THE UNIVALENCE AXIOM IN POSETAL MODEL CATEGORIES

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ABSTRACT. In this note we interpret Voevodsky's Univalence Axiom in the language of (abstract) model categories. We then show that any posetal locally Cartesian closed model category $\mathcal{Q}t$ in which the mapping $Hom^{(w)}(Z \times B, C) : \mathcal{Q}t \longrightarrow Sets$ is functorial in Z and represented in $\mathcal{Q}t$ satisfies our homotopy version of the Univalence Axiom, albeit in a rather trivial way. This work was motivated by a question reported in [Gar11], asking for a model of the Univalence Axiom not equivalent to the standard one.

1. INTRODUCTION

Though the notion of a categorical model of dependent type theory was known for quite some time now, it is only in recent years that it was realized that the extra categorical structure required to model the structure of equality in dependent type theory corresponds to the structure of weak factorization equivalence, occurring in Quillen's model categories ([Gar11, p.2]). This connection is the basis for V. Voevodsky project known as *univalent foundations* whose main objective is to give new foundations of mathematics based on dependent type theory, which is intrinsically homotopical, and in which types are interpreted not as sets, but rather as homotopy types (cf.). A central idea in Voevodsky's univalent foundations is the extension of Martin-Löf's dependent type theory by a "homotopy theory reflection principle", known as the *Univalence Axiom*. Roughly speaking, the Univalence Axiom is the condition that the identity type between two types is naturally weakly equivalent to the type of weak equivalences between these types (V. Voevodsky, talk at UPENN, May 2011).

Within the category (or, rather, the model category) of simplicial sets $sSets$, Voevodsky constructs ([KLV12]) a model of Martin-Löf dependent type theory, satisfying also the Univalence Axiom. The models constructed in this way are called the standard univalent models (cf. Definition 3.2). During a mini-workshop around these developments held in Oberwolfach in 2010 the following question was raised: "Does UA have models in other categories (e.g., 1-topoi) not equivalent to the standard one?", [Gar11, p.27]. Though this question is probably referring to a univalent universe (for type theory), it seems to be meaningful also if taken literally. It turns out that the Univalence Axiom can be given a precise meaning in the framework of Quillen's model categories (provided they are locally Cartesian closed). It is then meaningful to ask whether such a model category satisfies the Univalence Axiom.

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In this note we give what could well be the simplest possible example of a model category satisfying the univalence axiom. We start from an interpretation of the univalence axiom in abstract model categories (apparently, folklore among experts) and show that there are posetal model categories (i.e., posetal categories with an additional model structure) satisfying that interpretation of the axiom. Though the univalence axiom degenerates in posetal model categories (because all fibrations are automatically univalent), in our examples the extra structure needed for the axiom to make sense (e.g., internal *Hom*-objects for weak equivalences in slice categories) is meaningful, and the interpretation of the axiom in these categories is natural (though, as it turns out, its validity is automatic). It is our hope that, viewed this way, the present note will allow mathematicians not versed in the nomenclature of type theory to get some intuition regarding the meaning of the Univalence Axiom, and that our construction and possible variants thereof could serve as easy test cases in studying its properties and consequences.

There are two main parts to this note. In the first of these parts (Section 3) we give an interpretation of the notion of a univalent fibration in the language of abstract model categories. To formulate this notion we introduce, for a model category $\mathfrak{C}$, a correspondence $Hom^{(w)}(Z\times B, C) : \mathfrak{C} \longrightarrow Sets$, intended to capture the class of weak equivalences between given fibrant objects $B, C \in \mathcal{O}b\mathfrak{C}$. We then show that, given a fibration $p : C \xrightarrow{(f)} B$, if $Hom^{(w)}_{B\times B}(-\times B\times C, C\times B)$ is a representable functor (in the slice category $\mathfrak{C}/(B\times B)$), the "obvious" morphism (in $\mathfrak{C}/(B\times B)$) from the diagonal B_δ to $Hom_{B\times B}(B\times C, C\times B)$ factors "naturally" (and uniquely in that sense) through the object representing this functor, $((C \times B)^{B\times C})_w$. We can then define the fibration p to be *univalent* if the morphism $m : B_\delta \longrightarrow ((C \times B)^{B\times C})_w$ is a weak equivalence.

We observe that in a posetal model category where the above definition makes sense all fibrations are univalent. This is, of course, to be expected in view of Voevodsky's informal description of a univalent fibration as "...one of which every other fibration is a pullback in at most one way (up to homotopy)".

The second of the main parts of the paper (Section 4) is dedicated to a self-contained construction of a posetal locally (w/f)-Cartesian closed model (i.e., where the functor $Hom^{(w)}_{B\times B}(- \times B \times C, C \times B)$ is represented), QtNaamen$_c$. This construction is a special case of a more general construction introduced in [GH10]. Though, as mentioned above, the Univalence Axiom degenerates in any posetal model category, we aim for a closer analogy with Voevodsky's construction – one having a natural interpretation of all the key features in the construction, and we show – in addition to the structure needed to define the notion of a univalent fibration – that QtNaamen$_c$ admits a notion of smallness such that there exists a universal (univalent) fibration, of which all "small" fibrations are a pullback (in a unique way), [Voe10, Theorem 3.5].

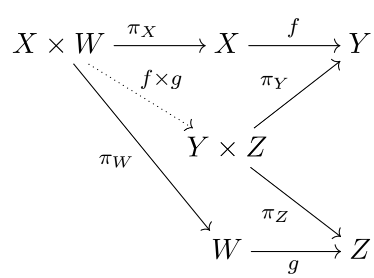
Admittedly, the model category QtNaamen$_c$ may be too simple an object to be of real interest. In [GH10] we suggest a construction of a (c)-(f)-(w)-labelled category analogous to that of QtNaamen$_c$ resulting in a non-posetal category whose slices are equivalent to those of QtNaamen$_c$. This category satisfies axioms (M1)-(M5) of Quillen's model categories, but does not have products (and co-products). We ask whether this richer category can be embedded in a model category, and whether such a model category would satisfy the Univalence Axiom, as formulated in this note.

It should be made clear that none of the authors of this note is familiar with type theory and its categorical models. Since most of the technical literature on Voevodsky's univalent foundations exists only in the form of Coq code, we decided to base our homotopy theoretic interpretation of the Univalence Axiom on the somewhat less formal presentation appearing, e.g., in [Voe10], [Gar11] and similar sources whose language is closer to the categorical language for which we were aiming. To keep track of our interpretation we used [KLV12] verifying that in the context of *sSets* our definitions conform with the ones appearing there. To compensate for the lack of precise references, we have taken some pains in giving a detailed formal account of our interpretation of those sources. M. Warren's comments and clarifications, [War], were of great help to us, but all mistakes, are – of course – ours.

A couple of words concerning terminology and notation are in place. In this text we refer to Quillen's axiomatization of model categories, as it appears in [Qui67]. Our usage of "Axiom (M0)...(M5)" refers to Quillen's enumeration of his axioms in that book. Our commutative diagram notation is pretty standard, and is explained in detail in [GH10]. The labeling of arrows, (c) for co-fibrations, (f) for fibrations and (w) for weak equivalences, is borrowed from N. Durov.

2. Cartesian closed posetal categories

We remind some category theoretic terminology that we need. Recall that a category $\mathfrak{C}$ is *Cartesian* if it is closed under finite Cartesian products (including the empty product, i.e., admitting a terminal object). Observe that if $\mathfrak{C}$ is Cartesian, $X \xrightarrow{f} Y$ and $W \xrightarrow{g} Z$, then there exists a unique arrow $X \times W \longrightarrow Y \times Z$ making the diagram commute:



We denote this arrow $f \times g$. More generally, the universal property of the Cartesian product assures that whenever $Z \xrightarrow{f} X$ and $Z \xrightarrow{g} Y$ there exists an arrow $Z \longrightarrow X \times Y$ such that the resulting diagram is commutative. In particular, if $X = Y = Z$ and $f = g = \mathrm{id}_X$ the resulting arrow $X \xrightarrow{\delta} X \times X$ is the diagonal morphism.

The Cartesian category $\mathfrak{C}$ is *Cartesian closed* if for any object X the functor $- \times X$ has a right adjoint, or, equivalently, $Hom(- \times X, Z)$ is representable for any objects X and

Z. This is equivalent to saying for any object Z there exists an object, Z^X, (also called the *internal Hom*) equipped with a morphism $Z^X \times X \xrightarrow{\epsilon_Z} Z$ inducing a bijection, natural in both Y and Z, between $Hom(Y \times X, Z)$ and $Hom(Y, Z^X)$, i.e., $\mathfrak{C}$ is Cartesian closed if for any object Y and morphism $Y \times X \xrightarrow{g}$ Z there is a unique arrow $Y \xrightarrow{\tilde{g}} Z^X$ such that $g = \epsilon_Z \circ (\tilde{g} \times \mathrm{id}_X)$.

Thus, a posetal category (where we let $X \longrightarrow Y$ if and only if $X \leq Y$) is Cartesian if it is a meet semi-lattice (i.e., that any two elements have an infimum). It is Cartesian closed if for all X, Z as above there exists an object, Z^X , such that for any object Y, $Y \times X \leq Z$ if and only if $Y \leq Z^X$. Indeed, if $\mathfrak{C}$ is posetal and Cartesian closed, then $Y \times X \leq Z$ implies straight from the definition that $Y \leq Z^X$, and $Y \leq Z^X$ implies (by the uniqueness of arrows) that $Y \times X \leq Z^X \times X$, so composing with ϵ we get $Y \times X \leq Z$. In the other direction, if $\mathfrak{C}$ is posetal and for all X, Z there exists an object Z^X such that for all Y, $Y \times X \leq Z$ if and only if $Y \leq Z^X$, then taking $Y = Z^X$ we get from $Z^X \leq Z^X$ that $Z^X \times X \leq Z$ (giving us the arrow $Z^X \times X \xrightarrow{\epsilon_Z} X$), and given $Y \times X \longrightarrow Z$ we get that $Y \leq Z^X$, with the commutativity of the resulting diagram following automatically.

Let us consider a Cartesian closed posetal model category, $\mathfrak{C}$, and let X be any object. Then the above condition tells us that $Y \times X \leq X$ (some object Y) if and only if $Y \leq X^X$. But the former condition is always true, by definition of the Cartesian product, so $Y \leq X^X$ for any object Y , i.e., for any object X the internal Hom, X^X , is the terminal object of $\mathfrak{C}$.

Given a category $\mathfrak{C}$ and $A \in \mathcal{O}b\mathfrak{C}$, the *slice* of $\mathfrak{C}$ over A, denoted $\mathfrak{C}/A$ is the category of arrows $B \longrightarrow A$: its objects are arrows $B \longrightarrow A$ in $\mathfrak{C}$ and an arrow from $B \longrightarrow A$ to $C \longrightarrow A$ is an arrow in $\mathfrak{C}$ making the triangular diagram commute. For a posetal category the slice $\mathfrak{C}/A$ can be identified with the full sub-category whose objects are all $B \in \mathcal{O}b\mathfrak{C}$ such that $B \longrightarrow A$. A category is locally Cartesian closed if $\mathfrak{C}/A$ is Cartesian closed for all $A \in \mathcal{O}b\mathfrak{C}$, [Awo10, Prop.9.20,p.206]. Observe that a posetal category with a terminal object is Cartesian closed if and only if it is locally Cartesian closed. Indeed, $\mathfrak{C}$ has a terminal object, $\top$ and $\mathfrak{C}/\top$ — which is, by assumption, Cartesian closed — is merely $\mathfrak{C}$, so locally Cartesian closed implies Cartesian closed. The other direction follows from the identification of $\mathfrak{C}/A$ with the full sub-category of elements smaller than A, and our characterisation of Cartesian closed posetal categories: if X, Z are objects in $\mathfrak{C}/A$ for some A, we can identify them with the corresponding objects in $\mathfrak{C}$, where we can find the object Z^X , and $Z^X \times A$ is the exponential object in $\mathfrak{C}/A$.

3. The Univalence Axiom

As explained in the introduction, the original formulation of the Univalence Axiom is given in the language of type theory (and, apparently, its precise formulation exists only in Coq code). The axiom asserts that, given a universe of type theory, the homotopy theory of the types in this universe should be fully and faithfully reflected by the equality on the universe. To prove that the universes of type theory he constructs in the category $sSets$ is univalent, Voevodsky proves, [Voe10, Theorem 3.5], that there is a fibration universal for the class of small fibrations, and that this fibration is univalent. Apparently, this statement is the right reformulation of the Univalence Axiom in the context of the model category of simplicial sets. Thus, we will be working with the following (re)formulation of Voevodsky's axiom:

Univalence Axiom: Let $\mathfrak{C}$ be a locally Cartesian closed model category, equipped with a notion of smallness. Then there exists a fibration, universal for the class of small fibration, and this fibration is univalent.

This section is dedicated to an explanation of the Univalence Axiom, as stated above. In Voevodsky's model of the Univalence Axiom within *sSets* a fibration is considered small if all its fibres are in the usual set theoretic sense (depending on the choice of the set theoretic setting, this could mean, e.g., "of cardinality smaller than λ", for some inaccessible λ). Lacking an intrinsic definition of smallness (i.e., one not referring to the internal structure of the objects of the category) our reformulation of the Univalence Axiom assumes that the model category $\mathfrak{C}$ comes equipped with its own notion of smallness, and we do not attempt to further clarify this aspect of the axiom. Therefore, this section is dedicated to the definition of the notion of a univalent fibration in arbitrary model categories.

Recall that Voevodsky's formulation of the Univalence Axiom takes place in the category of simplicial sets. Thus, in order to achieve our goal we have to build a dictionary between Voevodsky's terminology and the common terminology of model categories. Apparently, such a translation is folklore among experts, but since we were unable to find a precise formulation meeting both levels of generality and accuracy needed for this note, we give the details. Since there is no literature on the subject, our translation of this notion relies almost entirely on Voevodsky's notes, [Voe10], and some clarifications corresponded to us by Warren, [War].

Let us recall Voevodsky's definition of a univalent fibration in the category *sSets* of simplicial sets, [Voe10, p.71][1]:

> For any morphism $q : E \longrightarrow B$ consider the simplicial set $\underline{Hom}_{B\times B}(E \times B, B \times E)$. If q is a fibration then it contains, as a union of connected components, a simplicial subset $weq(E \times B, B \times E)$ which corresponds to morphisms which are weak equivalences. The obvious morphism from the diagonal $\delta : B \longrightarrow B \times B$ to $\underline{Hom}_{B\times B}(E \times B, B \times E)$ over $B \times B$ factors uniquely through a morphism $m_q : B \longrightarrow weq(E \times B, B \times E)$.
> [...] In this terminology the fibration $q : E \longrightarrow B$ is univalent if the morphism $m_q : B \longrightarrow weq(E \times B, B \times E)$ is a weak equivalence (cf. Definition 3.4 [ibid.])

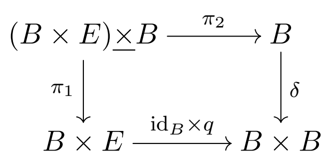
For the sake of clarity, we explain the above text word for word.

3.1. **The obvious morphism.** Let $\mathfrak{C}$ be a locally Cartesian closed model category, $E, B \in \mathcal{O}b\mathfrak{C}$ and $q : E \longrightarrow B$ a fibration. Thus, $E \times B \xrightarrow{q\times \mathrm{id}_B} B \times B$ and $B \times E \xrightarrow{\mathrm{id}_B \times q} B \times B$ are objects in the slice category $\mathfrak{C}/(B \times B)$. Since $\mathfrak{C}$ is locally Cartesian closed, we can identify

[1] In order to keep a clear distinction between Voevodsky's text and our interpretation of it, in the quote below we keep the original notation, though it does not conform with our own. Thus, below $\underline{Hom}_{B\times B}(E \times B, B \times E)$ denotes the internal *Hom*-object (in the slice category over $B \times B$) and not — as may be more common, the adjoint functor.

$\underline{Hom}_{B\times B}(E\times B, B\times E)$ with the object $((B\times E)^{E\times B})_{B\times B}$ (the internal Hom-object in the slice category over $B\times B$). So $((E\times B)^{B\times E})_{B\times B}$ and $B_\delta := B \xrightarrow{\delta} B\times B$ are objects in $\mathfrak{C}/(B\times B)$, so it is meaningful to have a morphism between them. To explain what the "obvious morphism" is recall that we have identified $Hom(X\times Y, Z)$ with $Hom(X, Z^Y)$, so in order to find a morphism in $Hom_{B\times B}(B_\delta, (E\times B)^{B\times E})$, it will suffice to find an "obvious morphism" in $Hom_{B\times B}(B_\delta \times (B\times E), E\times B)$.

To avoid confusion, let $X\underline{\times}Y$ denote the product in the category $\mathfrak{C}/(B\times B)$. By definition $B_\delta\underline{\times}(B\times E)$ is the pullback, in $\mathfrak{C}$, of the diagram $B \xrightarrow{\delta} B\times B \xleftarrow{\mathrm{id}_B\times q} B\times E$. So we get a commutative diagram:



In any Cartesian category, for any two objects A, C there is an isomorphism $A\times C \longrightarrow C\times A$, applying this observation twice we get an isomorphism of $\tau : (B\times E)\times B \longrightarrow B\times(E\times B)$, obtaining the following commutative diagram:

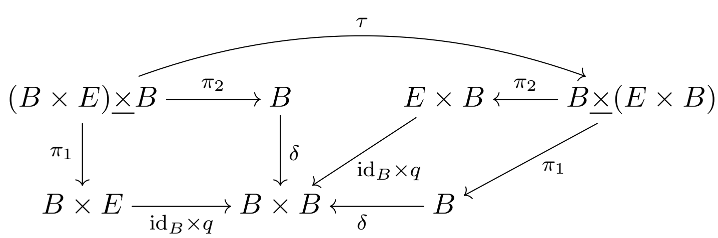


FIGURE 1

We have thus constructed, in $\mathfrak{C}/(B\times B)$ a morphism, between $B_\delta\underline{\times}(B\times E)$ and $E\times B$ (corresponding in the above diagram to the composition of $\pi_2\circ\tau$ and the isomorphism $B_\delta\underline{\times}(B\times E) \longrightarrow (B\times E)\underline{\times}B_\delta$ referred to above.). The corresponding morphism in $Hom_{B\times B}(B_\delta, (E\times B)^{B\times E})$ is the "obvious morphism".

Now that we understand what are $\underline{Hom}_{B\times B}(E\times B, B\times E)$ and the "obvious morphism" we turn to the object $weq(E\times B, B\times E)$ and to the factorisation of the obvious morphism through it.

3.2. **The Hom-object for weak equivalences.** Recall that, in Voevodsky's words, "[the object $weq(E\times B, B\times E)$] corresponds to morphisms [i.e., elements of $Hom_{B\times B}(E\times B, B\times E)$] which are weak equivalences. The class of weak equivalences is a sub-class of all morphisms. So, having identified (given objects B, C) $Hom(B, C)$ with the (representable) functor $Hom(- \times B, C)$, we are now looking for a representable sub-functor corresponding to weak equivalences:

Definition 1. Given $Z, B, C \in \mathcal{O}b\mathfrak{C}$, let

$$Hom^{(w)}(Z \times B, C) := \{h : Z \times B \longrightarrow C | (\pi_1 \times h) : Z \times B \xrightarrow{(w)} Z \times C\}.$$

To make the definition somewhat clearer, recall that $Hom(Z, C^B)$ is in a natural one to one correspondence with $Hom(Z \times B, C)$, which in turn is in one to one correspondence with those morphisms in $Hom(Z \times B, Z \times C)$ that do not change Z, i.e. $Hom_{/Z}(Z \times B \xrightarrow{\pi_1} Z, Z \times C \xrightarrow{\pi_1} Z)$ in the slice category over Z. However, weak equivalences are not preserved under these correspondences and it turns out, that the seemingly more natural choice of taking $Hom^{(w)}(Z \times B, C)$ to be the class of weak equivalences in $Hom(Z \times B, C)$ is not functorial in Z. Our definition corresponds to Voevodsky's (see, Lemma 22 of [KLV12], and the comment preceding Corollary 24 there) and though, in general, we cannot show that our definition is functorial, the following, is true, and will suffice for our needs:

Lemma 2. Let $\mathfrak{C}$ be a right proper Cartesian closed model category, B, C fibrant objects (i.e., $B \xrightarrow{(f)} \top$, $C \xrightarrow{(f)} \top$)). Then $Hom^{(w)}(Z \times B, C)$ is functorial in Z.

Proof. Recall that a model category is right proper if weak equivalences are stable under pullbacks along fibrations. Let $f : Z' \longrightarrow Z$ be any arrow. Then f induces an arrow $f_* : Hom(Z \times B, C) \longrightarrow Hom(Z' \times B, C)$. The mapping $Z \longrightarrow Hom(Z \times B, C)$ is, therefore, functorial, and our goal is to show that $Hom^{(w)}(- \times B, C)$ is a sub-functor, i.e., that if $g \in Hom^{(w)}(Z \times B, C)$ and $f : Z' \longrightarrow Z$ then $f_*(g) \in Hom^{(w)}(Z \times B, C)$. More precisely, we have to show that, in the following diagram,

$$\begin{array}{ccc} Z' \times B & \xrightarrow{f \times \mathrm{id}_B} & Z \times B \\ {\scriptstyle \pi_1 \times f_*(g)} \downarrow & & \downarrow {\scriptstyle \pi_1 \times g} \\ Z' \times C & \xrightarrow{f \times \mathrm{id}_C} & Z \times C \end{array}$$

FIGURE 2

if the right hand side arrow is a weak equivalence then so is the left hand side arrow. The key observation is that this diagram is a pullback. The proof of this observation is a standard and straightforward diagram chasing argument, but as it is lengthy we omit it. Assuming this, we can now proceed to the proof of the lemma.

Case I $Z' \xrightarrow{f} Z$ is a fibration.

Proof. It follows immediately from the definitions that

$$\begin{array}{ccc} Z' \times C & \xrightarrow{\pi_1} & Z' \\ {\scriptstyle f\times \mathrm{id}_C}\downarrow & & \downarrow{\scriptstyle (f)} \\ Z \times C & \xrightarrow{\pi_1} & Z \end{array}$$

is a pullback. Therefore, by the axioms of model categories, the arrow $f \times \mathrm{id}_C$ in the above diagram is a fibration, i.e., $Z' \times C \xrightarrow{(f)} Z \times C$. Therefore, in Figure 2 the bottom arrow is a fibration. Since, by properness, weak equivalences are preserved under such pullbacks, it follows that, in Figure 2, $Z' \times B \xrightarrow{\pi_1 \times f_*(g)} Z' \times C$ is a weak equivalence if $Z \times B \xrightarrow{\pi_1 \times g} Z \times C$ is, as claimed. $\square_{\text{Case I}}$

Case II $Z' \xrightarrow{g} Z$ is a (wc)-arrow.

Proof. The proof is quite similar to the one given in the first case. In fact, we will only be using the fact that g is a weak equivalence. Observe, first, that by definition

$$\begin{array}{ccc} Z \times X & \xrightarrow{\pi_1} & X \\ {\scriptstyle \pi_2}\downarrow & & \downarrow \\ Z & \longrightarrow & \top \end{array}$$

is a pullback (where $\top$ is the terminal object) for any objects X and Z. Thus, by the axioms of a model category, if X is a fibrant object (i.e., $X \xrightarrow{(f)} \top$), then the arrow $Z \times X \xrightarrow{\pi_2} Z$ is also a fibration. Since, by assumption, B, C are fibrant, it follows that $Z \times B \xrightarrow{\pi_2} Z$ and $Z \times C \xrightarrow{\pi_2} Z$ are both fibrations. Because, as above,

$$\begin{array}{ccc} Z' \times C & \xrightarrow{\pi_1} & Z' \\ {\scriptstyle f\times \mathrm{id}_C}\downarrow & & \downarrow{\scriptstyle (wc)} \\ Z \times C & \xrightarrow{\pi_1} & Z \end{array}$$

is a pullback diagram, and the bottom arrow is a fibration, properness implies $Z' \times C \xrightarrow{(w)} Z \times C$ and $Z' \times B \xrightarrow{(w)} Z \times B$. Applying Axiom 2-of-3 to the diagram of Figure 2 yields $Z' \times B \xrightarrow{(w)} Z' \times B$ for the arrow $\pi_1 \times f_*(g)$, as claimed. $\square_{\text{Case II}}$

In order to prove the lemma in the general case, let $Z' \xrightarrow{f^1} Z_{wc} \xrightarrow{f^2} Z$ be a $(wc) - (f)$ decomposition of f, and consider the following diagram:

$$\begin{array}{ccccc} Z' \times C_B & \xrightarrow{f^1\times \mathrm{id}_B} & Z_{wc}\times B & \xrightarrow{f^2\times \mathrm{id}_B} & Z\times B \\ \downarrow{\scriptstyle \pi_1\times f^1_*(f^2_*(g))} & & \downarrow{\scriptstyle \pi_1\times f^2_*(g)} & & \downarrow{\scriptstyle \pi_1\times g} \\ Z'\times C & \xrightarrow{f^1\times \mathrm{id}_C} & Z_{wc}\times C & \xrightarrow{f^2\times \mathrm{id}_C} & Z\times C \end{array}$$

Since f^2 is a fibration, by Case II, if $\pi_1 \times g$ is a weak equivalence then so is, $\pi_1 \times f^2_*(g)$, and by Case I (since f^1 is a weak equivalence) it follows that $\pi_1 \times f^1_*(f^2_*(g))$ too is a weak equivalence. But $f^1_*(f^2_*(g)) = f_*(g)$, so we are done. □

To simplify the discussion, we set:

Notation. If $Hom^{(w)}(- \times B, C)$ is a functor, and as such if it is represented in $\mathfrak{C}$ we let C^B_w denote the representing object. In particular we obtain a natural isomorphism of functors:

$$(**) \qquad Hom^{(w)}(Z \times B, C) \equiv Hom(Z, C^B_w).$$

Remark 3. In Voevodsky's text quoted above the object C^B_w is denoted $weq(B, C)$, and in [KLV12] it is denoted $Eq(B, c)$.

Definition 4. Let $\mathfrak{C}$ be a model category. Say that $\mathfrak{C}$ is (w/f)-Cartesian closed, if it is Cartesian closed and, in addition, $Hom^{(w)}(- \times B, C) : \mathfrak{C} \longrightarrow Sets$ is represented (in the sense of (∗∗) above) for all fibrant $B, C \in \mathcal{O}b\mathfrak{C}$ (i.e., $B \xrightarrow{(f)} \top$ and $C \xrightarrow{(f)} \top$). Say that $\mathfrak{C}$ is locally (w/f)-Cartesian closed, if for any $X \in \mathcal{O}b\mathfrak{C}$ the slice category $\mathfrak{C}/X$ is (w/f)-Cartesian closed.

Remark 5. Lemmas 21 and 22 of [KLV12] show that the model category sSets is (w/f)-Cartesian closed (and see also the remarks following Definition 23 and preceding Corollary 24 there).

We will now show that if $\mathfrak{C}$ is a (locally) Cartesian closed model category such that $Hom^{(w)}_{B\times B}(- \times (E \times B), B \times E)$ is represented in $\mathfrak{C}/(B \times B)$ then the object representing this functor satisfies the requirement in Voevodsky's text, namely, the "obvious" morphism from the diagonal to $(B \times E)^{E\times B}_{B\times B}$ factors uniquely through this representing object.

Referring back to Figure 1, we see that since τ is an isomorphism (and therefore a weak equivalence), as a morphism in $\mathfrak{C}/(B \times B)$ it is again a weak equivalence, so $B_\delta \underline{\times} (B \times E) \xrightarrow{\tau} B_\delta \underline{\times} (E \times B)$ is also a weak equivalence. By definition of the Hom-object for weak equivalences this means that, in the notation of Figure 1, $\pi_2 \circ \tau \in Hom^{(w)}_{B\times B}(B_\delta \underline{\times} (B \times E), E \times B)$. By (**) this gives rise to a morphism in $Hom_{B\times B}(B_\delta, ((E \times B)^{B\times E}_{B\times B})_w)$. Since, in general, C^B_w is a sub-functor of C^B this gives rise to a unique factorisation of the "obvious morphism" through $((B \times E)^{(E\times B)}_{B\times B})_w$, as claimed.

Remark 6. Observe that for any B if $q : E \longrightarrow B$ is a fibration then so is $q \times \mathrm{id}_B$, being the pullback of q along the projection $B \times B \longrightarrow B$. Applying this to the diagram of Figure 1, our usage of Lemma 2 in the last proof is legitimate.

3.3. **The Univalence Axiom in posetal model categories.** Having defined the object C_w^B for a locally right proper (w/f)-Cartesian closed model category $\mathfrak{C}$, we can define a fibration $p : E \longrightarrow B$ to be *univalent* if the morphism $\bar{m}_q : B_\delta \longrightarrow ((E\underline{\times}B)^{(B\underline{\times}E)})_w$ is a weak equivalence. The following observation is obvious:

Lemma 7. *Let $\mathfrak{C}$ be a locally right proper (w/f)-Cartesian closed posetal model category. Then every fibration is univalent.*

Proof. For any Cartesian category $\mathfrak{C}$ and $B \in \mathcal{O}b\mathfrak{C}$ there are morphisms $\pi : B \times B \longrightarrow B$, and $\delta : B \longrightarrow B \times B$. Thus, if $\mathfrak{C}$ is posetal $B \times B$ is isomorphic to B. So in the slice category over $B \times B$, the object B_δ is isomorphic to the terminal object, and by force, any arrow $B_\delta \longrightarrow Z$ is an isomorphism, and therefore a weak equivalence. □

Having seen that in posetal locally Cartesian closed model categories the notion of univalent fibrations degenerates, it remains to show that there exists a fibration p universal for the class of *small* fibrations. Of course, the notion of smallness in this context should be defined as well.

Definition 8. Let $\mathfrak{C}$ be a model category, Fix a morphism $\tilde{U} \xrightarrow{p} U$. A morphism $Y \xrightarrow{f} X$ is p-small if $Y \xrightarrow{f} X$ fits in a pull-back square:

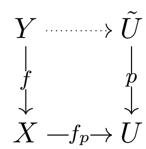


FIGURE 3. This is a pullback square if for any morphisms $Z \longrightarrow X$ and $Z \longrightarrow \tilde{U}$ making the diagram commute there is an arrow $Z \longrightarrow Y$ making the diagram commute.

Say that p is *universal* (with respect to a pre-defined class of *small* fibrations) if the class of p-small fibrations contains all small fibrations.

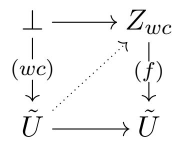
Observe that in a posetal category, given morphisms p and f as in the above definition, the morphism $X \xrightarrow{f_p} U$ is unique if it exists. Therefore, $Y \xrightarrow{f} X$ is p-small if and only if $X \longrightarrow U$ and Y is isomorphic to $\tilde{U} \times X$.

Lemma 9. *Let $\mathcal{Q}t$ be a posetal model category. Consider the unique morphism $\perp \longrightarrow \top$ and let $\tilde{U}$ be the unique object such that $\perp \xrightarrow{(wc)} \tilde{U} \xrightarrow{(f)} \top$. Let p denote the fibration $\tilde{U} \xrightarrow{(f)} \top$. Assume, in addition, that all morphisms in $\mathcal{Q}t$ are co-fibrations. Then a fibration $f : Y \longrightarrow X$ is p-small if and only if $\perp \xrightarrow{(wc)} Y$ (where $\perp$ is the initial object).*

Proof. The key to the proof is the following observation:

Claim If $Z \longrightarrow \tilde{U}$ then $Z \xrightarrow{(wc)} \tilde{U}$.

Proof. Let $Z \xrightarrow{(wc)} Z_{wc} \xrightarrow{(f)} \tilde{U}$ It will suffice to prove that $\tilde{U} \longrightarrow Z_{wc}$, since then Z_{wc} is isomorphic to $\tilde{U}$ (remember that $\mathcal{Q}t$ is posetal). This is immediate, using the following diagram:



$\square_{\text{Claim}}$

Now, if $\perp \xrightarrow{(wc)} Y \xrightarrow{(f)} X$, then $\perp \longrightarrow Y$ lifts with respect to $\tilde{U} \longrightarrow \top$ (in notation: $\perp \longrightarrow Y \curlywedge \tilde{U} \longrightarrow \top$), giving $Y \longrightarrow \tilde{U}$, and — since $Y \longrightarrow X$ — also $Y \longrightarrow X \times \tilde{U}$. It will suffice to show that this is an isomorphism. Let $Y \xrightarrow{(wc)} Y_{wc} \xrightarrow{(f)} X \times \tilde{U}$. By the above claim $X \times \tilde{U} \xrightarrow{(wc)} \tilde{U}$ and $Y_{wc} \xrightarrow{(wc)} \tilde{U}$. So by (M5):

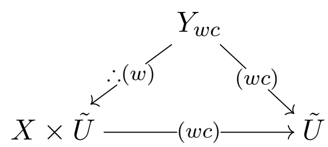


FIGURE 4. By (M5) the arrow $Y_{wc} \longrightarrow X \times \tilde{U}$ is a weak equivalence.

Since all morphisms in $\mathcal{Q}t$ are co-fibrations, we conclude that $Y_{wc} \xrightarrow{(wcf)} X \times \tilde{U}$, and since $\mathcal{Q}t$ is posetal, this must be an isomorphism. Therefore $Y \longrightarrow X \times \tilde{U} \curlywedge Y \longrightarrow X$, giving an arrow $X \times \tilde{U} \longrightarrow Y$, with the conclusion that Y is isomorphic to the product, as required.

In the other direction. If $Y \xrightarrow{(f)} X$ is p-small then $Y \longrightarrow \tilde{U}$, and by the claim $Y \xrightarrow{(wc)} \tilde{U}$. Similarly, if $\perp \xrightarrow{(wc)} Y_{wc} \xrightarrow{(f)} Y$ then $Y_{wc} \xrightarrow{(wc)} \tilde{U}$. So (M5), applied to the triangle $\tilde{U} \longleftarrow Y_{wc} \longrightarrow Y \longrightarrow \tilde{U}$, assures that $Y_{wc} \xrightarrow{(w)} Y$. Since, by assumption, all arrows are co-fibrations, we get $Y_{wc} \xrightarrow{(wcf)} Y$, with the conclusion that $\perp \xrightarrow{(wc)} Y$, as required. $\square$

We conclude that:

Proposition 10. *Let $\mathcal{Q}t$ be a posetal model category all of whose morphisms are co-fibrations. Let $\perp \xrightarrow{(wc)} \tilde{U} \xrightarrow{(f)} \top$, and define a fibration $Y \xrightarrow{(f)} X$ to be small if $\perp \xrightarrow{(wc)} Y$.*

Then the fibration $p : \tilde{U} \longrightarrow \top$ is universal. If, in addition, $\mathcal{Q}t$ is locally right proper (w/f)-Cartesian closed then $\mathcal{Q}t$ meets the Univalence Axiom, with respect to the above notion of small fibrations.

In the next section we give an example of a model category satisfying all the assumptions of Proposition 10, and whose model structure is not degenerate in the sense that not all morphisms are both fibrations and co-fibrations.

4. The model category $\mathrm{QtNaamen}_c$

In [GH10] we construct a posetal model category, QtNaamen, for set theory. Roughly speaking, this model category is obtained by constructing the simplest possible model category whose objects contain all sets, and where the notions of finiteness, countability and infinite equi-cardinality are to be reflected in the model structure. To our surprise this model category, with just a little extra spice from set theory, was able to extract meaningful set theoretic notions such as Shelah's covering numbers (from PCF theory), the notion of measurable cardinals and some more. The example we give in the present section is a full sub-category, $\mathrm{QtNaamen}_c$ of co-fibrant objects in QtNaamen.

With the possible exception of the fact that $\mathrm{QtNaamen}_c$ is (w/f)-Cartesian closed, all other properties of $\mathrm{QtNaamen}_c$ needed to apply Proposition 10 can be found in [GH10]. However, for the sake of completeness, we give the details. We do not attempt to justify the intuition behind the construction of $\mathrm{QtNaamen}_c$ (or its name[2]). Readers interested in the rational of the construction of $\mathrm{QtNaamen}_c$ that we are about to give are referred to [GH10].

To simplify the exposition, and in order to avoid irrelevant foundational issues, we give a slightly simplified version of the model category $\mathrm{QtNaamen}_c$. Let $\mathrm{QtNaamen}_c$ be the category whose objects are the non-empty directed members of $\{X : \emptyset \in X,\ \forall x \in X\ x \subseteq \mathbb{N},\ \ \forall x, y \in X\ x \cup y \in X\}$ and for $X, Y \in \mathcal{O}b\mathrm{QtNaamen}_c$ let $X \longrightarrow Y$ precisely when for every $x \in X$ there exists $y \in Y$ such that $x \subseteq y$. We leave it as an easy exercise for the reader to verify that this is indeed a (posetal) category with the initial object $\bot = \{\emptyset\}$ and terminal object $\top = \mathbb{P}(\mathbb{N})$.

Claim 11. The category $\mathrm{QtNaamen}_c$ has limits. Direct limits are given by the directed closure $X \vee Y = \{x \cup y : x \in X, y \in Y\}$ of the union $X \cup Y$, and inverse limits are given by pointwise intersection, namely $X \times Y = \{x \cap y : x \in X, y \in Y\}$. The same formulas hold for infinite limits.

Proof. This is straightforward. Assume, e.g. that we are given X, Y and $Z \longrightarrow X$, $Z \longrightarrow Y$. By definition, this means that for all $z \in Z$ there are $x \in X$, $y \in Y$ such that $z \subseteq x$ and $z \subseteq y$. This means that for all $z \in Z$ there are $x \in X$ and $y \in Y$ such that $z \subseteq x \cap y$. This proves that $X \times Y$ as defined above is the inverse limit of X and Y. The proof for direct limits is similar. □

[2]Indeed, for the latter there is no justification, except for the fact that StNaamen, our first attempt to construct a model category for set theory failed. Our next attempt was called QtNaamen.

Now we endow QtNaamen_c with a model structure. In order to meet the assumptions of Proposition 10, we must require that all morphisms are labelled (c). So we now proceed to the (w) and (f) labels. For the definition of weak equivalences it is convenient to denote for $X, Y \in \mathcal{O}b\text{QtNaamen}_c$, $X \longrightarrow^* Y$ if for all $y \in Y$ there exists $x \in X$ such that $|x \setminus y| < \aleph_0$. We now set $X \xrightarrow{(w)} Y$ if $X \longrightarrow Y$ and $Y \longrightarrow^* X$. This definition obviously satisfies Axiom (M5) (2 out of 3). Also, if $Z \xleftarrow{(wc)} X \longrightarrow Y$ then $Y \xrightarrow{(wc)} Z \vee Y$. Indeed, if $r \in Z \vee Y$ then $r = r_Z \cup r_Y$ where $r_Z \in Z$ and $r_Y \in Y$. There is $x \in X$ such that $|r_Z \setminus x| < \aleph_0$ but $X \longrightarrow Y$, so there is $y \in Y$ such that $x \subseteq y$ and thus $|r \setminus (y \cup r_Z)| \leq |r \setminus (x \cup r_Z)| < \aleph_0$. This shows that the (wc)-part of Axiom (M4) is met by this notation.

It remains to define the (f)-labelling: an arrow $X \longrightarrow Y$ is labelled (f) if and only if for every $x \in X \cup \{\emptyset\}$, $y \in Y$ and a finite subset $\{b_1, \dots, b_n\} \subseteq y$ there exists $x' \in X \cup \{\emptyset\}$ such that $(x \cap y) \cup \{b_1, ..., b_n\} \subseteq x'$.

First, we observe:

Claim 12. If $X \xrightarrow{(wcf)} Y$ then $Y \longrightarrow X$. If $X \longrightarrow Y$ and $Y \longrightarrow X$ then $X \xrightarrow{(wcf)} Y$.

Proof. Let $y \in Y$. We have to show that there exists $x \in X$ such that $y \subseteq x$. Let $x_0 \in X$ be such that $z := y \setminus x_0$ is finite, as provided by the (w)-label. So the (f)-label, applied for x_0, y and $z \subseteq y$ assures the existence of x with the desired property. □

This claim gives us, automatically, one part of (M1) — any arrow right-lifts with respect to an isomorphism — one part of (M2) — any arrow $X \xrightarrow{(c)} Y$ decomposes as $X \xrightarrow{(c)} Y \xrightarrow{(wf)} Y$ and (M3) (it remains only to verify that fibrations are stable under base-change). Axiom (M4) is also automatic. So we are left with the $(wc) \rightthreetimes (f)$ part of (M1), the (wc)-(f) decomposition of (M2) and the stability of fibrations under base change. All computations are trivial, so we will be brief.

Let $X \xrightarrow{(wc)} Y$ and $W \xrightarrow{(f)} Z$ be such that $X \longrightarrow W$ and $Y \longrightarrow Z$. We have to show that $Y \longrightarrow W$. So let $y \in Y$. Let $x \in X$ be such that $b := y \setminus x$ is finite. Let $w \in W$ be such that $x \subseteq w$. Let $z \in Z$ be such that $y \subseteq z$. Apply the definition of (f)-arrows with respect to w, z and b. Then there exists $w' \in W$ such that $(w \cap z) \cup b \subseteq w'$. So $y \subseteq w'$, as required. An essentially similar argument shows that fibrations are stable under base-change.

To prove (M2), let $X \longrightarrow Y$ be any arrow. Let

$$X_{wc} := \{x \cup y_0 : x \in X, (\exists y \in Y)(y_0 \subseteq y), y_0 \text{ finite}\}.$$

Then $X \xrightarrow{(wc)} X_{wc} \xrightarrow{(f)} Y$, as can be readily checked.

We conclude that QtNaamen_c is a posetal model category all of whose arrows are cofibrations. The morphism $\{\varnothing\} \xrightarrow{(wc)} X$ is not a fibration unless $X = \{\varnothing\}$, so not all morphisms are fibrations.

To show that $\mathrm{QtNaamen}_c$ is locally Cartesian closed, it suffices to show that it is Cartesian closed. Define, for $C, B \in \mathcal{O}b\mathrm{QtNaamen}_c$:

$$C^B := \bigvee\{Z : Z \times B \longrightarrow C\} = \bigcup\{\bigcup_{0<i<n} z_i \ : n \in \mathbb{N}, 0 < i < n, z_i \in Z_i, Z_i \times B \longrightarrow C\}.$$

This is an object in $\mathrm{QtNaamen}_c$, and the verification that it is a Hom-object for $Hom(B, C)$, i.e., that $C^B \times B \longrightarrow C$ and that for every object Z, if $Z \times B \longrightarrow C$ then $Z \longrightarrow C^B$, is obvious.

Remark 13. Note that the above shows that $\mathrm{QtNaamen}_c$ is, in particular, a logical model category in the sense of [GK95, Definition 23]. Consequently (cf. Theorem 26) $\mathrm{QtNaamen}_c$ admits a sound interpretation of the syntax of type theory (though the lack of non-trivial sections probably makes this interpretation trivial).

All of the above shows that $\mathrm{QtNaamen}_c$ is a posetal locally Cartesian closed model category. So in order to apply Proposition 10 it remains to show that it is locally (w/f)-Cartesian closed. We prove:

Claim 14. $Z \times B \xrightarrow{(wc)} Z \times C$ if and only if for all $\{z\} \longrightarrow Z$, $\{z\} \times B \xrightarrow{(wc)} \{z\} \times C$

Proof. The right to left direction is immediate from the definition of (wc)-arrows, so we prove the other direction. Without loss of generality, assume $z \in Z$. The arrow $Z \times B \xrightarrow{(wc)} Z \times C$ means that:

- for any $z \in Z$, $b \in B$ exists $z' \in Z$ and $c' \in C$ such that $\{z \cap b\} \longrightarrow \{z' \cap c'\}$; and
- for any $z'' \in Z$, $c'' \in C$ exists $z \in Z$, $b \in B$ such that $\{z'' \cap c''\} \longrightarrow^* \{z \cap b\}$.

Observe that the first bullet (for fixed $z \in Z, b \in B$) gives $z \cap b \subseteq z' \cap c'$, implying that $z \cap b \subseteq z \cap z' \cap c' \subseteq z \cap c'$, therefore $\{z\} \times B \longrightarrow \{z\} \times C$.

Analogously, for fixed $z'' \in Z, c'' \in C$ the assumption $\{z'' \cap c''\} \longrightarrow^* \{z \cap b\}$ implies $\{z'' \cap c''\} \longrightarrow^* \{z'' \cap z \cap b\} \longrightarrow \{z'' \cap b\}$. Combining these two observations we get $\{z\} \times B \xrightarrow{(wc)} \{z\} \times C$. □

Now, given $A \in \mathcal{O}b\mathrm{QtNaamen}_c$ and $B \longrightarrow A$, $C \longrightarrow A$, we define

$$(C^B_w)/A = \bigvee\{Z : Z \times B \xrightarrow{(w)} Z \times C\} \times A$$

and show that this is an object representing $Hom^{(w)}_A(- \times B, C)$ ($\mathrm{QtNaamen}_c$ is trivially right proper, and so are its quotients, hence this is indeed a functor). More precisely:

Claim 15. For all $Z \longrightarrow A$, we have $Z \longrightarrow (C^B_w)/A$ if and only if $Z \times B \xrightarrow{(w)} Z \times C$.

Proof. The right to left direction is immediate from the definition. So suppose $Z \longrightarrow (C^B_w)/A$. We need to show that $Z \times B \xrightarrow{(w)} Z \times C$. By Claim 14, this happens if for all $\{z\} \longrightarrow Z$, $\{z\} \times B \xrightarrow{(w)} \{z\} \times C$. But our assumption that $Z \longrightarrow (C^B_w)/A$ implies that $\{z\} \longrightarrow \cup_{0<i<n} z_i$, $z_i \in Z_i$ for some $n \in \mathbb{N}$ and Z_i such that $Z_i \times B \xrightarrow{(w)} Z_i \times C$. So by

Claim 14, $\{z_i\} \longrightarrow Z_i$ for $0 < i < n$, and, using that $(C^B_w)/A$ is closed under direct union, we are done. □

Note $\mathrm{Hom}\,(-xB, C)$ is representable even if B and C are not fibrant.

Combining everything together we get:

Theorem 16. There exists a non-trivial posetal model category satisfying the Univalence Axiom.

Recall from the introduction to the present section that $\mathrm{QtNaamen}_c$, is a full subcategory of the category of co-fibrant objects in the model category QtNaamen defined in [GH10]. The proof of the above theorem would work unaltered for the full category of all co-fibrant objects in QtNaamen. Of course this category captures the full homotopy structure of QtNaamen, and may – therefore – be a more interesting example. We remark also that there does not seem to be anything special about $\mathbb{N}$ or about $\aleph_0$ in the above construction (or in the more general construction of QtNaamen). Apparently, the exact same construction could be achieved for any regular cardinal λ (in place of $\aleph_0$) replacing, throughout "finite" by "less than λ". This gives an analogy with Voevodsky's notion of small fibrations: it is not unreasonable (see the following paragraph) to think of the morphisms in the resulting model category as a class of injections (satisfying certain compatibility conditions), our definition of smallness implies that a fibration is small precisely when every member of the class of these injections has a domain smaller than λ.

To conclude, let us consider the category $\mathfrak{C}$, whose objects are $\mathcal{O}b\mathrm{QtNaamen}_c$ and such that $\mathcal{M}or(X,Y)$ consists of the arrows $X \xrightarrow{\sigma} Y$ for $X, Y \in \mathcal{O}b\mathfrak{C}$ such that $\sigma : \bigcup X \longrightarrow \bigcup Y$ and $\sigma(X) \longrightarrow Y$ is an arrow in $\mathcal{M}or\mathrm{QtNaamen}_c$ (where $\sigma(X) := \{\{\sigma(a) : a \in x\} : x \in X\}$). The category $\mathfrak{C}$ is, on the one hand, obviously richer than $\mathrm{QtNaamen}_c$ (it is not posetal). But, on the other hand, it is readily seen that any slice of $\mathfrak{C}$ is (naturally) equivalent to the corresponding slice of $\mathrm{QtNaamen}_c$. This local model structure induces naturally a (c)-(f)-(w) labeling on $\mathcal{M}or(\mathfrak{C})$ (see [GH10] for the details) satisfying Quillen's axioms (M1)-(M5). But the category $\mathfrak{C}$ does not have products and co-products. So we ask:

Question: Is there a model category $\mathfrak{C}'$ such that the labeled category $\mathfrak{C}$ described above embeds in $\mathfrak{C}'$? Does $\mathfrak{C}'$ satisfy the Univalence Axiom?

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