EXERCICES DE STYLE: A HOMOTOPY THEORY FOR SET THEORY II

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Abstract. This is the second part of a work initiated in [6], where we constructed a model category, QtNaamen, for set theory. In the present paper we use this model category to introduce homotopy-theoretic intuitions to set theory. Our main observation is that the homotopy invariant version of cardinality is the covering number of Shelah's PCF theory, and that other combinatorial objects, such as Shelah's revised power function - the cardinal function featuring in Shelah's revised GCH theorem - can be obtained using similar tools. We include a small "dictionary" for set theory in QtNaamen, hoping it will help in finding more meaningful homotopy-theoretic intuitions in set theory.

> Every man is apt to form his notions of things difficult to be apprehended, or less familiar, from their analogy to things which are more familiar. Thus, if a man bred to the seafaring life, ... should take it into his head to philosophize concerning the faculties of the mind, it cannot be doubted, but he would draw his notions from the fabric of the ship, and would find in the mind, sails, masts, rudder, and compass.
>
> – Thomas Reid, "An Inquiry into the Human Mind", 1764

1. introduction

This is the second part of the paper [6], and continues the work initiated therein. In the first part of this paper we constructed a model category, QtNaamen, for set theory. Loosely speaking, QtNaamen can be thought of as the simplest model category for set theory modelling the notions of finiteness, countability and (infinite) equi-cardinality. From the purely category theoretic point of view QtNaamen is extremely simple (arrows are unique whenever they exist, so - e.g. - all diagrams commute), but as a model category the picture is slightly more complicated. On the one hand, most basic tools of model categories (such as the loop and suspension functors) degenerate in QtNaamen, but - on the other hand - as a model category QtNaamen does not seem to be such a trivial object (and the homotopy category associated with it is - at least to us - a new set theoretic object).

From the homotopy theoretic point of view, given the axioms of model categories and the set theoretic notions to be modelled the construction of QtNaamen is almost automatic (this is one of the main themes of [6]). Therefore, from that viewpoint QtNaamen should be an almost unavoidable (though somewhat degenerate) object. But, as far as we were able to ascertain, QtNaamen (or any close relative thereof) is not known (under the appropriate translation to set theoretic language, of course) to set theorists. On the face of it, it could be that the reason QtNaamen was not discovered by neither homotopy theorists nor set theorists is that it is too degenerate to be of interest. The aim of this paper is to show that this is, maybe, not entirely true. In the main result of the present paper we show that Shelah's covering numbers - one of the main objects of interest in PCF theory - discovered a century or so after Cantor's introduction of the notions of countability and cardinality, cannot be missed if one tries to study these notions from the homotopy theoretic point of view. Technically, we prove:

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Theorem 1. *Let λ be any cardinal. Then*

$$\mathbb{L}_c\text{card}\,(\{\lambda\}) = \text{cov}(\lambda, \aleph_1, \aleph_1, 2).$$

where $\mathbb{L}_c\text{card}$ *is the cofibrantly replaced left derived functor of the cardinality function (not functor (!))* card : $QtNaamen \to On^\top$.

The proof of Theorem 1 is, essentially, a triviality, but its formulation - at least for those not fluent in model category jargon - is far from obvious. The main part of this paper is dedicated to explaining the formulation of Theorem 1, and explaining - given the model category QtNaamen and the cardinality function card : $QtNaamen \to On^\top$ (where $On^\top$ is the class of ordinal augmented by a terminal object) - how to obtain the functor $\mathbb{L}_c\text{card}$. We then show how other covering numbers (such as Shelah's revised power function) can be recovered. This and similar constructions are discussed in Section 4.4.

It is intriguing that Shelah, in his book on Cardinal arithmetic, [14], and Kojman, in his survey of Shelah's PCF theory, [8], use algebraic topology as an analogy to explain the ideology and the usefulness of this theory, Kojman writes, rather directly that "This approach to cardinal arithmetic can be thought of as 'algebraic set theory' in analogy to algebraic topology" and Shelah, more by way of example mentions that: "... for a polyhedron v (number of vertices), e (number of edges) and f (number of faces) are natural measures, whereas $e + v + f$ is not, but from deeper point of view [the homotopy-invariant Euler characteristic] $v - e + f$ runs deeper than all...". Theorem 1 and its variants can be viewed as consolidating this analogy: they show that (some constructs of) PCF theory have an actual interpretation in terms of algebraic topology. But - at this stage - it is not clear whether this can be pushed much further, whether this connection with algebraic topology runs any deeper.

Of course, the covering numbers are not the only set theoretic notions that one can recover in QtNaamen. In [6] we saw that QtNaamen models finiteness, countability and equi-cardinality (at least to some extent). In the present paper we slightly enlarge the set theoretic dictionary of QtNaamen, giving some natural examples and non-examples (of set theoretic notions that QtNaamen cannot capture - e.g., the power set of a set). Notions such as a cardinal being measurable (Lemma 13) and intriguing possible connections with Jensen's covering lemma are discussed in Section 4.4.

As already mentioned, this is the second part of [6]. We expect readers of this paper to be familiar with the terminology and notation of [6], but for ease of reference we dedicate Section 2 to a a concise rendering of the main definitions and notational conventions. In Section 3 - as a warm up - we discuss various examples on how to use the model category QtNaamen in order to describe some basic notions of set theory, and describe (some of) its limitations. The statement and proof of Theorem 1 are given in Section 4: we explain the notions of derived functors and how to compute them in posetal categories. The last sub-section of Section 4 is dedicated to a brief overview of possible variants. We conclude the paper with some ideas for further investigation, emanating mainly form problems we identified in our construction: can we overcome the dependence of the derived functor of, say, cardinality on the choice of the model category (among equivalent model categories), can we find analogues for homotopy theory constructs (homotopy groups, long exact sequences etc.) in QtNaamen despite of it being "degenerate", can we actually prove set theoretic statements using QtNaamen (and the family of model structures QtNaamen_κ) and not only recover known concepts and definitions?

Remark 2 (Set-theoretic foundations)**.** It is often the case when working with categories that the category is *large*, namely that the objects and morphisms do not form a set, but rather a proper class. In the present work the objects of QtNaamen themselves are proper classes, and $\mathcal{O}b$QtNaamen is the collection of all classes. To avoid paradoxes one has to be careful, so some words concerning foundational issues may be in place.

There are many standard solutions for situations as described in the previous paragraph (most of them the authors are not familiar enough to say much about), and as we are using very little set theory, we believe that any of them could suit us with essentially no effect on the results. Probably, the easiest way to avoid foundational issues is (assuming the consistency of ZFC, of course) to choose and fix a transitive set-sized model $\underline{\mathcal{V}} = (\mathcal{V}, \in)$ of ZFC, and consider all constructions as taking place within $\underline{\mathcal{V}}$: read below a *class* as *a V-definable subset of* $\mathcal{V}$ (namely, the objects of our category are elements of $\mathbb{P}(V)$). This proves the consistency of our construction (relative to the consistency of ZFC). Stronger assumptions (e.g. large cardinal assumptions) could provide us with models whose notions of subset, ordinals, cardinals etc. coincides with the the corresponding notions in the "real" universe. We remark, moreover, that our construction seems to fit quite easily in set theories equi-consistent with ZFC, such as NBG.

2. Definitions of c-w-f arrows, notation and the construction of the model category.

We assume the reader familiar with the notation and terminology of [6], but for ease of reference we dedicate the present section to a concise rendering of the main definitions, notational conventions and results of [6].

Recall that a category $\mathfrak{C}$ is a pair $(\mathcal{O}b\mathfrak{C}, \mathcal{M}or\mathfrak{C})$ of *objects* and *morphisms* (or *arrows*) each carrying its own notion of equality. The arrows of a category can be composed whenever the composition makes sense (i.e. the arrows $X \longrightarrow Y$ and $Y' \longrightarrow Z$ can be composed whenever the objects Y and Y' are equal to produce and arrow $X \longrightarrow Z$). It is also required that to any object X there is a special morphism id_X, neutral with respect to left and right composition. Given a category $\mathfrak{C}$ it is often convenient (and we do it quite often in the present paper) to represent data in $\mathfrak{C}$ by means of a directed graph (possibly with loops and multiple edges between two vertices) whose nodes are objects and whose edges are morphisms. Such a diagram is *commutative* if the composition of arrows along any path in the graph depends only on the starting point and the ending point of the path, but not on the path itself.

A *labelled category* is a category where to each arrow is associated a (possibly empty) set of labels of a set S of labels. We require that for any label $s \in S$ the collection of all s-labelled arrows is itself a category. Namely, given a category $\mathfrak{C}$ the collection

$$\mathcal{O}b_s\mathcal{F} := \{X \in \mathcal{O}b\mathfrak{C} : \exists Y \in \mathcal{O}b\mathfrak{C} : Y \xrightarrow{(s)} X \vee X \xrightarrow{(s)} Y\}$$

with the collection $\mathcal{M}or_s\mathfrak{C}$ of all s-labelled arrows of $\mathfrak{C}$ is a category. Observe that in a labelled category the identity morphisms must carry all labels.

Before we explain what is a model category, it will be convenient to remind that given a category $\mathfrak{C}$ and arrows

$$\begin{tikzcd} X \arrow[r] \arrow[d] & W \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd}$$

Figure 1

The arrow $X \longrightarrow Y$ lifts with respect to the arrow $W \longrightarrow Z$ if for every commutative diagram as in Figure 1, there exists an arrow $Y \longrightarrow W$ making the resulting diagram commute. We denote this property by $X \longrightarrow Y \rightthreetimes W \longrightarrow Z$ and say that $X \longrightarrow Y$ left lifts with respect to $W \longrightarrow Z$ (or that $W \longrightarrow Z$ right lifts with respect to $X \longrightarrow Y$). Note that the notation $X \longrightarrow Y \rightthreetimes W \longrightarrow Z$ implicitly implies that $X \longrightarrow Y$ and $W \longrightarrow Z$) but not necessarily that $X \longrightarrow W$ and $Y \longrightarrow Z$. In particular, the lifting property $X \longrightarrow Y \rightthreetimes W \longrightarrow Z$ holds (vacuously) if there does not exist an arrow $X \longrightarrow W$ or if there does not exist an arrow $Y \longrightarrow Z$.

A model category is a $\{c, f, w\}$-labelled category, $\mathfrak{C}$, satisfying the following axioms:

- **(M0):** As a category $\mathfrak{C}$ is closed under (finite) direct and inverse limits.
- **(M1):** $(wc) \rightthreetimes (f)$ and $(c) \rightthreetimes (wf)$ (i.e., any appropriately labelled diagram as in Figure 1 has the lifting property).
- **(M2):** For any arrow $X \longrightarrow Y$ there are objects $X_{(wc)}$ and $X_{(wf)}$ such that $X \xrightarrow{(wc)} X_{(wc)} \xrightarrow{(f)} Y$ and $X \xrightarrow{(c)} X_{(wf)} \xrightarrow{(wf)} Y$ making the resulting diagrams commute.
- **(M3):** This is the axiom asserting that a model category is a labelled category.
- **(M4):** The pushforward of an arrow labelled (wc) and the pullback of an arrow labelled (wf) are both labelled (w).
- **(M5):** Given a triangle $X \longrightarrow Y \longrightarrow Z \longrightarrow X$, if any two of the arrows are labelled (w) so is the third.

If, in addition, the model category satisfies the requirement that any two of the labels determine the third, the model category is called *closed*.

2.1. The model category QtNaamen.

The model category QtNaamen whose construction is the main concern of [6] can be, roughly, thought of as the simplest model category modelling the notions of finiteness, countability and equi-cardinality. Below we give a combinatorial rendering of the $\{c, w, f\}$-labelling of our category. From this point of view, the construction may seem somewhat mysterious - so we start with a more "geometric" overview of the construction.

We start with the category $\mathrm{Sets}^{\subseteq}$, whose objects are sets, and whose arrows are inclusions:

- $A \longrightarrow_0 B$**:** A and B are sets and $A \subseteq B$.

$A \xrightarrow{(wc)}_0 B$**:** $A \longrightarrow_0 B$ and $B \setminus A$ is finite.

$A \xrightarrow{(c)}_0 B$**:** $A \longrightarrow_0 B$ and $\operatorname{card}(A) + \aleph_0 = \operatorname{card}(B) + \aleph_0$.

This category does not satisfy (M0) as it does not have a terminal object, so we add one formally, $\top$. But now the arrow $\varnothing \longrightarrow \top$ does not satisfy (M2): it is not too hard to see that $\varnothing_{(wc)}$ should be the "direct limit" of all finite sets - which can be identified with the (proper) class of all finite sets. So we replace the category we are working with. The objects, $\mathcal{O}b$StNaamen, are all classes of sets and the morphisms, $\mathcal{M}or$StNaamen are given by:

$X \longrightarrow Y$**:** For all $x \in X$ there exists $y \in Y$ such that $x \longrightarrow_0 y$.

We can naturally identify Sets$^{\subseteq}$ with a (full) sub-category of StNaamen, inducing a labelling (by the labels (c) and (wc)) of a class of arrows in $\mathcal{M}or$StNaamen. The full labelling on StNaamen is the one generated by this labelling (namely, the most economical labelling on StNaamen respecting the labelling of Sets$^{\subseteq}$ and satisfying (M2)). More precisely:

$X \xrightarrow{(f)} Y$**:** $X \longrightarrow Y$ and $(wc)_0 \rightthreetimes X \longrightarrow Y$.

$X \xrightarrow{(wf)} Y$**:** $X \longrightarrow Y$ and $(c)_0 \rightthreetimes X \longrightarrow Y$.

$X \xrightarrow{(c)} Y$**:** $X \longrightarrow Y$ and $X \longrightarrow Y \rightthreetimes (wf)$.

$X \xrightarrow{(wc)} Y$**:** $X \longrightarrow Y$ and $X \longrightarrow Y \rightthreetimes (f)$.

$X \xrightarrow{(w)} Y$**:** if $X \longrightarrow Y$ and there exists Z such that $X \xrightarrow{(wc)} Z \xrightarrow{(wf)} Y$.

The above labelling has a combinatorial interpretation (see [6, Proposition 16]):

Proposition 3. *The set theoretic interpretation of the last definition is:*

- (f) *an arrow $\mathcal{A} \longrightarrow \mathcal{B}$ is labelled (f) if and only if for every $A \in \mathcal{A} \cup \{\emptyset\}$, $B \in \mathcal{B}$ and a finite subset $\{b_1, \dots, b_n\} \subseteq B$ there exists $A' \in \mathcal{A} \cup \{\emptyset\}$ such that $(A \cap B) \cup \{b_1, ..., b_n\} \subseteq A'$.*
- (wf) *an arrow $\mathcal{A} \longrightarrow \mathcal{B}$ is labelled (wf) if and only if for every $A \in \mathcal{A} \cup \{\emptyset\}$, $B \in \mathcal{B}$ and subset $B' \subseteq B$ such that $\operatorname{card} B' \leqslant \operatorname{card}(A \cap B) + \aleph_0$, there exists $A' \in \mathcal{A} \cup \{\emptyset\}$ such that $B' \subseteq A'$.*
- (wc) *an arrow $\mathcal{A} \longrightarrow \mathcal{B}$ is labelled (wc) if and only if every $B \in \mathcal{B}$ is contained, up to finitely many elements, in some $A \in \mathcal{A} \cup \{\emptyset\}$ (i.e. $B \setminus A$ is finite for some $A \in \mathcal{A} \cup \{\emptyset\}$).*
- (c) *an arrow $\mathcal{A} \longrightarrow \mathcal{B}$ is labelled (c) if and only if for every $\{B\} \longrightarrow \mathcal{B}$ there exists $A \in \mathcal{A} \cup \{\emptyset\}$ such that $A \xrightarrow{\mathcal{B}} B$, where we define $\mathcal{A} \xrightarrow{\mathcal{B}} B$ if there exist $n \in \mathbb{N}$ and $\{B_0, \dots B_n\} \longrightarrow \mathcal{B}$ such that:*
  - (a) $\operatorname{card}(A \cap B_0) + \aleph_0 = \operatorname{card} B_0 + \aleph_0$,
  - (b) $\operatorname{card}(B_i \cap B_{i+1}) + \aleph_0 = \operatorname{card} B_{i+1} + \aleph_0$ *for all* $0 \leq i < n$, *and*
  - (c) $B = B_n$.
- (w) *an arrow $\mathcal{A} \longrightarrow \mathcal{B}$ is labelled (w) if and only if for every $A \in \mathcal{A}$, $B \in \mathcal{B}$ and subset $B' \subseteq B$ such that $\operatorname{card} B' \leqslant \operatorname{card}(A \cap B) + \aleph_0$, there exists $A' \in \mathcal{A}$ such that B' is contained in A' up to finitely many elements.*

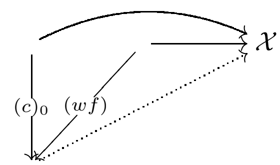
It turns out, using Proposition 3, that StNaamen with the above labelling satisfies (M0)-(M4). Moreover, the composition of two weak equivalences (i.e., the composition of two (w)-labelled arrows) is a weak equivalence (a fact that we will use freely). But StNaamen does not satisfy the full axiom (M5). The following counter example is relatively easy to come by:

$$
\begin{tikzcd}
{[A]^{<\aleph_1}} \arrow[r, "(wf)"] \arrow[dr, "(wf)"'] & {[A]^{<\aleph_1} \cup \{B\}} \arrow[d] & \{B\} \arrow[l] \arrow[dl, "(c)_0"] \\
& \{A\} &
\end{tikzcd}
$$

FIGURE 2. Let A, B be uncountable sets of the same cardinality and $[A]^{<\aleph_1}$ be the set of all countable subsets of A. Then the labelling of the arrows is as above, but $[A]^{<\aleph_1} \cup \{B\} \longrightarrow \{A\}$ will not, in general, be labelled (w).

This problem is addressed by restricting to a sub-category:

Definition 1. Say that an object $\mathcal{X}$ in StNaamen is *cute* if it satisfies the following diagram:



We let QtNaamen be the full labelled sub-category of cute objects of StNaamen.

FIGURE 3. The diagram reads: for every commutative diagram of solid arrows as above, there exists a dashed arrow such that the resulting diagram commutes.

The main result of [6] is:

Theorem 4. *The labelled category QtNaamen is a model category. The labelling is generated by co-fibrations between singletons.*

We remark that the category QtNaamen has limits and co-limits of arbitrary (not necessarily small) *well-defined* diagrams [A§3.1, Remark 9]; the notion of of "well-defined" depends on the set theoretic foundations used. E.g., if the objects of QtNaamen are *cute* classes, then the limits of any class of classes (i.e. any uniformly definable collection of classes) exist.

3. The expressive power of QtNaamen.

As already mentioned several times before, QtNaamen is a very simple model category. Intuitively, QtNaamen should be much simpler than set theory. To formulate this intuition somewhat more precisely, we observe that:

Lemma 5. *Let* $\mathbf{V}$ *be the universe of set theory and* σ *a bijective class function on* $\mathbf{V}$*. For a class* $X \subseteq \mathbf{V}$ *let* $\tilde{\sigma}(X) = \{\{\sigma(a) : a \in x\} : x \in X\}$*. Then* $\tilde{\sigma} : QtNaamen \longrightarrow QtNaamen$ *is a bijective functor on QtNaamen. Moreover,* $\tilde{\sigma}$ *preserves the model structure of QtNaamen.*

Proof. Because σ is a class function, if X is a class so is $\tilde{\sigma}(X)$ (hence, $\tilde{\sigma}$ is indeed a functor from QtNaamen to itself). The only non-trivial part is that $\tilde{\sigma}$ preserves the model structure, which is an immediate corollary of Proposition 3. □

Observe that in ZFC, given a set S, any $\sigma \in \mathrm{Sym(S)}$ extends to a class-bijection of $\mathbf{V}$ by setting $\sigma(x) = x$ for $x \notin S$. Therefore, the last lemma proves that the model structure on QtNaamen, while it must recognize the subset relation, does not respect - in a strong sense, the membership relation. For example, for a set X the set theoretic operation $X \mapsto \{X\}$ is not respected by QtNaamen, as can be inferred from the existence of an automorphism exchanging $\{\{\varnothing\}\}$ with $\{\{a\}\}$ (for any set a).

Even more trivially, since for any set S we have $\{S\} \longleftrightarrow \mathbb{P}(S)$, we see that QtNaamen cannot distinguish $\{S\}$ from the power set of S. Thus, despite of the fact that QtNaamen was constructed specifically to model the notion of equi-cardinality, it does so with limited success. Moreover, the notion of a set being a singleton is also a notion unknown to QtNaamen, as shows the above example.

In order to extract meaningful information from the model category QtNaamen we can - as is standard in mathematics - beside studying the structure of QtNaamen itself, study functors (and other "natural" set theoretic functions) from QtNaamen to other categories, and vice versa. The next, section, for example, is dedicated to the study of the cardinality function (not functor) card : QtNaamen $\longrightarrow On^{\top}$. In the present section we perform easier computations, showing that by imposing a little extra "natural" set theoretic structure on QtNaamen, more information can be obtained.

3.1. **Ordinals.** The first example we consider is more easily computed in StNaamen. The computations performed in this sub-section can be readily adapted to QtNaamen (with minor modifications), however, we were not able to find a natural set theoretic interpretation of these computations in the setting of QtNaamen.

Consider $Sets^{-}$, the full sub-category of StNaamen, whose objects are precisely those objects of StNaamen which happen to be sets. Consider the class function $S \mapsto S \cup \{S\}$ defined on $Sets^{-}$ (in fact, restricted to the category $Sets^{-}$ this is a functor). Indeed, an object of StNaamen is a set precisely if the operation $S \mapsto S \cup \{S\}$ is defined (in which case, of course, $S \longrightarrow S \cup \{S\}$ is a morphism in StNaamen, and therefore also in $Sets^{-}$). Let us label those arrows by (s).

Observe that the function $S \mapsto S \cup \{S\}$ on $Sets^{-}$ allows us to define *transitive* sets, namely: a set S is transitive precisely when $S \longrightarrow \{S\}$, or equivalently, if $S \cup \{S\} \longrightarrow S$, i.e., when the arrow $S \xrightarrow{(s)} S \cup \{S\}$ is invertible.

Indeed, $S \longrightarrow \{S\}$ if and only if $s \subseteq S$ for all $s \in S$, if and only if S is transitive. Thus, $S \in \mathcal{O}b\mathrm{StNaamen}$ is an ordinal if and only if $S \xrightarrow{(s)} S \cup \{S\} \longrightarrow S \longrightarrow On$, where On is the class of ordinals. We do not know whether the class On, as an object of StNaamen is definable (in some reasonable sense) in StNaamen, even when augmented by the (s)-labelling. We point out however, that at least on the face of it, since the membership relation is not recoverable in StNaamen, isolating the object On in QtNaamen allows us only to identify those objects of QtNaamen all of whose members are ordinals, but not necessarily ordinals themselves.

Note also that our (s)-labelling allows us only to identify arrows $S \longrightarrow S \cup \{S\}$. Given such an arrow, the object $\{S\}$ can be recovered as the complement of S in $S \cup \{S\}$, i.e. it is the unique object whose direct limit (in StNaamen) with $\{S\}$ is $S \cup \{S\}$ and whose inverse limit with S is $\varnothing$. Of course, the arrow $\{S\} \longrightarrow S$ never exists, as it would imply that $S \subseteq s$ for some $s \in S$, so that $s \in s$ contradicting the regularity axiom of ZFC.

In addition, by Proposition 3, if S is an ordinal then $\varnothing \xrightarrow{(wc)} S$ if and only if $S \leq \aleph_0$ and $\varnothing \xrightarrow{(c)} S$ if and only if $S \leq \aleph_1$. So these two cardinals can also be recovered in StNaamen (with the function $S \mapsto S \cup \{S\}$), as the direct limits of the classes of (wc) and (c)-arrows respectively. This is, of course, not surprising, since StNaamen was constructed to model the notions of finiteness and countability.

Of course, an ordinal α is limit precisely when $\beta \cup \{\beta\} \longrightarrow \beta \cup \{\beta \cup \{\beta\}\} \rightthreetimes \alpha \longrightarrow \top$ for all $\beta \in On$. It is a cardinal, precisely when for any ordinal β, if $\{\beta\} \xrightarrow{(c)} \{\alpha\}$ then $\{\alpha\} \xrightarrow{(c)} \{\beta\}$, which can be written as $\varnothing \longrightarrow \{\alpha\} \rightthreetimes \{\beta\} \xrightarrow{(c)} \{\alpha\}$. Finally, α is a regular cardinal precisely when (it is a cardinal and) $\alpha \xrightarrow{(wf)} \{\alpha\}$. To see this last claim, recall that, by construction, $\alpha \xrightarrow{(wf)} \{\alpha\}$ if and only if $\{A\} \xrightarrow{(c)} \{B\} \rightthreetimes \alpha \longrightarrow \{\alpha\}$ for all $\{A\} \xrightarrow{(c)} \{B\}$. But for sets, $\{A\} \xrightarrow{(c)} \{B\}$ if and only if $\mathrm{card}\, A + \aleph_0 = \mathrm{card}\, B + \aleph_0$. So the lifting property defining the (wf)-arrows assures that for any $B \subseteq \alpha$, if some $A \subseteq B$ of the same (infinite) cardinality satisfies $\{A\} \longrightarrow \alpha$ then $\{B\} \longrightarrow \alpha$. But $\{B\} \longrightarrow \alpha$ implies that there exists $\beta < \alpha$ such that $B \subseteq \beta$, so B is bounded in α. The other direction works in a similar way.

As explained above, the operation $S \mapsto S \cup \{S\}$ is "external" to QtNaamen (or StNaamen). As a side remark to this subsection we point out that some traces of it can be recovered in a more "geometric" way within these labelled categories. Consider, for example, the property $A = \{\{a\}\}$ for some set a. It is easy to see that if A is of this form then $\varnothing \longrightarrow Z \longrightarrow A$ implies that either $\varnothing \cong Z$ or $A \cong Z$. Conversely, if any decomposition $\varnothing \longrightarrow Z \longrightarrow A$ is such that $\varnothing \longrightarrow Z$ is an isomorphism or $Z \longrightarrow A$ is an isomorphism then $A = \{\{a\}\}$ for some set A. So we define,

Definition 1. An arrow $X \longrightarrow Y$ is *indecomposable* if whenever $X \longrightarrow Z \longrightarrow Y$ either $X \longrightarrow Z$ is an isomorphism or $Y \longrightarrow Z$ is an isomorphism.

With this definition the above observation can be stated as: A is of the form $\{\{a\}\}$ for some set a if and only if $\varnothing \longrightarrow A$ is indecomposable. Note also that while we do not know whether indecomposability can be expressed as a lifting property it is obviously invariant under graph automorphisms of QtNaamen (or StNaamen), and can therefore be thought of as an intrinsic property of these (labelled) categories.

It follows, for example, that with this in hand the property of A being *isomorphic* to a singleton (i.e., $A \cong \{a\}$ for some set a) can be stated as: for all X, Y if for all $\{\{a\}\}$, $\varnothing \longrightarrow \{\{a\}\} \rightthreetimes X \longrightarrow Y$ then $X \longrightarrow Y \rightthreetimes A \longrightarrow \top$. It will suffice, of course, to show that this statement is equivalent to $A \cong \{\bigcup A\}$. By definition, $\varnothing \longrightarrow \{\{a\}\} \rightthreetimes X \longrightarrow Y$ for all $\{\{a\}\}$ is equivalent to the statement that $\bigcup Y \subseteq \bigcup X$, in particular - for any set A - $\varnothing \longrightarrow \{\{a\}\} \rightthreetimes A \longrightarrow \{\bigcup A\}$ for all a. Therefore, if for all X, Y the assumption that $\varnothing \longrightarrow \{\{a\}\} \rightthreetimes X \longrightarrow Y$ for all a implies that $X \longrightarrow Y \rightthreetimes A \longrightarrow \top$ we can apply this with $A = X$ and $Y = \{\bigcup A\}$ to get $A \longrightarrow \{\bigcup A\} \longrightarrow A$, as required. The other direction is obvious, since the definition is invariant under changing A with an isomorphic object, and therefore, we may assume that A is a singleton.

3.2. **Cofinal and covering families.** A class A is $\subseteq$-cofinal in B if A is a sub-class of B and for all $b \in B$ there exists $a \in A$ such that $b \subseteq a$. In that case we also say that A *covers* B. By definition, this happens precisely when $B \longrightarrow A$. Since A is a sub-class of B we automatically get $A \longrightarrow B$, so that A is cofinal in B precisely when A is isomorphic to B, which happens if and only if $A \xrightarrow{(cwf)} B$. If B is a set, the cofinality of B is the minimal cardinality of a $\subseteq$-cofinal subset. In our notation, this can be expressed as:

$$\mathrm{cof}\,(B, \subseteq) = \min\{\mathrm{card}\, B' \,:\, B' \xrightarrow{(cwf)} B\}.$$

For a class B we have $\varnothing \xrightarrow{(c)} B$ if and only if every element of B is at most countable, and $\varnothing \xrightarrow{(wc)} B$ if and only if every element of B is finite. If S is a set then $\varnothing \xrightarrow{(c)} B \xrightarrow{(wf)} \{S\}$ if and only if B covers $[S]^{\leq \aleph_0}$, i.e., if the set of countable subsets of S is covered by B. In addition $\varnothing \xrightarrow{(c)} [S]^{\leq\aleph_0} \xrightarrow{(f)} \{S\}$ and $\varnothing \xrightarrow{(wc)} [S]^{<\aleph_0} \xrightarrow{(wf)} \{S\}$. Combining all of the above, we get that for a cardinal κ:

$$\operatorname{cov}(\kappa,\aleph_1,\aleph_1,2) = \min\{\operatorname{card} B' : \varnothing \xrightarrow{(c)} B' \xrightarrow{(wf)} \{\kappa\}\} = \inf\{\operatorname{card} B' : \varnothing \xrightarrow{(c)} B' \leftarrow B'' \xrightarrow{(wf)} \{\kappa\}\}$$

where $\operatorname{cov}(\kappa,\aleph_1,\aleph_1,2)$ is the minimal cardinality of a family of countable subsets of κ covering $[\kappa]^{\leq\aleph_0}$ (see Subsection 4.3 for more details). This shows that the covering number $\operatorname{cov}(\kappa,\aleph_1,\aleph_1,2)$ has a simple model categorical interpretation. The main goal of this paper is to show that, in fact, the right-most formula in the above equation is not only a simple translation of this set theoretic notion, but arises naturally from a model categorical study of QtNaamen. It is the *co-fibrantly replaced left-derived functor of the cardinality function from QtNaamen to* $On^{\top}$. This will be explained in detail in the next section.

3.3. **Some non-set theoretic concepts.** We conclude with a few simple non-set theoretic statements that can be expressed in QtNaamen. Consider, for example, $\mathcal{N}$ a monster model of some first order theory T. For a cardinal β let $\mathcal{N}_\beta$ be the set of all elementary sub-models of $\mathcal{N}$ of cardinality at most β. Then $\mathcal{N}_\beta \xrightarrow{(wf)} \{\mathcal{N}\}$ is the statement that the Lowehnheim-Skolem number of T is at most β. Namely, it states that every subset of N of cardinality at most β is contained in a model of size at most β. In particular, if T is countable then $\varnothing \xrightarrow{(c)} \mathcal{N}_{\aleph_0} \xrightarrow{(wf)} \{\mathcal{N}\}$ is the statement that every countable set is contained in a countable model. Of course, the objects $\mathcal{N}_\beta$ do not seem to be endemic to the model categorical setup.

Recall that if X is a topological space, then a set $A \subseteq X$ is closed if and only if $\mathsf{acc}\,(A) \subseteq A$, namely, if A contains all its accumulation points. Thus, a topological space can be given (instead of giving a collection of closed sets) by giving, to any subset S of X the collection $\mathsf{acc}\,(S)$. Therefore, a topological space X gives rise to a functor acc : QtNaamen $\longrightarrow$ $Sets^{-}$ by $\mathcal{S} \mapsto \{\mathsf{acc}\,(S \cap X) : S \in \mathcal{S}\}$, and the topology on X can be recovered from the functor acc only - namely, this gives a purely category theoretic definition of the topology.

4. Functors and derived functors.

The idea of "forgetting structure" is, of course, a central theme in mathematics. The pigeon-hole principal is about forgetting all the structure but cardinality, the dimension of algebraic extensions of fields is defined by forgetting the field structure on the larger field, keeping only the linear structure etc. In some sense, one could argue that algebra and topology (when applied in the solution of mathematical problems in other fields) act as powerful tools for forgetting irrelevant information. In homotopy theory, it is common to forget information that is not homotopy invariant. Thus, e.g., interesting homology theories are obtained by restricting scalars - the functor taking R-modules to S-modules for S a sub-ring of R, or by considering the functor of global sections, which given a sheaf, forgets all its information but its global part. In model categories, Quillen's axiomatization of homotopy theory, structure is forgotten by *deriving* functors, i.e. given a functor F the (left) derived functor $\mathbb{L}F$ is the homotopy invariant functor "closest" to F (from the left). Here "closest to F" is interpreted as being universal among the *homotopy invariant* functors such that there exists a natural transformation (also known as morphism of functors) $G \Rightarrow F$, i.e., if $\mathbb{L}F$ is the derived functor of F there exists a natural transformation $\mathbb{L}F \Rightarrow F$ and any natural transformation $G \Rightarrow F$ from a homotopy invariant functor G, factors uniquely via $\mathbb{L}F \Rightarrow F$.

Homotopy invariance is defined with the help of the *homotopy category.* The homotopy category, $\mathrm{Ho}\mathfrak{C}$, associated with a model category $\mathfrak{C}$ is the category obtained from $\mathfrak{C}$ by formally inverting all weak equivalence so that weak equivalences, and only weak equivalences, become isomorphisms in $\mathrm{Ho}\mathfrak{C}$. A functor F from a model category $\mathfrak{C}$ is *homotopy invariant* if it factors trough $\mathrm{Ho}\mathfrak{C}$.

Another means of forgetting structure in a model category $\mathfrak{C}$ is obtained by restricting to the sub-category of co-fibrantions (i.e., by forgetting all arrows that are not (c)-labelled. Note that since a model category is a labelled category, if we want - in addition - the resulting category to have an initial object, we have to restrict ourselves further to the category of co-fibrant objects (namely those objects X such that $\varnothing \xrightarrow{(c)} X$), and whose only morphisms are co-fibrations. It turns out that some constructions are well behaved only when restricted to this category of co-fibrant objects.

Observe that by axiom (M2), given a model category $\mathfrak{C}$ any object X is isomorphic in the homotopy category $\mathrm{Ho}\mathfrak{C}$ to a co-fibrant object, $X_{(wf)}$ such that $\varnothing \xrightarrow{(c)} X_{(wf)} \xrightarrow{(wf)} X$. Thus, from the homotopy category point of view, every object can be replaced with a co-fibrant object, a process known as the co-fibrant replacement. For example, in the model structure on the chain complexes or sheaves this corresponds to replacing a module or a sheaf by its projective (resp. injective) resolution before computing cohomology.

4.1. **Functors and derived functors in quasi partial orders.** As a category, QtNaamen is extremely simple: it is *posetal*, namely, arrows are unique whenever they exist (so, e,g,, all diagrams commute). We call such categories posetal, for an obvious reason: the relation $X \leq Y$ defined by $X \longrightarrow Y$ is a partial quasi order encoding fully the category structure. In this sub-section we explain the details of the discussion of the previous paragraphs in the special case of posetal categories.

Recall that given two categories $\mathfrak{C}$ and $\mathfrak{C}'$ a *covariant functor* (or simply a functor) is a mapping $F := (F_1, F_2) : (\mathcal{O}b\mathfrak{C}, \mathcal{M}or\mathfrak{C}) \longrightarrow (\mathcal{O}b\mathfrak{C}', \mathcal{M}or\mathfrak{C}')$ sending the arrow $X \xrightarrow{f} Y$ to the arrow $F_1(X) \xrightarrow{F_2(f)} F_1(Y)$ and respecting commutative diagrams (namely, F_2 respects the composition of arrows). If both $\mathfrak{C}$ and $\mathfrak{C}'$ are posetal a functor $F : \mathfrak{C} \longrightarrow \mathfrak{C}'$ is merely an order preserving mapping.

If F, G are such functors between (arbitrary) categories, $\mathfrak{C}, \mathfrak{C}'$, let $\epsilon(X) \in \mathcal{M}or\mathfrak{C}'$ be an arrow $F(X) \longrightarrow G(X)$ (if such a morphism exists). The collection of arrows $\{\epsilon(X) : X \in \mathcal{O}b\mathfrak{C}\} \subseteq \mathcal{M}or(\mathfrak{C}')\}$ is a *natural transformation* from F to G if $\epsilon(X)$ is defined for all $X \in \mathcal{O}b\mathfrak{C}$ and it preserves commutative diagrams, i.e. all diagrams involving F, G and ϵ, that exits purely for formal reasons, are necessarily commutative; in this case it means only that $G(f) \circ \epsilon(X) = \epsilon(Y) \circ F(f)$ for every $X \xrightarrow{f} Y \in \mathcal{M}or\mathfrak{C}$:

$$\begin{tikzcd}
& X \arrow[r, "f"] \arrow[dl] \arrow[d] & Y \arrow[d] \arrow[dr] & \\
F(X) \arrow[r, "\epsilon(X)"] \arrow[rr, bend right, "F(f)"'] & G(X) \arrow[rr, bend right, "G(f)"'] & F(Y) \arrow[r, "\epsilon(Y)"] & G(Y)
\end{tikzcd}$$

FIGURE 4. There are two possible paths from $F(X)$ to $G(Y)$. If ϵ is a natural transformation then the composition of morphisms along those two paths are the same.

For posetal categories $\mathfrak{C}, \mathfrak{C}'$ and functors $F, G : \mathfrak{C} \longrightarrow \mathfrak{C}'$ the arrow $\epsilon(X)$ exists if and only if $F(X) \leq G(X)$. Since the collection of functors from $\mathfrak{C}$ to $\mathfrak{C}'$ is itself quasi-partially ordered by pointwise domination, namely, for $F, G : \mathfrak{C} \longrightarrow \mathfrak{C}'$ write

$$F \leq_N G \iff F(X) \leq_{\mathfrak{C}'} G(X) \text{ for all } X \in \mathcal{O}b\mathfrak{C}.$$

Thus, given two functors $F, G : \mathfrak{C} \longrightarrow \mathfrak{C}'$ there is a natural transformation from F to G precisely when $F \leq_N G$. Moreover, if such a natural transformation exists, it is unique. Thus, for posetal $\mathfrak{C}, \mathfrak{C}'$ the functors $F, G : \mathfrak{C} \longrightarrow \mathfrak{C}'$ are *naturally equivalent* precisely when $F \leq_N G \leq_N F$, which - by definition - happens precisely when $F(X)$ is isomorphic (in $\mathfrak{C}'$) to $G(X)$ for all $X \in \mathcal{O}b\mathfrak{C}$.

As mentioned in the introductory paragraphs of this section, homotopy theory is interested in data only up to "homotopy equivalence". In the context of model categories, $\mathfrak{C}, \mathfrak{C}'$, this means that those functors $F : \mathfrak{C} \longrightarrow \mathfrak{C}'$ a homotopy theorist is interested in are the ones respecting homotopy equivalence, i.e., those mapping (w)-labelled arrows to (w)-labelled arrows. Given a functor $F : \mathfrak{C} \longrightarrow \mathfrak{C}'$ which is not necessarily homotopy invariant, we may want to replace F with a homotopy invariant relative, and naturally, we want this relative to be as close to F as possible. We will now explain how this is done when $\mathfrak{C}'$ carries a trivial model structure.

Recall that any category $\mathfrak{C}'$ can be given a model structure by labelling all arrows (cf) and identifying the label (w) with isomorphisms. Call such a model structure trivial, and observe that any functor between model categories carrying a trivial model structure is homotopy invariant. Now assume that $\mathfrak{C}, \mathfrak{C}'$ are model categories with $\mathfrak{C}'$ trivial. Then $F : \mathfrak{C} \longrightarrow \mathfrak{C}'$ is homotopy invariant precisely when it maps weak equivalences to isomorphisms. Since the homotopy category $\mathrm{Ho}\mathfrak{C}$ is the localization of $\mathfrak{C}$ at the class of weak equivalences, this means precisely that F is homotopy invariant if an only if F factors through γ, where $\gamma : \mathfrak{C} \longrightarrow \mathrm{Ho}\mathfrak{C}$ is the localization functor.

Thus, under the same assumptions, given any functor $F : \mathfrak{C} \longrightarrow \mathfrak{C}'$, the homotopy invariant version of F we are looking for can be identified with a a functor $\mathbb{L}^\gamma_F : \mathrm{Ho}\mathfrak{C} \longrightarrow \mathfrak{C}'$ such that the composition $\mathbb{L}^\gamma_F \circ \gamma$ is "closest" to F. Formally, this last condition is interpreted as the existence of a natural transformation from $\mathbb{L}^\gamma_F \circ \gamma$ to F, and such that if $G : \mathrm{Ho}\mathfrak{C} \to \mathfrak{C}'$ is any functor such that there is a natural transformation from $G \circ \gamma$ to F then there is also a natural transformation from G to $\mathbb{L}^\gamma_F \circ \Gamma$. In the case that $\mathfrak{C}, \mathfrak{C}'$ are posetal this reduces to:

Definition 2. Let $\mathfrak{C}$ be a posetal model category, and $\mathfrak{C}'$ any posetal category. Given a functor $F : \mathfrak{C} \to \mathfrak{C}'$ the left derived functor of F is given by

$$\mathbb{L}^\gamma F(X') = \inf\{F(X) : X' \leqslant_{\mathrm{Ho}\mathfrak{C}} \gamma(X),\ X \in \mathcal{O}b\mathfrak{C}\}.$$

In particular, the left derived functor exists if and only if the right hand side is well-defined.

Observe that the definition of a derived functor $F : \mathfrak{C} \longrightarrow \mathfrak{C}'$ depends on $\mathfrak{C}$ being a model category only in as much as the homotopy category $\mathrm{Ho}\mathfrak{C}$ is the category through which we want to factor (an approximation) of F. In general, given a category $\mathfrak{D}$ and a functor $\gamma : \mathfrak{C} \longrightarrow \mathfrak{D}$, we can still (left) derive any functor $F : \mathfrak{C} \longrightarrow \mathfrak{C}'$ with respect to the functor γ. If all categories involved are posetal, the formula in the above definition still gives the left-derived functor with respect to γ.

Example 6 (co-limits of commutative diagrams as derived functors.)**.** A commutative diagram $D : A \longrightarrow \mathfrak{B}$ is a functor where A is a (usually finite) partial order. Let $\gamma : A \longrightarrow \{\bullet\}$ be the functor sending A to the category $\mathfrak{P} = \{\bullet\}$ consisting of a single object and a single morphism. By definition $\mathbb{L}^\gamma D$ is an object of $\mathfrak{B}$. For simplicity, we abuse notation and denote this object $\mathbb{L}^\gamma_D$. We claim that $\mathbb{L}^\gamma_D$ (if it exists) is the colimit of the diagram D. Indeed, given a a functor $G : \{\bullet\} \longrightarrow \mathfrak{B}$, a natural transformation from $G \circ \gamma$ to D is a collection of arrows $\{\epsilon(a) : a \in \mathcal{O}bA\}$ such that $G(\bullet) \xrightarrow{\epsilon(a)} D(a)$ and such that if $a \xrightarrow{f} b$ (some $a, b \in \mathcal{O}bA$) then $\epsilon(b) = D(f) \circ \epsilon(a)$. The universality property of $\mathbb{L}^\gamma_D$ implies that given such a functor G and a natural transformation as above, there exists a unique natural transformation from G to $\mathbb{L}^\gamma_D$, i.e., there exists a unique arrow from $G(\bullet)$ to $\mathbb{L}^\gamma_D(\bullet)$, making the whole diagram commute. The latter is the defining property of the (direct) co-limit of D.

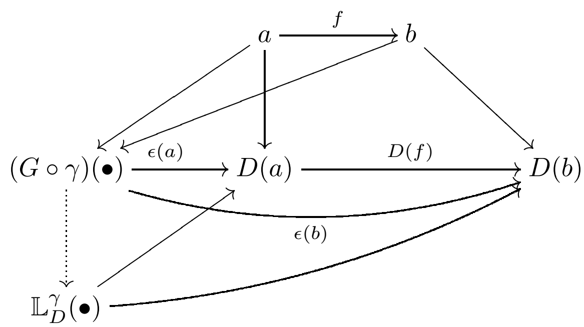


FIGURE 5. $\mathbb{L}^\gamma_D$ is the direct colimit of the diagram D

Similarly, the co-limit of a commutative diagram D is the right derived functor of the diagram viewed as a functor.

Note that the localization functor $\gamma : A \longrightarrow \{\bullet\}$, up to equivalence, corresponds to the degenerate model structure on A where every arrow is labelled (wc) and only isomorphisms are labelled (f) (or, alternatively, every arrow is labelled (wf) and only isomorphisms are labelled (c)). For this model structure every arrow in the homotopy category of A is an isomorphism and this category is equivalent to $\{\bullet\}$.

Example 7 (A category theoretic view of ordered sets[1])**.** We have seen that (quasi) partially ordered sets can be viewed as a category. In this way any subset of such a partial order determines uniquely a commutative diagram. The colimit of such a diagram is the least upper bound of its vertices, and its limit is the greatest lower bound - if they exist.

Say that the limit or colimit of a diagram D is *degenerate* if it is isomorphic to one of its vertices. It is then easy to check that a partially ordered set is *linear* if and only if either all finite limits or all finite colimits (exist and) are degenerate, (and, obviously, one implies the other, i.e. the limit of every finite diagram is degenerate if and only if the colimit of each degenerate diagram is degenerate). A linearly ordered set is *complete* if and only if every diagram has a limit if and only if every diagram has a colimit. A partially ordered class is *well-ordered* if and only if limits always exist and are always degenerate.

We conclude with a category theoretic characterisation of degenerate limits. Obviously, any functor preserves all degenerate limits and colimits. We will show that, conversely, a limit preserved under all functors is degenerate. More precisely, if A is a quasi-partial order, D a commutative diagram, if for any functor $F : A \longrightarrow A'$ it holds that $\lim F(D)$ and $F(\lim D)$ are isomorphic (and in particular the former exists if and only the latter does), then the limit of D is degenerate. Indeed, To see this, consider the category A', $\mathcal{O}bA' = \mathcal{O}bA \cup L$, and for $X, Y \neq L$ $\mathcal{M}or_{A'}(X, Y) = \mathcal{M}or_A(X, Y)$, $\mathcal{M}or_{A'}(X, L) = \mathcal{M}or_A(X, \lim D)$. Finally, set $\mathcal{M}or_{A'}(L, X) \neq \varnothing$ if and only if

[1]The authors thank Marco Porta for these observations.

for some vertex D_i in D it holds $\mathcal{M}or_A(D_i, X)$. We leave it is as a trivial exercise to check that A' is indeed a (posetal) category and that L and $\lim D \in \mathcal{O}bA'$ are not isomorphic unless the limit of D is degenerate.

In particular, a quasi partially ordered class A is well-ordered if and only if for every diagram D and every functor $F : A \longrightarrow B$, it holds $F(\lim D) = \lim F(D)$, i.e. $\lim D$ exists if and only if $\lim F(D)$ exists, and if they both exists, the formula holds.

For what follows it is crucial to observe that the formula of Definition 2 is meaningful whenever we are given a function $F : \mathfrak{C} \longrightarrow \mathfrak{C}'$ - not necessarily a functor. Thus, for example. if $\mathfrak{C}'$ is well ordered or if $\mathfrak{C}'$ is Dedekind complete then any function $F : \mathfrak{C} \longrightarrow \mathfrak{C}'$ can be derived from the left.

Let $On^{\top}$ be the posetal category of ordinals (i.e., given ordinals α, β there is an arrow $\alpha \longrightarrow \beta$ if and only if $\alpha \leq \beta$) augmented by a terminal object $\top$. The following definition sums up the discussion of the previous paragraphs, with an extra edge:

Definition 3. Let $\mathfrak{C}$ be a posetal model category. For a *function* $F : \mathfrak{C} \longrightarrow On^{\top}$ we let the *co-fibrantly replaced left derived functor* of F be:

$$\mathbb{L}_c F(X) = \min \left\{ F(Y) : \begin{tikzcd} & X_1 & & X_3 & & X_n \arrow[r, dashed] & Y \\ X \arrow[ur, dashed] & & X_2 \arrow[ul, dashed, "(w)"'] \arrow[ur, dashed] & & \dots \arrow[ul, dashed, "(w)"'] \arrow[ur, dashed] & & \bot \arrow[u, dashed, "(c)"'] \end{tikzcd} \right\}$$

where the minimum is taken over all finite sequences of the same form.

Observe that given $X, Y \in \mathcal{O}b\mathfrak{C}$ a sequence of the form

$$(\diamond) \qquad X \longrightarrow X_1 \xleftarrow{(w)} X_2 \longrightarrow X_3 \xleftarrow{(w)} \dots \longrightarrow X_n \longrightarrow Y$$

exists for some $n \in \mathbb{N}$ if an only if there exists $g \in \mathcal{M}or\mathrm{Ho}\mathfrak{C}$ such that $X \xrightarrow{g} Y$. Thus, we can write:

$$(\clubsuit) \qquad \mathbb{L}_c F(X) = \min\{F(Y) : X \longrightarrow_h Y, \bot \xrightarrow{(c)} Y\},$$

where $X \longrightarrow_h Y$ means that that there is an arrow from X to Y in the homotopy category. Using this notation we immediately see that $\mathbb{L}_c F$ is a homotopy invariant (because it factors through the homotopy category) *functor* (because $X \longrightarrow Y$ implies that $\mathbb{L}_c F(X) \geq \mathbb{L}_c F(Y)$) depending only on the values F takes on co-fibrant objects.

Note that if $F : \mathfrak{C} \longrightarrow On^{\top}$ is a functor then for any $X \in \mathcal{O}b\mathfrak{C}$, letting $\bot \xrightarrow{(c)} X_{(wf)} \xrightarrow{(wf)} X$ we see that $F(X_{(wf)}) \leq F(X)$, but $\mathbb{L}_c F$ is functorial so $\mathbb{L}_c F(X_{(wf)}) \leq \mathbb{L}_c F(X)$. By what we have just said $X_{(wf)} \longrightarrow X$ implies that $\mathbb{L}_c F(X_{(wf)}) \geq \mathbb{L}_c F(X)$, so - in the case F is a functor:

$$\mathbb{L}_c F(X) = \min\{F(Y) : X \longrightarrow_h Y\} = \mathbb{L}^{\gamma} \circ \gamma(X).$$

Thus, the co-fibrantly replaced left-derived functor generalises the definition of (left) derived functors (but the two definitions need not agree if F is not a functor).

Remark 8. Let $\mathfrak{C}$, $\mathfrak{C}'$ be equivalent model categories, witnessed by the functors $F : \mathfrak{C} \to \mathfrak{C}'$ and $G : \mathfrak{C}' \to \mathfrak{C}$. Assume that $f : \mathfrak{C} \to On^{\top}$ is any function, then there is no reason to expect that $\mathbb{L}_c f(G(Y))) = \mathbb{L}_c(f \circ G)(Y)$. This is, of course, not the case if f is a functor. In other words, the price for deriving arbitrary functions is that the process is not invariant under equivalence of model categories. This is discussed further in Section 5.

We will discuss the co-fibrant replacement a little more later on. Our discussion up to this point should convince the reader that - at least for functors - the left derived co-fibrant replacement is a natural means for forgetting non-homotopy-invariant information.

4.2. **Examples of derived functors.**

4.3. **The covering number of $\aleph_\omega$ as a value of a derived functor.** In this sub-section we prove the main result of this paper, we show that the covering number of $\aleph_\omega$ is the value of the co-fibrantly replaced left-derived functor of the cardinality function $\mathrm{card} : \mathrm{QtNaamen} \longrightarrow On^{\top}$. Cardinality is certainly one of the most natural functions anyone studying set theory is bound to run into. Possibly, it is the simplest set theoretic function not arising directly from purely logical operations (in the way the union and intersection operations do). To adapt the notion of cardinality to our setting we define a function $\mathrm{card} : \mathrm{QtNaamen} \longrightarrow On^{\top}$ such that $X \longmapsto \mathrm{card}(X)$ if X is a set and $X \longmapsto \top$ otherwise. Observe that cardinality is not a functor on QtNaamen. Indeed $\{X\} \longrightarrow \mathcal{P}(X) \longrightarrow \{X\}$ but $\mathrm{card}(\{X\}) = 1 < \mathrm{card}(\mathcal{P}(X)) > 1$ for all non-empty X. Similarly,

$$\{\{\bullet_1\},\{\bullet_1,\bullet_2\}\} \xrightarrow{(wcf)} \{\{\bullet_1,\bullet_2\}\}$$

is an isomorphism but $2 = \text{card}\,\{\{\bullet_1\},\{\bullet_1,\bullet_2\}\} > \text{card}\,\{\{\bullet_1,\bullet_2\}\} = 1$ are non-isomorphic.

However, cardinality is a natural function and the homotopy ideology discussed in the introduction to this section suggests (despite of the fact it is not a functor) that we try and find a homotopy invariant approximation to cardinality. As discussed above, any function from a model category to $On^\top$ can be derived. Unfortunately, as we will see later, deriving the cardinality function (according to the formula in Definition 2) gives us an uninteresting result. So we take the co-fibrantly replaced left derived functor of cardinality, as in Definition 3. The resulting function, $\mathbb{L}_c\text{card}$, can be viewed, as homotopy theory yoga suggests, as the homotopy invariant version of cardinality.

Interestingly, the homotopy invariant version of cardinality has a purely set theoretic interpretation ($\mathbb{L}_c\text{card}\,(\{\aleph_\alpha\}) = \text{cov}(\aleph_\alpha,\aleph_1,\aleph_1,2)$ - where $\text{cov}(\aleph_\alpha,\aleph_1,\aleph_1,2)$ is the *covering number* to be discussed in detail below). The construction of this function uses fairly little set theory: the only notions needed in an essential way to construct it are $A \subseteq B$, finiteness, countability and infinite equi-cardinality. Thus, $\mathbb{L}_c\text{card}$ will remain meaningful in any set theory where those notions keep their meaning. More importantly, $\mathbb{L}_c\text{card}$ is considerably tamer, say, than the power function, and can be effectively bounded in ZFC (but these are deep results in PCF theory, and we do not claim that they can be identified, let alone proved - using homotopy theoretic tools). For example, Shelah's famous inequality

$$(\aleph_\alpha)^{\aleph_0} \leqslant \text{cov}(\aleph_\alpha,\aleph_1,\aleph_1,2) + 2^{\aleph_0}$$

can be interpreted (paraphrasing Shelah) as a decomposition of $(\aleph_\alpha)^{\aleph_0}$ into a "noise" component (wild and highly independent on ZFC) and a "homotopy invariant" part, which can be well understood within ZFC.

Another curious feature of the function $\mathbb{L}_c\text{card}$ is that it is non-trivial only on singular cardinals. Thus, from a homotopy theoretic view point singular cardinals present themselves almost immediately as a natural object of interest in set theory (compare with, [9], describing the early and spectacular appearance of singular cardinals on the mathematical stage, and their immediate disappearance for several decades). We now proceed with a detailed exposition of the discussion of the last paragraphs.

By definition, the *covering number*

$$\text{cov}(\lambda,\kappa,\theta,\sigma)$$

is the *least size of a family $X \subseteq [\lambda]^{<\kappa}$ of subsets of λ of cardinality less than κ, such that every subset of λ of cardinality less than θ, lies in a union of less than σ subsets in X.*

Theorem 9. *(the covering number as a derived functor).* For any cardinal λ

$$\mathbb{L}_c\text{card}\,(\{\lambda\}) = cov(\lambda,\aleph_1,\aleph_1,2)$$

Proof. First, assume that $\mathcal{Y}$ is a covering family for λ witnessing $\text{cov}(\lambda,\aleph_1,\aleph_1,2) = \kappa$. Then, by definition of the covering number $\varnothing \xrightarrow{(c)} \mathcal{Y}$. We claim that $\mathcal{Y} \xrightarrow{(w)} \{\lambda\}$, which will prove $\mathbb{L}_c\text{card}\,(\lambda) \leq \kappa$. By Proposition 3 we only have to show that any countable subset of λ is contained in an element of $\mathcal{Y}$, which is merely the definition of $\mathcal{Y}$ being a covering family. To prove the other inequality, observe that:

Claim I: If $\mathcal{X} \longrightarrow_h \mathcal{Y}$ in QtNaamen, with $\mathcal{X} := \mathcal{X}_0,\mathcal{X}_1,\ldots,\mathcal{X}_n =: \mathcal{Y}$ witnessing it (as in ($\diamond$)) then for every $i \leq n$, every countable subset L with $\{L\} \longrightarrow \mathcal{X}$ is contained, up to finitely many elements, in some $\{X\} \longrightarrow \mathcal{X}_i$.

Proof. For X_0 there is nothing to prove, and for X_1 this follows from the definition of $\mathcal{X} \longrightarrow \mathcal{X}_1$. For $\mathcal{X}_2 \xrightarrow{(w)} \mathcal{X}_1$ this is a special case of Proposition 3, and as the condition is transitive, induction gives this observation. $\square_{\text{Claim I}}$

The proof of the theorem now follows from the following:

Claim II: Let $\varnothing \xrightarrow{(c)} \mathcal{Y}$ be such that $\{L\} \longrightarrow^* \mathcal{Y}$ for every countable set, $L \subseteq \lambda$. Then there exists a covering family $\mathcal{Z}$ of λ whose cardinality is at most that of $\mathcal{Y}$.

Proof. Let $\mathcal{Y}_0$ be the inverse limit of $\mathcal{Y}$ and $\{\lambda\}$. Then $\text{card}\,\mathcal{Y}_0 \leq \text{card}\,\mathcal{Y}$ (by definition of the inverse limit in QtNaamen), and $\varnothing \xrightarrow{(c)} \mathcal{Y}_0$. By assumption, $\varnothing \xrightarrow{(c)} \mathcal{Y}_0 \xrightarrow{(wc)} \lambda_{(wf)}$, where $\varnothing \xrightarrow{(c)} \lambda_{(wf)} \xrightarrow{(wf)} \{\lambda\}$. By Lemma 35 of [6] there is a set Λ and some $\mathcal{Y}'$ such that the following diagram is true in StNaamen:

$$\begin{tikzcd}
 & & \mathcal{Y}_0 \arrow[d, "(wc)"] & \mathcal{Y}' \arrow[l] \arrow[r, "(wf)"] & \{\Lambda\} \arrow[dl, "(wc)"] \\
\varnothing \arrow[urrr, bend left, "(c)"] \arrow[r, bend right, "(c)"] & & \lambda_{(wf)} \arrow[r, "(wf)"] & \{\lambda\} &
\end{tikzcd}$$

Since QtNaamen is a full sub-category, to show that this diagram is also true in QtNaamen it suffices to verify that all objects in the diagram are objects in QtNaamen. This amounts to checking that $\{\lambda\}$ and $\mathcal{Y}'$ are in QtNaamen, which is obvious since all singleton sets and all co-fibrant objects are.

Let $\mathcal{Y}''$ be the inverse limit of $\mathcal{Y}_0$ and $\{\Lambda\}$, by Claim 33 of [6] (and this also follows readily from the definition), this is simply $\{y \cap \Lambda : y \in \mathcal{Y}_0\}$. By definition of the inverse limit we get $\mathcal{Y}' \longrightarrow \mathcal{Y}''$. Since $\mathcal{Y}' \xrightarrow{(wf)} \{\Lambda\}$ it follows (e.g., by Proposition 3) that $\mathcal{Y}'' \xrightarrow{(wf)} \{\Lambda\}$. Since all elements in $\mathcal{Y}'$ are countable so are all the elements in $\mathcal{Y}''$. By Proposition 3 these two facts together mean precisely that $\mathcal{Y}''$ is a covering family for Λ.

Finally, since $\{\Lambda\} \xrightarrow{(wc)} \{\lambda\}$ we get (again, using Proposition 3), that $\lambda \setminus \Lambda$ is a finite set, say, C. Let $\mathcal{Z} := \{y \cup C : y \in \mathcal{Y}''\}$. Then $\mathcal{Z}$ is a co-fibrant object, and is therefore an object of QtNaamen. All elements in $\mathcal{Z}$ are countable, and every countable subset of λ is contained in an element of $\mathcal{Z}$. So $\mathcal{Z}$ is a covering family for λ. Observe that $\operatorname{card} \mathcal{Z} \leq \operatorname{card} \mathcal{Y}'' \leq \operatorname{card} \mathcal{Y}_0 = \operatorname{card} \mathcal{Y}$. Thus, $\mathcal{Z}$ witnesses that $\mathbb{L}_c\mathrm{card} \geq \mathrm{cov}(\lambda, \aleph_1, \aleph_1, 2)$. $\square_{\text{Caim II}}$

This completes the proof of the theorem. $\square$

We conclude with a summary, in our notation, of some of Shelah's results concerning PCF bounds:

Theorem 10 (Shelah)**.** *The following inequalities are true in ZFC:*

(i) *if $\aleph_\alpha$ is regular cardinal, then*

$$\mathbb{L}_c(\{\aleph_\alpha\}) = \mathbb{L}_c(2^{\aleph_\alpha}) = \mathrm{cov}(\aleph_\alpha, \aleph_1, \aleph_1, 2) = \aleph_\alpha$$

(ii) $\mathbb{L}_c(\{\aleph_\omega\}) = \mathbb{L}_c(2^{\aleph_\omega}) = \mathrm{cov}(\aleph_\omega, \aleph_1, \aleph_1, 2) < \aleph_{\omega_4}$

(iii) *If $\aleph_\delta$ is a singular cardinal such that $\delta < \aleph_\delta$, then $\mathbb{L}_c(\{\aleph_\delta\}) = \mathbb{L}_c(2^{\aleph_\delta}) = \mathrm{cov}(\aleph_\delta, \operatorname{card} \delta^+, \operatorname{card} \delta^+, 2) < \aleph_{\operatorname{card} \delta^{+4}}$*

(iv) *(Shelah's Revised GCH). If θ is a strong limit uncountable cardinal, then for every $\lambda \geqslant \theta$, $\kappa_0 \leqslant \kappa < \theta$ $\lambda^{[\kappa]} = \lambda$*

Proof. (i) is immediate by induction; (ii) is a particular case of (*iii*); (iii) is Theorem 7.2 of [Handbook of Set theory, page 1209] ; (iv) Theorem 8.1 of [Handbook of Set theory, page 1210]

Note that we do not say anything about the *fixed points* $\alpha = \aleph_\alpha$ of $\aleph_\bullet$-function. (todo: is there an explanation)

4.4. **Other model categories and covering numbers.** Simple variations on the theme leading us to "rediscover" the covering number $\mathrm{cov}(\lambda, \aleph_1, \aleph_1, 2)$ result in other covering numbers. Since most of the details are quite similar, we will be brief.

For an object A of QtNaamen, let QtNaamen^A be the full sub-category of arrows $A \longrightarrow X$ with the induced model structure, i.e., the full subcategory of QtNaamen consisting of those objects X such that $A \longrightarrow X$ with the labelling induced from QtNaamen. This is, trivially, a model category. Applying Definition 2 for QtNaamen^A and the function card : QtNaamen$^A \longrightarrow On^T$ to obtain the functor

$$\mathbb{L}_c^{Qt^A}\mathrm{card} : \mathrm{QtNaamen}^A \longrightarrow On^\top,$$

the cofibrantly replaced left-derived functor of cardinality (on the model category QtNaamen^A). We obtain:

Theorem 11. *Let $\beta \leq \alpha$ be ordinals. Let $\aleph_\beta^* := [\aleph_\beta]^{<\aleph_\beta}$. Then, with the above notation, if $\aleph_\beta$ is regular then*

$$\mathbb{L}_c^{Qt^{\aleph_\beta^*}}\mathrm{card}\,(\{\aleph_\alpha\}) = \mathrm{cov}(\aleph_\alpha, \aleph_\beta, \aleph_\beta, 2).$$

In particular, if $\aleph_\alpha < \aleph_{\aleph_\alpha}$ and $\aleph_\beta = (\mathrm{cof}\,\aleph_\alpha)^+$ then

$$\mathbb{L}_c^{Qt^{\aleph_\beta^*}}\mathrm{card}\,(\{\aleph_\alpha\}) = \mathrm{pp}_{\mathrm{cof}\,\aleph_\alpha}(\aleph_\alpha) = \mathrm{pp}(\aleph_\alpha)$$

$$\{\aleph_\alpha\} \longrightarrow \{\aleph_\alpha\} \xleftarrow{(w)} \mathcal{Y} \times \{\aleph_\alpha\} \longrightarrow \mathcal{Y} \xleftarrow{(c)} \perp_\beta .$$

Thus, $\mathbb{L}_c^{Qt^{\aleph_\beta^*}}\operatorname{card}(\{\aleph_\alpha\}) \leq \kappa$. So we now turn to the proof of the other inequality. Let $\{\aleph_\alpha\} \longrightarrow_h \mathcal{Y}$ for some co-fibrant object $\mathcal{Y}$ of minimal cardinality. We first prove:

Claim I Let $L \subseteq \aleph_\alpha$ be any set with $\operatorname{card} L < \aleph_\beta$. Then there exists some $L' \subseteq L$ such that $L \setminus L'$ is finite and such that $\{L'\} \longrightarrow \mathcal{Y}$. We denote this property $[\aleph_\alpha]^{<\aleph_\beta} \longrightarrow^* \mathcal{Y}$.

Proof. By Claim I of the previous theorem we know that every countable subset of L is contained, up to a finite set, in some element of $\mathcal{Y}$. That is, if $L_0 \longrightarrow L_c$ (where $\varnothing \xrightarrow{(c)} L_c \xrightarrow{(wf)} \{L\}$) then $L_0 \longrightarrow^* \mathcal{Y}$. Letting $\mathcal{Y}_0$ be the inverse limit of $\mathcal{Y}$ and L this means that $\mathcal{Y}_0 \xrightarrow{(wc)} L_c$. Therefore, as in the proof of Claim II of the previous theorem, we may apply Claim 33 of [6] to obtained L' satisfying the requirements. $\square_{\text{Claim I}}$

To conclude the proof of the theorem we need one additional combinatorial fact, a generalisation of Lemma 35 of [6]:

Claim II Assume that $[\aleph_\alpha]^{<\aleph_\beta} \longrightarrow^* \mathcal{Y}$. Then there is a finite set B such that $\aleph_\alpha^{<\aleph_\beta} \longrightarrow \mathcal{Y}_B$, where $\mathcal{Y}_B$ is the set $\{Y \cup B : Y \in \mathcal{Y}\}$.

Proof. Assume not. Then for any finite $b \subseteq \aleph_\beta$ there exist $L_b \in [\aleph_\alpha]^{<\aleph_\beta}$ and $L \not\longrightarrow \mathcal{Y}_b$. Let $L_0 = L_\varnothing$. Define by induction:

$$L_{i+1} = L_i \cup \{L_b : b \in [L_i]^{<\aleph_0}, L_b \not\longrightarrow \mathcal{Y}_b\}.$$

Let $L = L_\omega := \bigcup_{i<\omega} L_i$. Because $\aleph_\beta$ is regular, $\operatorname{card} L < \aleph_\beta$. But there is no finite set $b \subseteq L$ such that $L \longrightarrow \mathcal{Y}_b$. Indeed, if b were such a set, then $b \subseteq L_i$ for some $i < \omega$. So $\{L_b\} \longrightarrow \{L_{i+1}\}$ and $\{L_b\} \not\longrightarrow \mathcal{Y}_b$. Since $\{L_{i+1}\} \longrightarrow \{L\}$ it follows that $L \not\longrightarrow \mathcal{Y}_b$, a contradiction. $\square_{\text{Claim II}}$

Let B be a finite set as in Claim II, then $\mathcal{Y}_B$ covers $\aleph_\alpha^{\aleph_\beta}$, and $\operatorname{card}\mathcal{Y}_B = \operatorname{card}\mathcal{Y}$. Because B is finite and $\mathcal{Y} \in \mathcal{O}b\mathrm{QtNaamen}$ it follows immediately from the definition that $\mathcal{Y}_B \in \mathcal{O}b\mathrm{QtNaamen}$, with the desired conclusion. $\square$

The construction of the co-slice category, $\mathrm{QtNaamen}^A$, for an object $A \in \mathcal{O}b\mathrm{QtNaamen}$ is standard in category theory. We proceed now to a slightly different construction, to our taste quite natural from the set theoretic point of view, but not entirely obvious from on the category theoretic side:

Let X be a class of sets, fix a (regular) cardinal κ and denote $\bigcup_{<\kappa} X := \{\bigcup S : S \subseteq X, \operatorname{card} S < \kappa\}$. Call a class X of sets κ-directed if $\bigcup_{<\kappa} X \longrightarrow X$, namely if any collection of less than κ members of X has a common upper bound (with respect to $\subseteq$)) in X. Let $\mathrm{StNaamen}_\kappa$ be the full subcategory of StNaamen consisting of κ-directed classes.

Let $\mathrm{StNaamen}_\kappa^+$ be a category that has the same object as QtNaamen, $\mathcal{O}b\mathrm{StNaamen}_\kappa^+ = \mathcal{O}b\mathrm{QtNaamen}$, and $X \longrightarrow Y$ in $\mathrm{StNaamen}_\kappa^+$ if and only if $\bigcup_{<\kappa} X \longrightarrow \bigcup_{<\kappa} Y$. Given $X \in \mathcal{O}b\mathrm{StNaamen}_\kappa^+$ denote $F(X) := \bigcup_{<\kappa} X$. It is clear that $F : \mathrm{StNaamen}_\kappa^+ \to \mathrm{StNaamen}_\kappa$ is a functor. Moreover the inclusion mapping $G : \mathrm{StNaamen}_\kappa \to \mathrm{StNaamen}_\kappa^+$ given by $G(X) = X$ is a functor (as for any $X, Y \in \mathcal{O}b\mathrm{StNaamen}_\kappa$ if $X \longrightarrow Y$ then $\bigcup_{<\kappa} X \longrightarrow \bigcup_{<\kappa} Y$). By definition, for $X \in \mathcal{O}b\mathrm{StNaamen}$, $X \longleftrightarrow \bigcup_{<\kappa} X$, so the functors F and G show that $\mathrm{StNaamen}_\kappa$ is equivalent to $\mathrm{StNaamen}_\kappa^+$.

It is easy to check that for regular κ the category $\mathrm{StNaamen}_\kappa$ equipped with the following labelling satisfies Quillen's axioms (M1)-(M4) and (M6):

Definition 2.
(1) $X \longrightarrow Y$ iff $\forall x \in X \exists y \in Y\, x \subseteq y$
(2) $X \xrightarrow{(wc)} Y$ iff $\forall y \in Y \exists x \in X\, (\operatorname{card}(y \setminus x) < \kappa)$ (and $X \longrightarrow Y$)
(3) $X \xrightarrow{(c)} Y$ iff $\forall x \in X \exists y \in Y (\operatorname{card} y \leqslant \operatorname{card} x + \kappa)$ (and $X \longrightarrow Y$)
(4) $X \xrightarrow{(f)} Y$ iff $\forall x \in X \forall y' \subseteq y \in Y \exists x' \in X (\operatorname{card} y' < \kappa \implies x \cup y' \subseteq x')$ (and $X \longrightarrow Y$)
(5) $X \xrightarrow{(wf)} Y$ iff $\forall x \in X \forall y' \subseteq y \in Y \exists x' \in X (\operatorname{card} y' \leqslant \operatorname{card} x + \kappa \implies y' \subseteq x')$ (and $X \longrightarrow Y$)
(6) $X \xrightarrow{(w)} Y$ iff $\forall x \in X \forall y' \subseteq y \in Y \exists x' \in X \operatorname{card}(y' \setminus x') < \kappa)$ (and $X \longrightarrow Y$)

Remark 12. Observe that $X \xrightarrow{(wc)} Y$ ($X \xrightarrow{(wf)} Y$) if and only if $X \xrightarrow{(c)} Y$ ($X \xrightarrow{(f)} Y$) and $X \xrightarrow{(w)} Y$. Moreover, $X \xrightarrow{(w)} Y$ if and only if there exists Z such that $X \xrightarrow{(wc)} Z \xrightarrow{(wf)} Y$.

To turn $\mathrm{StNaamen}_\kappa$ into a model category, as with StNaamen, let $\mathrm{QtNaamen}_\kappa$ be the full sub-category of *cute* objects of $\mathrm{StNaamen}_\kappa$, namely, those objects satisfying the diagram of Figure 1 (with respect to the labelling in the above definition). Now one defines, for $X \in \mathcal{O}b\mathrm{StNaamen}_\kappa$, $\tilde{X}$ to be the product of all cute $Y \in \mathcal{O}b\mathrm{StNaamen}_\kappa$ such that $X \longrightarrow Y$ (we leave it as an exercise to verify that this is indeed an object in StNaamen). It is then easy to verify that $\tilde{X}$ is cute and that if $\varnothing \xrightarrow{(c)} X$ then $X = \tilde{X}$ and that $\widetilde{\{S\}} = \{S\}$ for any set S. So $\mathrm{QtNaa}\underbrace{\mathrm{men}}_\kappa$ satisfies Axiom (M0) (inverse limits are simply products, and the direct limit $\{X_1, \ldots, X_k\}$ is simply $\Sigma_{i=1}^k X_i$, where ΣX_i is the limit of the X_i in $\mathrm{StNaamen}_\kappa$). That the remaining axioms are satisfied in $\mathrm{QtNaamen}_\kappa$ can be proved precisely as in [6], with the obvious adaptations (replacing "countable" there with "of cardinality at most κ" and "finite" there with "of cardinality smaller than κ", and see also Remark 37 in [6] for the fixed point argument needed for the proof the analogue of Lemma 35).

Recall that, as pointed out above, $\mathrm{StNaamen}^+_\kappa$ is equivalent to $\mathrm{StNaamen}_\kappa$. This equivalence can be used to label $\mathrm{StNaamen}^+_\kappa$ uniquely to make the two categories equivalent as labelled categories. Since the definition of $\mathrm{QtNaamen}_\kappa$ is given strictly in terms of the labelling of $\mathrm{StNaamen}_\kappa$, we obtain a full sub-category, $\mathrm{QtNaamen}^+_\kappa$, of $\mathrm{StNaamen}^+_\kappa$, equivalent as a labelled category to $\mathrm{QtNaamen}_\kappa$ ($\mathrm{QtNaamen}^+_\kappa$ is both the image of $\mathrm{QtNaamen}_\kappa$ under the functor mapping $\mathrm{StNaamen}_\kappa$ into $\mathrm{StNaamen}^+_\kappa$ and the full sub-category of cute objects of $\mathrm{StNaamen}^+_\kappa$ as a labelled category). Thus, $\mathrm{QtNaamen}^+_\kappa$ is a model category equivalent to $\mathrm{QtNaamen}_\kappa$.

As $\mathrm{QtNaamen}_\kappa$ is equivalent (as a model categories) to $\mathrm{QtNaamen}^+_\kappa$ so are their associated homotopy categories. Computing the homotopy category of $\mathrm{QtNaamen}_\kappa$ is rather simple: objects are $<_\kappa$-directed classes with arrows $X \longrightarrow Y$ if and only if for all $x \in X$ there exists $y \in Y$ such that $\mathrm{card}\,(x \setminus y) < \kappa$ (this follows immediately Definition 2 and the fact that $\mathrm{HoQtNaamen}^+_\kappa$ is obtained by inverting all (w)-arrows in $\mathrm{QtNaamen}^+_\kappa$).

It is now straightforward to verify that the left derived functor of $\mathrm{card} : \mathrm{QtNaamen}^+_\kappa \longrightarrow On^\top$ is Shelah's revisited power function:

$$\mathbb{L}_c\mathrm{card}\,(\{\lambda\}) = \lambda^{[\kappa]} := \mathrm{cov}(\lambda, \kappa^+, \kappa^+, \kappa).$$

Indeed, $\mathrm{cov}(\lambda, \Delta, \theta, \sigma)$ is the least size of a family $X \subseteq [\lambda]^{<\Delta}$, such that every subset of λ of cardinality smaller than θ, lies in a union of less than σ subsets in X. In our notation, taking $\Delta = \kappa^+ = \theta$ and $\sigma = \kappa$, the condition on the family X can be stated as: $X \longrightarrow [\lambda]^{\leq\kappa}$ and $[\lambda]^{\leq\kappa} \longrightarrow \bigcup_{<\kappa} X$. Now, the first of these conditions is precisely $\varnothing \xrightarrow{(c)} X \longrightarrow \{\lambda\}$, whereas the second condition is $\bigcup_{\leq\kappa} X \longleftarrow Y \xrightarrow{(wf)} \{\lambda\}$ for some Y. But in $\mathrm{StNaamen}^+_\kappa$ (and therefore in $\mathrm{QtNaamen}^+_\kappa$), $\bigcup_{<\kappa} X \longleftrightarrow X$. Therefore, this last condition is equivalent to $X \longleftarrow Y \xrightarrow{(wf)} \{\lambda\}$. Combining everything together we get that $\mathrm{cov}(\lambda, \kappa^+, \kappa^+, \kappa) \geq \mathbb{L}_c\mathrm{card}\,(\{\lambda\})$. The proof of the other direction is similar (modulo the obvious adaptations) to the proof of the analogous fact in Theorem 9.

The model category $\mathrm{QtNaamen}^+_\kappa$ allows us to formulate quite easily the notion of the cardinal κ being (non) measurable. Recall that a cardinal κ is *measurable* if it is uncountable and admits a $< k$-complete non-principal ultrafilter, or, equivalently, 0-1 valued probability countably additive measure such that every subset is measurable. Such an ultrafilter exists on he cardinal $\kappa = \omega$, as any filter is $< \omega$-complete. We prove that:

Lemma 13. *The following are equivalent for a regular cardinal $\kappa > \omega$:*

(1) *κ is not measurable.*
(2) *For all $X \in \mathcal{O}b QtNaamen^+_\kappa$ if $X \xrightarrow{(i)} \{\kappa\}$ then $X \xrightarrow{(w)} \{\kappa\}$.*
(3) *$X \xrightarrow{(i)} Y \xleftarrow{(c)} \perp$ implies $X \xrightarrow{(wc)} Y$.*
(4) *In $\mathrm{Ho}QtNaamen^+_\kappa$ if $X \xrightarrow{(i)} Y$ then $Y \longrightarrow X$ for all X, Y.*

where $X \xrightarrow{(i)} Y$ means that $X \longrightarrow Y$ is an indecomposable arrow.

Proof. First, we observe that if $X \in \mathcal{O}b\mathrm{StNaamen}^+_\kappa$ and $\bar{X} = \bigcup_{<\kappa} X$ then $\bar{X} \longleftrightarrow X$ (in $\mathrm{StNaamen}^+_\kappa$). In particular, $X \in \mathcal{O}b\mathrm{QtNaamen}^+_\kappa$ if and only if $\bar{X}$ is. Note that if $X \in \mathcal{O}b\mathrm{StNaamen}^+_\kappa$ and X is a non-empty set then $\bar{X}$ is a κ-complete ideal on $\bigcup X$. Indeed, it is closed under unions of size less than κ, and by definition $\bar{X}$ is downward closed.

Thus, if $X \in \mathcal{O}b\mathrm{StNaamen}^+_\kappa$ then the statement $X \longrightarrow \{\kappa\}$ is equivalent to $\bar{X} \longrightarrow \{\kappa\}$ and since $\{\bigcup X\} \in \mathrm{QtNaamen}^+_\kappa$ we get $X \longrightarrow \{\bigcup X\} \longrightarrow \{\kappa\}$. If, in addition, $X \xrightarrow{(i)} \{\kappa\}$ then either $\{\bigcup X\} \leftrightarrow X$ or $\bigcup X = \kappa$. But if $\bigcup X \neq \kappa$ then $X \longrightarrow \{\kappa\}$ is not indecomposable (take $X \cup \{y\}$ for any $y \in \{\kappa\} \setminus \bigcup X\}$). So $X \xrightarrow{(i)} \{\kappa\}$ is equivalent to $\bar{X}$ being a maximal ideal on κ which is also κ-complete.

It remains, therefore, to ascertain when is such an ideal principal. On the one hand, it is obvious that if X is a maximal principal (κ-complete) ideal on λ then $X \xrightarrow{(wc)} \{\kappa\}$. Now assume that X is a maximal ideal on

$\{\kappa\}$ which is κ-complete. Then $X \longrightarrow \{\kappa\}$, which is, by Definition 2, equivalent to $X \xrightarrow{(c)} \{\kappa\}$, and assume that $X \xrightarrow{(w)} \{\kappa\}$. Then $X \xrightarrow{(wc)} \{\kappa\}$, which - by definition - means some $x \in X$ satisfies $\operatorname{card}(\kappa \setminus x) < \kappa$, and since X is k-complete this means that $\bigcup X \neq \{\kappa\}$ (if it were, then already a small union would cover everything). Maximality implies that in that case X is principal.

The above shows the equivalence of (1) and (2) above, as well as (3) $\Rightarrow$ (1). So it remains to prove (2) $\Rightarrow$ (3). Indeed, assume that X, Y are as in (3). We may assume that $X = \bar{X}$. We may also assume that there exists some $y \in Y$ such that $\operatorname{card}(y) = \kappa$ (otherwise $X \xrightarrow{(wc)} Y$ is automatic from Definition 2). So fix any $y \in Y$. It will suffice to show that $X_y := \{x \in X : x \subseteq y\}$ is a maximal k-complete ideal on y. This will be enough since then, by (2) this ideal is principal, and as $y \in Y$ was arbitrary of cardinality κ, we will be done, by Definition 2. So it remains to show that X_y is a maximal ideal on y. That is it is a k-complete ideal is proved exactly as above. So we only have to verify its maximality, which is immediate from the indecomposability of the arrow $X \longrightarrow Y$.

To see the equivalence with (4) we may assume that X and Y are co-fibrant. Further, note that $X \xrightarrow{(i)}_h Y$ if there exists a sequence as in ($\diamond$) in which all but one of the arrows is a (w)-arrow, and this arrow is indecomposable. But, if

$$\perp \xrightarrow{(c)} X \xrightarrow{(w)} X_1 \xleftarrow{(w)} X_2 \xrightarrow{(w)} X_3 \xleftarrow{(w)} \dots X_i \xrightarrow{(i)} X_{i+1} \xrightarrow{(w)} X_n \longrightarrow Y \xleftarrow{(c)} \perp$$

then in $\mathrm{HoQtNaamen}_\kappa$ the object X is isomorphic to X_i and Y is isomorphic to X_{i+1} and X_{i+1} is co-fibrant. Thus, (3) $\Longleftrightarrow$ (4). $\square$

We point out that, since $\mathrm{QtNaamen}^+_\kappa$ is a closed model category (i.e., it satisfies axiom (M6)), condition (3) above can be expressed as a lifting property:

$$\perp \xrightarrow{(c)} Y \Rightarrow X \xrightarrow{(i)} Y \not\prec X' \xrightarrow{(f)} Y'.$$

Thus, the notion of κ being a (non)-measurable cardinal has an essentially model category-theoretic interpretation.

The following gives another intriguing set theoretic angle to the model category $\mathrm{QtNaamen}^+_\kappa$. Let L denote, as usual, Gödel's constructible model of set theory. Recall, e.g., Theorem [, Theorem 13.9], that L is the least transitive class (i.e., $L \longrightarrow \{L\}$) closed under all Gödel operations, and universal in the sense that $X \subseteq L$ implies $X \subseteq Y \in L$. Let $\mathrm{StNaamen}^L$ be the full sub-category of StNaamen whose objects are sub-classes of L definable within L. In other words, viewing L as a model of ZFC and ignoring the ambient universe V, we let $\mathcal{O}b\mathrm{StNaamen}^L$ be all sub-classes of L. Note that as L itself is a class element in $\mathcal{O}b\mathrm{StNaamen}^L$ is, in fact an object of StNaamen.

Remark 14. It is well known (and easy to prove) that if there is no measurable cardinal below κ then κ is measurable if and only if it admits a σ-complete ultrafilter, in other words, a countably additive measure such that every subset has measure either 0 or 1. Thus, the statement "there is no measurable cardinal" is equivalent to the statement "the only indecomposable arrows in $\mathrm{StNaamen}_{\aleph_1}$ are (wc)-arrows".

Obviously, by the remark concluding the previous paragraph, since $L \models ZFC$, we can perform the construction of [6] in $\mathrm{StNaamen}^L$. However, since notions of countability and (infinite) equicardinality are not absolute, the labelling obtained in this way will, in general, not coincide with the labelling induced on $\mathrm{StNaamen}^L$ from StNaamen. Indeed, the labelling induced on $\mathrm{StNaamen}^L$ from StNaamen will not (in general) satisfy axiom (M2) (whereas, the the labelling following the construction of [6] does).

In the above, it seems clear that the labelling induced on $\mathrm{StNaamen}^L$ from StNaamen is not the "right" one. The situation is less clear when trying to give sub-categories of $\mathrm{StNaamen}^L$ a model structure. As mentioned above, carrying out the construction of [6] we can construct $\mathrm{QtNaamen}^L$ and the associated model categories $\mathrm{QtNaamen}^+_\kappa(L)$ (where κ is a cardinal in L). But it seems that under certain set-theoretic assumptions other model structures can also be constructed.

Let κ be a (regular) cardinal in V. Let $\mathrm{QtNaamen}^+_\kappa(L)$ be the full labelled sub-category of $\mathrm{QtNaamen}^+_\kappa$ whose objects are also in $\mathcal{O}b\mathrm{StNaamen}^L$. It is not hard to check that $\mathrm{QtNaamen}^+_\kappa(L)$ is closed under (small) limits, and that, being a full sub-category of $\mathrm{QtNaamen}^+_\kappa$ it also satisfies $(M1)$ and $(M3)$-$(M6)$ (though $(M6)$ requires a small calculation). Recall that the $(M2)$-decomposition of each arrow in $\mathrm{QtNaamen}^+_\kappa$ is unique (up to $\mathrm{QtNaamen}^+_\kappa$-isomorphism), so in order to check whether $\mathrm{QtNaamen}^+_\kappa(L)$ satisfies $(M2)$ we have to show that if either $X \xrightarrow{(c)} Z \xrightarrow{(wf)} Y$ or $X \xrightarrow{(wc)} Z \xrightarrow{(f)} Y$ with $X, Y \in \mathcal{O}b\mathrm{QtNaamen}^+_\kappa(L)$ then Z is isomorphic to an element in $\mathcal{O}b\mathrm{QtNaamen}^+_\kappa(L)$. Now, recall that, by definition Z is the class of all $z \subseteq y \in Y$ such that $\operatorname{card} z \leq \operatorname{card} x + \kappa$ for some $x \in X$, or Z is the class of all $z \subseteq y \in Y$ such that $\operatorname{card}(z \setminus x) < \kappa$ for some $x \in X$. Observe that, since X and Y are definable within L so is the class $Z_L := Z \cap L$ of all constructible members of Z. Thus, it will suffice to show that $Z_L \longleftrightarrow Z$. Of course, in general, there is no reason for this to be true. But if $\kappa > \aleph_1$ then this statement is equivalent to the conclusion of Jensen's covering lemma (for κ).

Thus, e.g., if $0^\#$ does not exist and $\kappa > \aleph_1$ then $\mathrm{QtNaamen}^+_\kappa(L)$ is a model category. It is an easy exercise to check that there is no cardinal $\lambda \in L$ such that $\mathrm{QtNaamen}^+_\kappa(L)$ is precisely the model category $\mathrm{QtNaamen}^+_\lambda$ constructed within L. We do not know whether $\mathrm{QtNaamen}^+_\kappa(L)$ could be equivalent to $\mathrm{QtNaamen}^+_\lambda$ for some λ (with the latter constructed within L).

5. Suggestions for future research

Among the possible objections to the work presented in the present paper and its predecessor there are two which we view as most intriguing. These are the coherence and usefulness of work.

The problem which we call *coherence* is that, as explained in Section XYZ, if $f : \mathfrak{C} \longrightarrow On^\top$ is any function on the posetal model category $\mathfrak{C}$, then the value of the (cofibrantly replaced) left-derived functor of f is not necessarily invariant under the equivalence of model categories. Namely, if $\mathfrak{C}' \equiv \mathfrak{C}$ (as model categories) and $F : \mathfrak{C}' \to \mathfrak{C}$ is a witness (of one direction of) this equivalence then $\mathbb{L}_c(F \circ f)(x)$ is not necessarily the same as $\mathbb{L}_c f(F(x))$.

This is most obvious in our calculation of Shelah's revised power function. In deriving the cardinality function on $\mathrm{QtNaamen}^+_\kappa$ we obtain the desired result, but if we tried doing the same on the equivalent model category $\mathrm{QtNaamen}_\kappa$, we would have obtained a different answer. The same situation would have occurred if trying to derive cardinality in QtNaamen we worked with the (equivalent) full subcategory of "downward closed" objects.

Because the derivation of a function $f : \mathfrak{C} \longrightarrow On^\top$ on a posetal model category can be viewed as a minimization operation (of $f(x)$ over all $x' \in \mathcal{O}b\mathfrak{C}$ homotopy equivalent to x) our (informal) approach to this problem was that the "correct" derivation is the one giving the minimal results, i.e., if $\mathfrak{C}' \equiv \mathfrak{C}$ witnessed by the functor $F : \mathfrak{C} \longrightarrow \mathfrak{C}'$ which is injective on $\mathcal{O}b\mathfrak{C}$ then the "correct" function to derive is $(f \circ F)$, rather than F. The first problem for future research is, therefore

Problem 15. Let $\mathfrak{C}$ be a posetal model category, $f : \mathfrak{C} \to On^T$ any function. Find a functor $\tilde{\mathbb{L}}_f f : \mathfrak{C} \longrightarrow On^\top$ such that

(1) $\tilde{\mathbb{L}}_f(x) \leq \mathbb{L}_f(x)$ for all x.
(2) $\mathbb{L}_f$ is not trivial (unless, say, $L_{(f \circ F)}$ is trivial for every functor $F : \mathfrak{C}' \to \mathfrak{C}$ with $\mathfrak{C}' \equiv \mathfrak{C}$.
(3) $\tilde{\mathbb{L}}_f$ is invariant under the equivalence of model categories (in the sense explained above).

In other words, extend the notion of the left derived functor of a functor $f : \mathfrak{C} \longrightarrow On^\top$ to a larger class of function with the result as invariant as possible under equivalence of model categories.

The second objection to the present work relates to its usefulness. Here is a list of problems, a positive answer to some of which could indicate of the usefulness of the new tools developed in the present work:

Problem 16. Are there more combinatorial concepts that can be captured by our suggested formalism, e.g., closed unbounded sets, stationary sets, Fodor's lemma, diamond, square etc..

As a somewhat speculative special case of the previous problem consider the fact that there are no measurable cardinals in L. As we have seen in Remark 14 the statement "there are no measurable cardinals" can be restated in our geometric language. Thus it is natural to ask:

Problem 17. Can it be proved using (mainly) the language of model categories that there are no measurable cardinals in L. In other words, can an analogue of Scott's theorem [12] stating that if there are measurable cardinals then $V \neq L$ be proved using our geometric language?

As we do not have any "geometric" characterisation of L (unlike, e.g., the set theoretic characterisation of L being the smallest inner model, i.e., the smallest submodel of V containing all ordinals, or the smallest transitive universal class closed under Gödel operations), the above question is somewhat speculative. As in our treatment of ordinals in Section 3.1, it seems reasonable to use some auxiliary notions such as naming $On \in \mathcal{O}b\mathrm{StNaamen}$ to address this problem.

Problem 18. Apparently, given a model structure on a category $\mathfrak{C}$, the computation of homotopy limits (i.e. the computation of limits in the associated homotopy category) gives in many cases important information on the category $\mathfrak{C}$. In the case of QtNaamen, one can easily give an explicit combinatorial interpretation of the limit (at least for set-sized diagrams). Are these objects of set theoretic significance? More generally, is there a set theoretic significance to the class of *cute* objects? To the homotopy category itself? Are there other derived functors defining invariants of models of ZFC that, say, can be bounded in ZFC?

In classical homotopy theory, homotopy groups (by themselves) and the associated structures (such as long exact sequences) are powerful tools allowing many calculations. In (pointed) model categories analogues of such constructions exist, such as the groupoid of homotopy classes between any two objects A, B (where A is a co-fibrant object and B is a fibrant object) as well as other constructs, analogous of other classical homotopical tools such as the suspension and loop functors, fibration sequences and more. In posetal model categories these constructions degenerate, and much of the computational power of the associated homotopy structure is lost. This may be one of the reasons that while we were able to recover homotopical interpretations of important and non-trivial set theoretic objects we were unable to prove any of their properties using the model category structure on QtNaamen.

In view of the above it is interesting to look for other constructions in QtNaamen (or HoQtNaamen), which may serve as analogues of the above mentioned model categorical constructions. One possible such construction is the sequence of model categories $\mathrm{QtNaamen}_\kappa$ when κ ranges over all cardinals.

First, recall that we were able to give the category $\mathrm{QtNaamen}_\kappa$ a model structure only under the assumption that κ is a regular cardinal. A first problem is, therefore, to construct a similar model category for singular κ. It seems that such a model category can be constructed inductively (assuming $\mathrm{QtNaamen}_\lambda$ was constructed for all $\lambda < \kappa$) by taking an appropriate "limiting" process. For example, one could define

$$\mathcal{O}b\mathrm{StNaamen}_\kappa := \left\{ \bigcup_{\lambda<\kappa} X_\lambda : X_\lambda \in \mathcal{O}b\mathrm{StNaamen}_\kappa, X_\lambda \subseteq X_{\lambda'} \text{ if } \lambda < \lambda' \right\}$$

with the additional requirement that if $X = \bigcup_{\lambda<\kappa} X_\lambda$ as above then the X_λ are uniformly definable (this is required in order to assure that X is, indeed, a class). And the labelling

$$X \xrightarrow{(*)}_\kappa Y \iff (\forall^* \lambda)(y \in Y \Rightarrow X \times \{y\} \xrightarrow{(*)}_\lambda Y$$

where $\forall^*\lambda$ means "for all large enough $\mu < \kappa$". Passing to the full sub-class of *cute* objects (with respect to this labelling) we apparently get a model category $\mathrm{QtNaamen}_\kappa$. It is unclear to us, however, whether this construction is the "correct" one.

There is also an obvious functor $F_\kappa : \mathrm{StNaamen}_\kappa \longrightarrow \mathrm{StNaamen}_{\kappa^+}$ given by $X \mapsto \bigcup_{<\kappa^+} X$ for $X \in \mathcal{O}b\mathrm{StNaamen}_\kappa$. Indeed, this is a functor of model categories: this is obvious for (c) and (f) arrows, and not much harder for (w)-arrows, with the conclusion following from Remark 12. On the level of the associated homotopy categories, it is clear that, $\gamma(F_\kappa(X)) = \perp_{\mathrm{QtNaamen}^+_\kappa}$ for any co-fibrant $X \in \mathcal{O}b\mathrm{QtNaamen}_\kappa$ (where, as above, $\gamma : \mathrm{QtNaamen}_\kappa \longrightarrow \mathrm{HoQtNaamen}_\kappa$ is the localization functor). Since the co-fibrant objects of any model category suffice to determine the associated homotopy category, it follows that the homotopy category associated with the image of $\mathrm{QtNaamen}_\kappa$ under F_κ is trivial. This gives the sequence of categories $\mathrm{QtNaamen}_\kappa$ a certain flavour of "exactness", which seem to require some further research.

5.1. **Looking back.** We conclude these notes looking back to the original motivation leading to the development of the model category QtNaamen, i.e., the goal of developing a homotopy structure for the class of models of an uncountably categorical theory and, more generally, to (quasi-minimal) excellent abstract elementary classes (see, e.g., [2] for the details). The need for homotopy theoretic tools in this contexts arose through the study, by Zilber and his school, of categoricity problems of model theoretic structures such as pseudo-exponentiation [15] and covers of semi-Abelian varieties [3], [4]. The model theoretic analysis needed to show the (uncountable) categoricity of the natural examples studied in the above mentioned references uses known number theoretic and algebro-geometric results and conjectures nowadays understood as being of essentially cohomological character, and formulated in functorial language. Such statements are, e.g., particular cases of André's generalized Grothendieck conjectures on periods of motives ([5], [1, 7.5.2.1 Conjecture],[10, §4.2 Conjecture,§1.2 Conjecture]), the Mumford-Tate conjecture on the image of Galois action on the first étale cohomology ([13]), Kummer theory ([11]), and more. This does not seem to be entirely coincidental, as the definition of Shelah's, so called, Excellent classes - the model theoretic machinery employed in this study - and in particular the requirement that there exists a unique prime model over maximally independent tuples of countable (sub) models (and that this requirement makes sense) reminds, at least superficially, some of the axioms of a model category.

However, the common model theoretic language does not seem to have the means to incorporate this functorial language in its full power and generality. Thus, in order to be applied in addressing the above mentioned categoricity problems "old-fashioned" reformulations of these conjectures, deprived of their functorial language and homological character had to be used — Schanuel's conjecture and its cognates explicated by Bertolin [5] derived from the generalized Grothendieck conjecture on periods, Bashmakov's original formulations of Kummer theory for elliptic curves ([3], [7]), and Serre's explicit description of the image of the Galois action on the Tate module as a subgroup of the profinite group $\mathrm{GL}_2(\hat{\mathbb{Z}})$.

It the first author's belief that the inability of common model theoretic language to digest these statements in their full power and generality is a major obstacle in further exploring these intriguing connections between Shelah's excellent classes and deep algebro-geometric conjectures. The homotopy theoretic approach to set theory discussed in the present paper and in [6] is a toy example exploring the ways in which homotopy theoretic language could be introduced into the realm of model theory.

Unfortunately, we were unable to use the model category QtNaamen to associate such a homotopy structure to those classes of models. In fact, it is not even clear to us when this could be done:

Problem 19. Let $\mathfrak{K}$ be a (quasiminimal) excellent abstract elementary class (e.g. algebraically closed fields of characteristic p, models of pseudo-exponentiation). Let QtNaamen($\mathfrak{K}$) be the sub-category of QtNaamen whose objects are elements of $\mathfrak{K}$ and such that for $\mathcal{M}, \mathcal{N} \in \mathcal{O}b\mathfrak{K}$ there is an arrow $\mathcal{M} \longrightarrow \mathcal{N}$ if $\mathcal{M} \prec \mathcal{N}$. Are there natural model theoretic conditions under which QtNaamen($\mathfrak{K}$) is a model category? What about $\mathrm{QtNaamen}_\kappa(\mathfrak{K})$? Is there a similiar construction associating a model category to the class $\mathfrak{K}$?

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