (*transitivity*). Finally, a type p forks over A iff it divides over A* (equivalence of forking and dividing).

Definition 3.5 A set $C \subseteq \mathfrak{C}$ will be called *strongly independent* over A if for any $a \in C$, the type $\mathrm{tp}(a, A \cup (C \setminus \{a\}))$ is the unique extension in $\mathbf{S}(A \cup (C \setminus \{a\}))$ of $\mathrm{tp}(a, A)$ which does not fork over A.

Definition 3.6 We say that a sequence $\langle \mathcal{I}_\alpha : \alpha < \gamma \rangle$ (where $\mathcal{I}_\alpha \neq \varnothing$ and $\alpha < \gamma$) is a *strongly independent decomposition of M of length γ* if for all $\alpha < \gamma$, it holds that $\mathcal{I}_\alpha$ is strongly independent over $\mathcal{I}_{<\alpha}$ (in M), and that $|M| = \mathcal{I}_{<\gamma}$, where $\mathcal{I}_{<\alpha} = \bigcup_{\beta<\alpha} \mathcal{I}_\beta$, of course.

Lemma 3.7 (Symmetry of strong independence) *Let $a_1, a_2 \in \mathfrak{C}$, $A \supseteq B_1, B_2$ such that $\mathrm{tp}(a_i, A \cup \{a_{3-i}\})$ does not fork over B_i and $\mathrm{tp}(a_i, A)$ is the unique non-forking extension of $\mathrm{tp}(a_i, B_i)$ in $\mathbf{S}(A)$ for $i = 1, 2$. Then $(*)_1 \Leftrightarrow (*)_2$, where:*

$$\textit{The type } \mathrm{tp}(a_i, B_i) \textit{ has a unique extension to } A \cup \{a_{3-i}\} \textit{ which is non-forking.} \qquad (*)_i$$

Proof. By symmetry it suffices to show that $\neg(*)_2 \Rightarrow \neg(*)_1$. Assume that $p := \mathrm{tp}(a_2, B_2)$ has two different non-forking extensions $p_1, p_2 \in \mathbf{S}(A \cup \{a_1\})$. Since both types are complete, there exists a formula $\varphi = \varphi(x, a_1, \overline{c})$ with $\overline{c} \subseteq A$ such that $\varphi \in p_1$, $\neg\varphi \in p_2$. Let b_1, b_2 realize p_1, p_2, respectively.

So, $\mathrm{tp}(b_i, A) = p_i \restriction A$ is a non-forking extension of p, by uniqueness it follows that $p_1 \restriction A = p_2 \restriction A$. Hence, for $i < 2$ there exist elementary maps F_i in $\mathfrak{C}$ such that $F_i \restriction A = \mathrm{id}_A$, $F_i(b_i) = a_2$.

Let $q_i \in \mathbf{S}(A \cup \{b_i\})$ be a non-forking extension of $\mathrm{tp}(a_1, B_1)$. Then $F_i(q_i) \in \mathbf{S}(A \cup \{a_2\})$ is a non-forking extension of $\mathrm{tp}(a_1, B_1)$ (since $F_i \restriction A = \mathrm{id}_A$, and non-forking is preserved under elementary maps).

Now, note that $\models \varphi(b_1, a_1, \overline{c}) \wedge \neg\varphi(b_2, a_1, \overline{c})$ which implies $\varphi(a_2, x, \overline{c}) \in F_1(q_1)$ and also $\neg\varphi(a_2, x, \overline{c}) \in F_2(q_2)$. This implies that $F_1(q_1)$, $F_2(q_2)$ are distinct extensions of $\mathrm{tp}(a_1, B_1)$, as needed. □

An *order-preserving refinement* is a partition $\langle \mathcal{J}_\alpha : \alpha < \gamma' \rangle$ which refines $\langle \mathcal{I}_\alpha : \alpha < \gamma \rangle$ such that for all $\alpha < \beta < \gamma$, if $\alpha', \beta' < \gamma'$, $\mathcal{I}_\alpha \supseteq \mathcal{J}_{\alpha'}$ and $\mathcal{I}_\beta \supseteq \mathcal{J}_{\beta'}$, then $\alpha' < \beta'$.

Proposition 3.8 *If $\langle \mathcal{I}_\alpha : \alpha < \gamma \rangle$ is a strongly independent decomposition of M, then every order-preserving refinement of this partition is also a strongly independent decomposition of M.*

Proof. Use the basic properties of non-forking. □

Theorem 3.9 *Assume T is stable, and let $A \subset B$ such that for every formula φ over B which is almost over A, φ is equivalent (in T) to a formula over A. If $p, q \in \mathbf{S}(B)$ are distinct and non-forking over A, there exists a $\varphi_*(x, \overline{c})$ over A such that $p \vdash \varphi_*$ and $q \vdash \neg\varphi_*$.*

Proof. By Theorem 1.9, there exists an equivalence relation $E \in \mathbf{FE}(A)$ such that $p(x) \cup q(y) \vdash \neg E(x, y)$. Let $\{b_i : i < n(E)\} \subseteq \mathfrak{C}$ enumerate representatives for all the distinct equivalence classes of E and let

$$w := \left\{ i < n(E) : p(x) \cup \{E(x, b_i)\} \text{ is consistent} \right\}.$$

Without loss of generality, assume that b_i realizes p for all $i \in w$. Let $\varphi(x) := \bigvee_{i\in w} E(x, b_i)$. It can be easily verified that $p(x) \vdash \varphi(x)$ and similarly, $q(x) \vdash \neg\varphi(x)$. We will show that $\varphi(x)$ is preserved by every $f \in \mathrm{Aut}(\mathfrak{C}/B)$:

Since p is over B and E is a formula over B, they are preserved by f and so, we have:

1. $p(x) \cup \{E(x, b_i)\} \Leftrightarrow p(x) \cup \{E(x, f(b_i))\}$ holds for all $i < n(E)$.
2. $\neg E(b_i, b_j)$ holds for every $i, j < n(E)$ with $i \neq j$ and hence also $\neg E(f(b_i), f(b_j))$.

Hence, f can be regarded as a permutation on $\{b_i/E : i \in w\}$, the equivalence classes of E in $\mathfrak{C}$, and therefore

$$f(\varphi(\mathfrak{C})) = f(\bigcup_{i \in w} b_i/E) = \bigcup_{i \in w} f(b_i)/E = \varphi(\mathfrak{C}).$$

Consequently, $\models \varphi(x) \equiv f(\varphi(x))$. Now use Theorem 1.14, which gives the required equivalent formula. □

Lemma 3.10 *Let $\mu = |T|^+$. For a stable T, for each model $M \models T$ there exists a strongly independent decomposition of length $\gamma \le \mu$.*

Since our definition of a decomposition does not allow empty sets, we use ordinals $\gamma \le \mu$. Such a decomposition is trivial for $\|M\| \le \mu$ (e.g., by taking an enumeration of $|M|$ and converting to singletons).

P r o o f . We choose by induction a sequence $\langle \mathcal{I}_\alpha : \alpha < \mu \rangle$ such that $\mathcal{I}_\alpha$ is strongly independent over $\mathcal{I}_{<\alpha}$, and is also maximal in $|M| \backslash \mathcal{I}_{<\alpha}$ with respect to this property for every $\alpha < \mu$. Indeed this process may end within at most μ steps, so we define $\gamma = \min(\{\alpha < \mu : \mathcal{I}_\alpha = \varnothing\} \cup \{\mu\})$.

Assume towards contradiction that the elements of M were not exhausted after μ iterations, then there exists an $a \in M \backslash \mathcal{I}_{<\mu}$. Recall that for a stable theory $\kappa(T) \le \mu$ (cf. [4, III.3.2 & III.3.3]) and so, by the definition of $\kappa(T)$ there exists a set $B \subseteq \mathcal{I}_{<\mu}$ with $|B| < \kappa(T) \le \mu$ such that $p(x) := \mathrm{tp}(a, \mathcal{I}_{<\mu})$ is non-forking over B, and by regularity of μ there exists $\alpha_0(*) < \mu$ such that $\mathcal{I}_{<\alpha_0(*)} \supseteq B$. Now, recalling Theorem 1.6, let

$$\Gamma := \left\{ \varphi(x; \bar{c}) : \varphi(x, \bar{c}) \text{ almost over } B,\ \varphi(x; \bar{y}) \in \mathcal{L},\ \bar{c} \in^{\mathrm{lh}(\bar{y})} \mathcal{I}_{<\mu} \right\},$$

and choose $\Gamma_* \subseteq \Gamma$ a minimal set of representatives up to logical equivalence from Γ. By Theorem 1.6 we have $|\Gamma_*| \le |B| + |T| < \mathrm{cf}(\mu)$, and since μ is regular, there exists $\alpha_1(*) < \mu$ such that $\bar{b} \subseteq \mathcal{I}_{<\alpha_1(*)}$ for all $\varphi(x, \bar{b}) \in \Gamma_*$. Let $\alpha(*) = \max\{\alpha_0(*),\ \alpha_1(*)\}$. Let $B' = \bigcup(\bar{b} : \varphi(\bar{x}, \bar{b}) \in \Gamma_*) \cup B$ be such that $B' \subseteq \mathcal{I}_{<\alpha(*)}$, $|B'| \le |T|$. We will now prove that the type $p \restriction \mathcal{I}_{\le\alpha(*)}$ is the unique extension in $\mathbf{S}(\mathcal{I}_{\le\alpha(*)})$ of $p \restriction \mathcal{I}_{<\alpha(*)}$.

Clearly $p \restriction \mathcal{I}_{\le\alpha(*)}$ extends $p \restriction \mathcal{I}_{<\alpha(*)}$, now assume that $q \in \mathbf{S}(\mathcal{I}_{\le\alpha(*)})$ extends $p \restriction \mathcal{I}_{<\alpha(*)}$, and does not fork over $\mathcal{I}_{<\alpha(*)}$.

By the transitivity of non-forking, q does not fork over B. Assume towards contradiction that $p \ne q$. By Theorem 1.9 there exists $E \in \mathbf{FE}(B)$ such that $q(x) \cup p(y) \vdash \neg E(x, y)$, and particularly $q(x) \vdash \neg E(x, a)$.

The formula $E(x, a)$ is almost over B, so by the choice of $\alpha_1(*)$ ($q \vdash \neg E(x, a)$ implies that $\neg E(x, a)$ is logically equivalent to a formula in x over $\mathcal{I}_{<\mu}$), there exist a $\bar{b} \subseteq \mathcal{I}_{<\alpha(*)}$ and $\varphi(x, \bar{b})$ logically equivalent to $E(x, a)$. Now since $E(a, a)$ holds, we also get $\models \varphi(a, \bar{b})$, and since $\bar{b} \subseteq \mathcal{I}_{<\alpha(*)}$, we get $\varphi(x, \bar{b}) \in \mathrm{tp}(a, \mathcal{I}_{<\alpha(*)}) = q \restriction \mathcal{I}_{<\alpha(*)}$, a contradiction.

So we have proved that $\mathrm{tp}(a, \mathcal{I}_{\le\alpha(*)})$ is the unique non-forking extension of $\mathrm{tp}(a, \mathcal{I}_{<\alpha(*)})$ in $\mathbf{S}(\mathcal{I}_{\le\alpha(*)} \setminus \{b\})$. Recall from the choice of $\mathcal{I}_{\alpha(*)}$ that for all $b \in \mathcal{I}_{\alpha(*)}$, $\mathrm{tp}(b, \mathcal{I}_{\le\alpha(*)} \setminus \{b\})$ is the unique extension in $\mathbf{S}(\mathcal{I}_{\le\alpha(*)} \setminus \{a\})$ which does not fork over $\mathcal{I}_{<\alpha(*)}$. Also, Lemma 3.7 implies that $\mathrm{tp}(b, \mathcal{I}_{\le\alpha(*)} \setminus \{b\} \cup \{a\})$ is the unique non-forking extension of $\mathrm{tp}(b, \mathcal{I}_{<\alpha(*)})$ in $\mathbf{S}(\mathcal{I}_{\le\alpha(*)} \setminus \{b\} \cup \{a\})$.

From the last two sentences it follows that the condition in Definition 3.5 holds for $\mathcal{I}_{\alpha(*)} \cup \{a\}$ over $\mathcal{I}_{<\alpha(*)}$, which contradicts the maximality of $\mathcal{I}_{\alpha(*)}$.

Theorem 3.11 *A stable first order theory T is representable in* $\mathrm{Ex}^1_{|T|^+,|T|}(\mathfrak{k}^{\mathrm{eq}})$.

P r o o f . Let $M \models T$. By lemma 3.10 we get a strongly independent decomposition $\langle \mathcal{I}_\alpha : \alpha < \gamma \le |T|^+ \rangle$ of M. By Proposition 3.8, we can assume without loss of generality that $|\mathcal{I}_1| = |\mathcal{I}_0| = 1$.

Define $\mathbf{I}^+ = \langle F_i^*,\ P_\alpha,\ F_{\varphi(x,\bar{y}),j} : \alpha < \gamma,\ i < |T|,\ \varphi(x, \bar{y}) \in \mathcal{L},\ j < k_{\varphi(x,\bar{y})} \rangle \in \mathfrak{k}^{\mathrm{eq}}$ as follows: $|\mathbf{I}^+| = |M|$, $P_\alpha^{\mathbf{I}^+} = \mathcal{I}_\alpha$ for all $\alpha < \gamma$.

By Lemma 1.11 we can find some $B(a) \subseteq \mathcal{I}_{<\alpha}$ for all $a \in \mathcal{I}_\alpha$, with the following properties:

1. $|B(a)| \le |T|$,
2. for every formula $\varphi(\bar{x}, \bar{c})$ over $\mathcal{I}_{<\alpha}$ which is almost over B(a), there exists a formula $\vartheta(\bar{x}, \bar{d}) B(a)$ such that $\models \forall \bar{x}(\vartheta(\bar{x}, \bar{d}) \leftrightarrow \varphi(\bar{x}, \bar{c}))$, (✠)
3. if $\alpha > 0$, then $\mathcal{I}_0 \subseteq B(a)$, and
4. $\mathrm{tp}(a, \mathcal{I}_{<\alpha})$ does not fork over $B(a)$.

Now, we define the functions $\langle (F_i^*)^{\mathbf{I}^+} : i < |T|\rangle$ on $\mathrm{dom}(F_i^*) = |M|\backslash(\mathcal{I}_0 \cup \mathcal{I}_1)$ for all $i < |T|$. Fix $\alpha > 1,\ a \in \mathcal{I}_\alpha$. We define $F_i^*(a) = b_i(a)(i < |T|)$ for some enumeration $\langle b_i(a) : i < |T|\rangle$ of $B(a)$, possibly with repetitions.

Now, let ψ_φ be a formula which is guaranteed to exist from Theorem 1.8 and define the partial unary functions $\{F^{\mathbf{I}^+}_{\varphi(x,\overline{y}),j}(x) : j < k_{\varphi(x,\overline{y})}\}$ as follows: Let $a \in \mathrm{dom}(F^{\mathbf{I}^+}_{\varphi(x,\overline{y}),j}) = |M|\backslash(\mathcal{I}_0 \cup \mathcal{I}_1)$, and let $\alpha < \mu$ be such that $a \in \mathcal{I}_\alpha$. Since $|\mathcal{I}_{<\alpha}| \geq 2$ for all $2 \leq \alpha < \mu$, it follows from the definition of ψ_φ (recall Theorem 1.8) that there exists $\overline{c}_a \in^{\mathrm{lh}(\overline{z}_\varphi)} \mathcal{I}_{<\alpha}$ such that $\models \varphi[a, \overline{b}] \Leftrightarrow \models \psi_\varphi[\overline{b}, \overline{c}_a]$. Now we define $F^{\mathbf{I}^+}_{\varphi(x,\overline{y}),j}(a) := (\overline{c}_a)_j$, for all $j < \mathrm{lh}(\overline{z}_\varphi)$. Thus we have defined $\mathbf{I}^+$ and we define $f : M \to \mathbf{I}^+$ as $f(a) = a$ for all $a \in |M|$. Now, to prove that f is indeed a representation, it follows by Proposition 3.12 that it would suffice to prove that $\mathrm{tp}(h(\overline{a}), \varnothing, M) = \mathrm{tp}(\overline{a}, \varnothing, M)$, for every $\overline{a} \subseteq \mathrm{dom}(h)$ and partial automorphism h of $\mathbf{I}^+$ with domain and range closed under functions.

Let $D_\alpha = \mathcal{I}_\alpha \cap \mathrm{dom}(h),\ R_\alpha = \mathcal{I}_\alpha \cap \mathrm{ran}(h)$. It is easily verified that for $\alpha < \gamma$, $h\restriction D_\alpha$ is a partial automorphism of $\mathbf{I}^+$ from D_α onto R_α. We will prove by induction on $\alpha < \gamma$ that

$$h(\mathrm{tp}(\overline{a}, D_{<\alpha}, M)) = \mathrm{tp}(h(\overline{a}), R_{<\alpha}, M) \text{ for all } \overline{a} \in D_\alpha \text{ of length } n \text{ that do not have repetitions,} \qquad (\boxtimes_{\alpha,n})$$

holds for all $n < \omega$.

Now, $\boxtimes_{\alpha,n}$ holds for $\alpha < 2$ since by the definition, $|\mathcal{I}_\alpha| = 1$. Now let $\alpha \geq 2$ and assume that $\boxtimes_{\beta,n}$ holds for all $n < \omega$, $\beta < \alpha$. We prove by induction on $n < \omega$ that $\boxtimes_{\alpha,n}$ holds.

First, $\boxtimes_{\alpha,1}$: Let $a \in \mathcal{I}_\alpha$, $\varphi = \varphi(x, \overline{c})$ a formula over $D_{<\alpha}$. Without loss of generality, assume $\models \varphi[a, \overline{c}]$. By the definition of the functions it follows $\models \psi_\varphi[\overline{c}, F_{\varphi,0}(a) \dots F_{\varphi,\mathrm{lh}(\overline{z}-1)}(a)]$. This formula contains only constants from $D_{<\alpha}$, so by the induction hypothesis, $\models \psi_\varphi[h(\overline{c}), h(F_{\varphi,0}(a)) \dots h(F_{\varphi,\mathrm{lh}(\overline{z}-1)}(a))]$ holds. Since h is a partial automorphism (with closed range and domain) of $\mathbf{I}^+, h$ commutes with the functions on $\mathbf{I}^+$ so $\models \psi_\varphi[h(\overline{c}), F_{\varphi,0}(h(a)) \dots F_{\varphi,\mathrm{lh}(\overline{z}-1)}(h(a))]$ holds. By the definitions of $F_{\varphi,j}$ $(j < \mathrm{lh}(\overline{z}))$, ψ_φ we get $M \models \varphi[h(a), h(\overline{c})]$, as needed.

For $n > 1$ we continue by induction, but first we state the following property of $\mathbf{I}^+$ (to be proven later):

> If $A \subseteq \mathbf{I}^+$ is closed under the partial functions in the vocabulary $\tau_{\mathbf{I}^+}$, then $A \cap \mathcal{I}_\alpha$ is strongly independent over $A \cap \mathcal{I}_{<\alpha}$. $\qquad (*)$

Now, let $\overline{a} \in D_\alpha$ of length n and $b \in D_\alpha \backslash \overline{a}$. By the induction hypothesis (on n), it follows that $h\restriction(D_{<\alpha} \cup \overline{a})$ is elementary. By (*), D_α is strongly independent over $D_{<\alpha}$. Hence, $\mathrm{tp}(b, D_{\leq\alpha} \setminus \{b\})$ does not fork over $D_{<\alpha}$ and particularly $\mathrm{tp}(b, D_{<\alpha} \cup \overline{a})$ does not fork over $D_{<\alpha}$.

By the induction hypothesis, $h\restriction(D_{<\alpha} \cup \overline{a})$ is elementary, and so $q := h(\mathrm{tp}(b, D_{<\alpha} \cup \overline{a}))$ does not fork over $h(D_{<\alpha}) = R_{<\alpha}$. Note that $q \supseteq h(\mathrm{tp}(b, D_{<\alpha}))$ and by $\boxtimes_{\alpha,1}$ (cf. above) $h(\mathrm{tp}(b, D_{<\alpha})) = \mathrm{tp}(h(b), R_{<\alpha})$ holds. Hence, q extends $\mathrm{tp}(h(b), R_{<\alpha})$ to a type over $R_{<\alpha} \cup \overline{a}$ and does not fork over $R_{<\alpha}$. Therefore there exists an extension $q \subseteq q' \in \mathbf{S}(R_{\leq\alpha} \backslash h(b))$ which does not fork over $R_{<\alpha}$.

Since R_α is closed under the partial functions, it follows from (*) above that R_α is strongly independent over $R_{<\alpha}$, meaning that $q' = \mathrm{tp}(h(b), R_{\leq\alpha} \setminus \{h(b)\})$. Now we reduce both types to the domain $R_{<\alpha} \cup h(\overline{a})$ to get

$$\mathrm{tp}(h(a), R_{<\alpha} \cup h(\overline{a})) = h(\mathrm{tp}(b, D_{<\alpha} \cup \overline{a})$$

and the induction step on n:

$$\mathrm{tp}(h(b^\frown\overline{a}), R_{<\alpha}) = h(\mathrm{tp}(b^\frown\overline{a}, D_{<\alpha}))$$

Hence, f is a representation.

We now prove (*) from the proof of Theorem 3.11:

Proof of (*). Let $A_\alpha = \mathcal{I}_\alpha \cap A, a \in A_\alpha$, and recall that $B(a) = \{F_i^*(a) : i < |T|\}$. We prove that $\mathrm{tp}(a, A \cap \mathcal{I}_{\leq\alpha} \setminus \{a\})$ is the unique non-forking extension of $\mathrm{tp}(a, A \cap \mathcal{I}_{<\alpha})$ in $\mathbf{S}(A \cap \mathcal{I}_{\leq\alpha} \setminus \{a\})$.

Since A is closed under the F_i^*, it follows that $B(a) \subseteq A$, and consequently $B(a) \subseteq A_{<\alpha}$. Also, $\mathrm{tp}(a, \mathcal{I}_{<\alpha})$ does not fork over $B(a)$ (Recall (✠) above). By transitivity of non forking $\mathrm{tp}(a, \mathcal{I}_{\leq\alpha} \setminus \{a\})$, which is a non-forking extension of $\mathrm{tp}(a, \mathcal{I}_{<\alpha})$, does not fork over $B(a)$ either. By the definition of $B(a)$, we also get that every formula over $\mathcal{I}_{<\alpha}$ which is almost over $B(a)$ is equivalent to a formula over $B(a)$ (again, cf. (✠)).

Now, by monotonicity of non-forking we get that $\mathrm{tp}(a, A_{\leq\alpha} \setminus \{a\}) \subseteq \mathrm{tp}(a, \mathcal{I}_{\leq\alpha} \setminus \{a\})$ does not fork over $A_{<\alpha}$.

To prove uniqueness, let $q_0 \in \mathbf{S}(A_{\leq\alpha} \setminus \{a\})$ be a non-forking extension of $\mathrm{tp}(a, A_{<\alpha})$. The type q_0 has a non-forking extension $q \in \mathbf{S}(\mathcal{I}_{\leq\alpha} \setminus \{a\})$. By transitivity, q does not fork over $B(a)$. Recall that the functions F_i^* are defined so that every formula over $\mathcal{I}_{<\alpha}$ and almost over $B(a)$ is equivalent to a formula over $B(a)$. The types $q\restriction\mathcal{I}_{<\alpha}$, $\mathrm{tp}(a, \mathcal{I}_{<\alpha})$ are both non-forking over $B(a)$. Since q extends $\mathrm{tp}(a, A_{<\alpha}) \supseteq \mathrm{tp}(a, B(a))$ we get that $q\restriction\mathcal{I}_{<\alpha}$, $\mathrm{tp}(a, \mathcal{I}_{<\alpha})$ (both non-forking over $B(a)$) agree on all formulas over $B(a)$, and by Theorem 3.9 this implies $q\restriction\mathcal{I}_{<\alpha} = \mathrm{tp}(a, \mathcal{I}_{<\alpha})$. Now, since q is a non-forking extension of $\mathrm{tp}(a, \mathcal{I}_{<\alpha})$ and $\mathcal{I}_\alpha$ is strongly independent over $\mathcal{I}_{<\alpha}$ we get that $q = \mathrm{tp}(a, \mathcal{I}_{\leq\alpha} \setminus \alpha)$ and so $q_0 = q\restriction(A_{\leq\alpha} \setminus \{a\}) = \mathrm{tp}(a, A_{\leq\alpha} \setminus \{a\})$, as required.

Proposition 3.12 *Let $\mathbf{I}^+ \in \mathrm{Ex}^1_{\mu,\kappa}(\mathfrak{k}^{\mathrm{or}})$ and $f : M \to \mathbf{I}^+$. Suppose that for every partial automorphism h of $\mathbf{I}^+$ with domain and range which are closed under the partial functions, and sequences $\overline{a}, \overline{b} \in M$, we have that $h(f(\overline{a})) = f(\overline{b})$ implies that $\mathrm{tp}(\overline{a}, \varnothing, M) = \mathrm{tp}(\overline{b}, \varnothing, M)$. Then f is a representation.*

P r o o f . Let f be as described above. Now assume towards contradiction that f is not a representation. Therefore there exist $\overline{a}, \overline{b} \in M$ which have different types in M such that the map $f(\overline{a}) \mapsto f(\overline{b})$ is a partial automorphism of $\mathbf{I}^+$. It is possible to extend this partial automorphism to one with domain and range closed under the partial functions, contrary to the definition of f. □

4 $\aleph_0$-stable theories

In this section we will prove the following result:

Theorem 4.1 (Characterization of $\aleph_0$-stable theories) *For a complete, countable first-order theory T, the following conditions are equivalent:*

1. *T is $\aleph_0$-stable.*
2. *T is representable in $\mathrm{Ex}^2_{\aleph_0,\aleph_0}(\mathfrak{k}^{\mathrm{eq}})$.*
3. *T is representable in $\mathrm{Ex}^1_{\aleph_0,2}(\mathfrak{k}^{\mathrm{eq}})$.*
4. *T is representable in $\mathrm{Ex}^{0,\mathrm{lf}}_{\aleph_0,2}(\mathfrak{k}^{\mathrm{eq}})$ (cf. Definition 2.9)*

P r o o f . Theorem 4.4 gives 1.⇒2. and Observation 2.18 gives 2.⇒3. 3.⇒4. is immediate since $\mathrm{Ex}^1_{\aleph_0,2}(\mathfrak{k}^{\mathrm{eq}}) \subseteq \mathrm{Ex}^{0,\mathrm{lf}}_{\aleph_0,2}(\mathfrak{k}^{\mathrm{eq}})$ by Observation 2.14. 4.⇒1. is the content of Proposition 4.2.

Proposition 4.2 *If T is representable in $\mathrm{Ex}^{0,\mathrm{lf}}_{\aleph_0,2}(\mathfrak{k}^{\mathrm{eq}})$ then T is $\aleph_0$-stable.*

P r o o f . To prove $\aleph_0$-stability it suffices to show that $|S(B, M)| \leq \aleph_0$ for every model $M \models T$ and countable $B \subseteq |M|$. Now, suppose that $f : M \to \mathbf{I}$ is a representation, then for any $a, b \in M$ such that $\mathrm{tp}(a, B, M) \neq \mathrm{tp}(b, B, M)$, it also holds that $\mathrm{tp}_{\mathrm{qf}}(f(a), f(B), \mathbf{I}) \neq \mathrm{tp}_{\mathrm{qf}}(f(b), f(B), \mathbf{I})$, so it suffices to show that $|\mathbf{S}_{\mathrm{qf}}(\mathbf{I}, A)| \leq \aleph_0$ for every structure $\mathbf{I} \in \mathrm{Ex}^{0,\mathrm{lf}}_{\aleph_0,2}(\mathfrak{k}^{\mathrm{eq}})$, and countable $A \subseteq I = |\mathbf{I}|$. Let $\mathbf{I}, A$ be as above, then without loss of generality A is closed under the functions of $\mathbf{I}$. Furthermore, $\mathrm{tp}_{\mathrm{qf}}(\overline{a}, \mathbf{I}) \in \mathbf{S}_{\mathrm{qf}}(\mathbf{I})$ is determined by formulas of the types $P_\alpha(\sigma(b))$ (for $b \in \overline{a}$), $\sigma_1(b_0) = \sigma_2(b_1)$ (for $b_0, b_1 \in \overline{a}$), and $\sigma_1(b_0) = b_1$ (for $b_0 \in \overline{a}$ and $b_1 \in A$) for terms $\sigma, \sigma_1, \sigma_2 \in \tau_{\mathbf{I}}$, and so necessarily unary. Moreover, since $\mathbf{I}$ is locally finite, $\mathrm{tp}_{\mathrm{qf}}(\overline{a}, \mathbf{I})$ is determined by a finite subset of these formulas. So, the number of unary types over A is at most $|A|^{<\omega} \leq \aleph_0$ □

We assume for the rest of this section that T is stable in $\aleph_0$.

Lemma 4.3 *Let $M \models T$ and $\mathcal{I}_0 \subseteq |M|$. There exists a sequence of sets $\langle \mathcal{I}_n : 0 < n < \omega \rangle$ such that*

1. *for all $a \in \mathcal{I}_n$, $n < \omega$ there exists a finite $B_a \subseteq \mathcal{I}_{<n}$ such that $\mathrm{tp}(a, \mathcal{I}_{\leq n} \setminus \{a\})$ is the unique non-forking extension of $\mathrm{tp}(a, B_a)$ in $\mathbf{S}(\mathcal{I}_{\leq n} \setminus \{a\})$,*
2. *$\mathcal{I}_n \cap \mathcal{I}_{<n} = \varnothing$, and*
3. *$\mathcal{I}_{<\omega} = |M|$.*

In particular, $\langle \mathcal{I}_n : n < \omega \rangle$ is a strongly independent decomposition of M.

Proof. We choose the sequence $\mathcal{I}_n$ by induction on $n \geq 0$. The case $n = 0$ is given in the assumptions. For $n > 0$ assume that we have $\mathcal{I}_n$ and choose $\mathcal{I}_{n+1}$ as a set $\mathcal{I}_{n+1} \subseteq M \setminus \mathcal{I}_{\leq n}$ which is maximal (possibly empty) under the requirements 1., 2. above. Now, assume towards contradiction that $a \in M \setminus \mathcal{I}_{<\omega}$. By Corollary 1.13, there exists a finite $B_a \subseteq \mathcal{I}_{<\omega}$ such that $\mathrm{tp}(a, \mathcal{I}_{<\omega})$ is the unique non-forking extension of $\mathrm{tp}(a, B_a)$ in $\mathbf{S}(\mathcal{I}_{<\omega})$. Similarly, for any $b \in \mathcal{I}_n$, let $B_b \subseteq \mathcal{I}_{<n}$ be such that $\mathrm{tp}(b, \mathcal{I}_{<n})$ is the unique non-forking extension of $\mathrm{tp}(b, B_b)$ in $\mathbf{S}(\mathcal{I}_{<n})$. Let $0 \leq n_* < \omega$ be minimal such that $B_a \subseteq \mathcal{I}_{<n_*}$.

Now, for any $b \in \mathbf{S}(\mathcal{I}_{n_*})$ it holds that $\mathrm{tp}(b, B_b)$ has a unique non-forking extension in $\mathbf{S}(\mathcal{I}_{\leq n_*} \setminus \{b\})$. Also, $\mathrm{tp}(a, B_a)$ has a unique non-forking extension in $\mathbf{S}(\mathcal{I}_{\leq n_*})$ (since it has a unique non-forking extension in $\mathbf{S}(\mathcal{I}_{<\omega})$). Now by the symmetry of strong independence (Lemma 3.7) it follows that $\mathcal{I}_{n_*} \cup \{a\}$ is strongly independent over $\mathcal{I}_{<n_*}$, which contradicts the maximality of $\mathcal{I}_{n_*}$.

Theorem 4.4 *Let $M \models T$, $\lambda = \|M\|$, $\mathcal{I}_0$ a set of indiscernibles in M. Then M can be represented in $\mathcal{M}_{\aleph_0,\aleph_0}(\mathcal{I}_0 \cup \lambda) \in \mathrm{Ex}^2_{\aleph_0,\aleph_0}(\mathfrak{k}^{\mathrm{eq}})$ by an extension of the identity function on $\mathcal{I}_0$.*

Proof. Let $\langle \mathcal{I}_n : n < \omega \rangle$ be as in Lemma 4.3. Let $g : \|M\| \to \lambda$ be a one-to-one function. The theory T is $\aleph_0$-stable and so $\mathbf{S}^m(\varnothing)$ is countable for all $m < \omega$. For convenience we use the symbols $\left\{F_{p,n} : n < \omega,\ p \in \mathbf{S}^{<\omega}(\varnothing)\right\}$ as the function symbols of $\mathcal{M}_{\aleph_0\aleph_0}(\mathcal{I}_0 \cup \lambda)$, such that for each m-type p, $F_{p,n}$ is an m-ary function symbol.

We define an increasing sequence of one-to-one functions $f_i : \mathcal{I}_{\leq i} \to \mathcal{M}_{\aleph_0,\aleph_0}(\mathcal{I}_0 \cup \lambda)$ by induction on $n < \omega$: Define f_0 as the identity on $\mathcal{I}_0$. Assume that f_n was defined and now define $f_{n+1} \supseteq f_n$ as follows. For each $a \in \mathcal{I}_{n+1}$ recall B_a from Lemma 4.3. Let $\overline{c}_a \in {}^{\ell}(\mathcal{I}_{\leq n})$ enumerate B_a. Now define $p := \mathrm{tp}(a^\frown \overline{c}_a, \varnothing, M) \in \mathbf{S}^{\ell+1}(\varnothing)$ and $f_{n+1}(a) := F_{p,n}(f_n(\overline{c}_a), g(a))$. Let $f = \bigcup_{n<\omega} f_n$. We will use (proof is omitted) an analogue of Proposition 3.12 to show that f is a representation:

Proposition 4.5 *Let $f : M \to \mathcal{M}(S)$. Suppose that for all $\overline{a}, \overline{b} \in M$ and every partial automorphism h of $\mathcal{M}(S)$ whose domain and range are closed under subterms, we have that $h(f(\overline{a})) = f(\overline{b})$ implies that $\mathrm{tp}(\overline{a}, \varnothing, M) = \mathrm{tp}(\overline{b}, \varnothing, M)$. Then f is a representation.*

First note that $a \in \mathcal{I}_n$ and also $f(a) = F_{p,n}(f(\overline{c}_a), g(a))$, so $p = \mathrm{tp}(a^\frown \overline{c}_a, \varnothing, M)$ and $\mathrm{tp}(a, \mathcal{I}_{\leq n} \setminus \{a\})$ is the unique non-forking extension of $\mathrm{tp}(a, \overline{c}_a)$.

We now show that f fulfills the conditions of the Proposition. Let h a partial automorphism of $\mathcal{M}(\mathcal{I}_0 \cup \lambda)$ with domain and range closed under the functions. Fix $n < \omega$ and sequences $\overline{a}, \overline{b} \in \mathcal{I}_{\leq n}$ such that $h(f(\overline{a})) = f(\overline{b})$. Since f is one-to-one, without loss of generality, $\overline{a}, \overline{b}$ are without repetition. We prove that $\mathrm{tp}(\overline{a}, \varnothing, M) = \mathrm{tp}(\overline{b}, \varnothing, M)$ by induction on n:

For $n = 0$, the proposition holds since $\mathcal{I}_0$ is an indiscernible set. For $n = m + 1$, we prove the claim by induction on $\ell = |\overline{a} \cap \mathcal{I}_n| = |\overline{b} \cap \mathcal{I}_n|$ (the latter equality is easy to verify).

For $\ell = 0$, this is the proposition of the induction hypothesis (on n). For $\ell = \ell_0 + 1$, let $a_0, b_0 \in \mathcal{I}_n$, $\overline{a}_1, \overline{b}_1 \in {}^{\ell}\mathcal{I}_n$, $\overline{b}_1, \overline{b}_2 \in {}^{<\omega}\mathcal{I}_{<n}$ such that $h(f(a_0\overline{a}_1\overline{a}_2)) = f(b_0\overline{b}_1\overline{b}_2)$. By the definition there exist $\overline{c}_{a_0}, \overline{c}_{b_0}$ such that $f(a_0) = F_{p,n}(f(\overline{c}_{a_0}), g(a_0))$, $f(b_0) = F_{p',n}(f(\overline{c}_{b_0}), g(b_0))$ for some sequences and types. Since $\mathrm{dom}(h)$ is closed under subterms we get:

$$F_{p',n}(f(\overline{c}_{b_0}), g(b_0)) = f(b_0) = h(f(a_0)) = h(F_{p,n}(f(\overline{c}_{a_0}), g(a_0))) = F_{p,n}(h(f(\overline{c}_{a_0})), h(g(a_0)))$$

and by the definition of the free algebra $p' = p$ and $h(f(\overline{c}_{a_0})) = f(\overline{c}_{b_0})$. The induction hypothesis implies that the map G defined by $G(\overline{a}_1) = \overline{b}_1$, $G(\overline{a}_2) = \overline{b}_2$, and $G(\overline{c}_{a_0}) = \overline{c}_{b_0}$ is elementary. Now, let $q = \mathrm{tp}(a_0, \overline{a}_1 \cup \overline{a}_2 \cup \overline{c}_{a_0})$. Since $\mathrm{tp}(a_0^\frown \overline{c}_{a_0}) = p = p' = \mathrm{tp}(b_0^\frown \overline{c}_{b_0})$ holds, it follows that $G(q) \restriction \overline{c}_{b_0} = \mathrm{tp}(b_0, \overline{c}_{b_0})$. The definition of $\mathcal{I}_n$ implies that $\mathrm{tp}(a_0, \mathcal{I}_{\leq n} \setminus \{a_0\})$ is non-forking over $\overline{c}_{a_0}$, and so is $\mathrm{tp}(a_0, \overline{a}_1 \cup \overline{a}_2 \cup \overline{c}_{a_0})$. On the other hand, since G is elementary, $G(q)$ does not fork over $\overline{c}_{b_0}$. Let $\mathbf{S}(\mathcal{I}_{\leq n} \setminus \{b_0\}) \ni q' \supseteq G(q)$ a non-forking extension. Since $\mathrm{tp}(b_0, \mathcal{I}_{\leq n} \setminus \{b_0\})$ is the unique non-forking extension of $\mathrm{tp}(b_0, \overline{c}_{b_0})$, and by transitivity q' is also a non-forking extension, it follows that $q' = \mathrm{tp}(b_0, \mathcal{I}_{\leq n} \setminus \{b_0\})$ and after reduction ($\overline{b}$ is without repetitions, so $b_0 \notin \overline{b}_1$ and $\overline{b}_1 \cup \overline{b}_2 \cup \overline{b}_{a_0} \subseteq \mathcal{I}_{\leq n} \setminus \{b_0\}$), we get that $G(q) = q' \restriction \overline{b}_1 \cup \overline{b}_2 \cup \overline{c}_{b_0} = \mathrm{tp}(b_0, \overline{b}_1 \cup \overline{b}_2 \cup \overline{b}_{a_0})$. Hence, $G \cup \left\{(a_0, b_0)\right\}$ is elementary and the proof is complete.

Acknowledgements The authors thank Assaf Hasson for his remarks on early versions of this paper. The first author would like to thank the Israel Science Foundation for partial support of this research via grant 710/07.

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