Draft: Exercises de style: A homotopy theory for set theory. Part II.

notes by Misha Gavrilovich[*,†]

if a man bred to the seafaring life ... and if he should take it into his head to philosophize concerning the faculties of the mind, it cannot be doubted, but he would draw his notions from the fabric of the ship, and would find in the mind, sails, masts, rudder, and compass.

As a scholar, meantime, he was trivial, and incapable of labor.

Abstract:

We observe that the notion of two sets being equal up to finitely many elements is a homotopy equivalence relation in a model category, a common axiomatic formalism for homotopy theory introduced by Quillen to cover in a uniform way a large number of arguments in homotopy theories that were formally similar to well-known ones in algebraic topology. We show the same formalism covers some arguments in (naive) set theory, and a well-known set-theoretic invariant, the covering number $cf([\aleph_\omega]^{\leqslant aleph_0})$, of PCF theory. Further we observe a similarity between homotopy theory ideology/yoga and that of PCF theory, and briefly discuss conjectural connections with model theory and arithmetics and geometry. We argue that the formalism is curious as it suggests to look at a homotopy-invariant variant of *Generalised Continuum Hypothesis* which has less independence of ZFC, and first appeared in PCF theory independently but with a similar motivation.

This is part II. In part I we construct the model category (in the sense of Quillen) for set theory starting form a couple of arbitrary, but natural, conventions, as the simplest category satisfying our conventions and modelling the notions of finiteness, countability and infinite equi-cardinality.

We also argue that from the homotopy theory point of view our construction is, essentially, automatic following basic existing methods, and so is (almost all) the verification that the construction works.

notes by Misha Gavrilovich. Department of mathematics,Ben Gurion University of the Negev, Be'er Sheva, Israel. Parts of these notes, especially those connecting Quillens model categories with Shelahs approach to cardinal arithmetic, arose in the course of a joint work with Assaf Hasson, and will eventually appear in the form of a joint paper. Any help in proofreading is appreciated.

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1. Let us now briefly review the main construction and the main theorem.

1.1. **Introduction.** *The structure of the paper.*

In §2 we give brief but complete defintions of the main objects we are concerned with. as well as some examples. Notation used in §2 suggests an interpretation of these objects in terms of homotopy theory as model categories but formally nothing beyond naive set theory is required. The rest of the paper introduces the necessary definitions and discusses this interpretation in detail. In §3 we briefly summarise Part A and define the notions of a category and a model category in a computational form convenient for us. The main theorem of Part A is stated in §3.4. The notion of a derived functor $\mathbb{L}^\gamma F : HoA \longrightarrow B$ is introduced in §4; as an example we show that the limit of a commutative diagram is a derived functor.

In §4.5 we make a critial observation that the derived functor $\mathbb{L}F$ is well-defined for F an arbitrary *function* between quasi partial orders. In §4.6-7 we introduce the notion of a model category and that of a derived functors from a model category. In §4.8-10 we then give a characterisation of a well-order and a limit of a commutative diagram in these terms. Finally, in 4.11-13 we apply these notions to cardinality and introduce the notion of cofibrant replacement.

In §5 we sketch our approach to the simplest covering number $\mathrm{cov}(\aleph_\omega, \aleph_1, \aleph_1, 2)$. In §6 we discuss more complicated covering numbers and interpret them in terms of the cofibrantly replaced left derived functors.

In §7 we conjecture a similar construction gives a model category associated to a (not necessarily first order) uncountably categorical theory, and discuss algebro-geometric applications. In §8 we make some remarks and indicate directions for further research. Appendices give some examples and a complete definition of a model category.

The reader uncomfortable with model category theory may find our exposition is not smooth: we give a lot of definitions and skip many checks that consist only of a careful unwinding of the defitinions.

1.2. *Set-theoretic interlude*. We have to be careful about foundations of set theory. We work in naive set theory and expressly ignore the hindrance of a particular axiomatisation of set theory; a reader may assume that we work either in a Grothendieck universe, a model of ZFC or a set theory with a universal set. cf. §3.8.3. It appears that most of our category theoretic arguments by their nature may be formalised using only stratified formulae and therefore in NF.

Our definitions deal with classes and a few words on foundational issues are in order. As stated, these definitions are not in ZFC but rather in NBG: we quantify over all *classes*. However, NBG is a conservative extension of ZFC, and thus equiconsistent. Within NBG, it is best to interpret these definitions as a *notation*: we write a labelled arrow to denote a particular first-order formulae with two free variables. Recall that in ZFC (and NBG) working with classes requires some care: the class of all classes is meaningless whereas the class of all sets and the union of all elements of a class are meaningful. The main distinction between sets and classes are that a class cannot be someone's element.

Alternatively if we wish to avoid being careful with the classes, we may choose and fix a transitive model $\underline{\mathcal{V}} = (\mathcal{V}, \in)$ of ZFC, and consider all constructions as taking place within $\underline{\mathcal{M}}$: read below a class as *a subset of* $\mathcal{V}$. The main difference between these two approaches is that in the latter one we have to assume a large cardinal axiom equivalent to consistency of ZFC.

2. **Definitions of c-w-f arrow notation and the construction of the model category. Examples.**

In $n°2.2.1$ and $n°2.2.2$ below we give a set-theoretic take on the definitions of $StNaamen$ and $QtNaamen$ of Part A. $n°2.2.3$ contains a definition not mentioned in Part A.

2.1. *Definition (lifting property).* Let $\longrightarrow$ denote a reflective binary relation. We write $A \longrightarrow B \ \rightthreetimes\ X \longrightarrow Y$ iff for any $A{\longrightarrow}X$ and $B{\longrightarrow}Y$ there exists $B{\longrightarrow}X$ and $A{\longrightarrow}Y$. We read $A{\longrightarrow}B \rightthreetimes X \longrightarrow$

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Y as: the morphism (or arrow) $A \longrightarrow B$ lifts with respect to $X \longrightarrow Y$.

2.1.1. $StNaamen$ *partial quasi partial order* $\longrightarrow$. We introduce the following notation:

$(\rightarrow)_0$ $\{A\} \longrightarrow \{B\}$ iff $A \subseteq B$

$(\rightarrow)$ $X \longrightarrow Y$ iff $\forall x \in X\ \exists y \in Y\ (x \subseteq y)$

Further we introduce the following notation:

$(wc)_0$ $\{A\} \xrightarrow{(wc)_0} \{B\}$ iff $A \subseteq B$ and the difference $B \setminus A$ is finite

$(c)_0$ $\{A\} \xrightarrow{(c)_0} \{B\}$ iff $A \subseteq B$ and either $\operatorname{card} A = \operatorname{card} B$ or $\operatorname{card} B \leqslant \aleph_0$ and $A \subseteq B$

(f) $X \xrightarrow{(f)} Y$ iff $X \longrightarrow Y$ and $\{a\} \longrightarrow \{b\} \ \rightthreetimes\ X \longrightarrow Y$ whenever $\{a\} \xrightarrow{(wc)_0} \{b\}$

(wf) $X \xrightarrow{(wf)} Y$ iff $X \longrightarrow Y$ and $\{a\} \longrightarrow \{b\} \ \rightthreetimes\ X \longrightarrow Y$ whenever $\{a\} \xrightarrow{(c)_0} \{b\}$

(c) $A \xrightarrow{(c)} B$ iff $A \longrightarrow B$ and $A \longrightarrow B \ \rightthreetimes\ X \longrightarrow Y$ whenever $X \xrightarrow{(wf)} Y$

(wc) $A \xrightarrow{(wc)} B$ iff $A \longrightarrow B$ and $A \longrightarrow B \ \rightthreetimes\ X \longrightarrow Y$ whenever $X \xrightarrow{(f)} Y$

(w) $A \xrightarrow{(w)} Y$ iff there exists B such that $A \xrightarrow{(wc)} B$ and $B \xrightarrow{(wf)} Y$

2.1.2. *Arrow and c-w-f label notation.* An dual's dual argument [A§3.2.1,§3.2.2] and a further check [A§3.2.3] shows that $A \xrightarrow{(wc)} B$ iff both $A \xrightarrow{(w)} B$ and $A \xrightarrow{(c)} B$, and that $X \xrightarrow{(wf)} Y$ iff both $X \xrightarrow{(w)} Y$ and $X \xrightarrow{(f)} Y$. We also extend the notation and use $X \xrightarrow{(cwf)} Y$ denote $X \xrightarrow{(c)} Y$ and $X \xrightarrow{(w)} Y$ and $X \xrightarrow{(f)} Y$.

This allows us to think of of $(c), (w)$ and (f) as independent labels on arrows: items $(wc)_0 - (w)$ provide an inductive definition of a notation defining a *labelling on arrows by symbols* $(c), (w), (f)$: an arrow $X \longrightarrow Y$ carries a label (x) iff this follows from *rules* $(wc)_0 - (w)$ in finitely many steps. We write $A \xrightarrow{(wc)} B \xrightarrow{(f)} Y$ to denote that both $A \xrightarrow{(wc)} B$ and $B \xrightarrow{(f)} Y$, thus it means that the arrow $A \longrightarrow B$ exists and carries labels (w) and (c), and the arrow $B \longrightarrow Y$ exists and carries label (f); similarly for other arrows and labels. This notation is motivated by category theory; we say more in the next §.

2.1.3. Call an *arbitrary* set or class X *cute* iff either of the following equivalent conditions holds: for any class M and sets a and b and arrows $A \xrightarrow{(c)} B$, $B' \xrightarrow{(wf)} B$, $X \longrightarrow Y$:

(Qt_1) $A \xrightarrow{(c)} B \rightthreetimes X \longrightarrow Y$ or $B' \xrightarrow{(wf)} B \rightthreetimes X \longrightarrow Y$

(Qt_1') $\cup\{X'' : X \dashleftarrow X_0 \overset{(c)}{\dashleftarrow} X'' \overset{(wf)}{\dashrightarrow} X' \dashrightarrow X\} \dashrightarrow X$

(Qt_2) if $\{a\} \rightarrow X$, $B' \longrightarrow X$ and $\{a\} \xrightarrow{(c)} \{b\}$ and $B' \xrightarrow{(wf)} \{b\}$, then $\{b\} \longrightarrow X$

(Qt_3) $M^{\leqslant \aleph_0} \longrightarrow X$ implies $M^{\leqslant \operatorname{card}(x \cap M)} \longrightarrow X$ for any $x \in X$ (where $M^{\leqslant \lambda} := \{L \subseteq M : \operatorname{card} L \leqslant \lambda\}$)

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A rather tedious but straightforward check shows

$$(Qt_3) \overset{a=x,B'=M^{\leqslant\aleph_0},}{\Longleftrightarrow} (Qt)_2 \overset{A=\{a\},B=\{b\}}{\Longleftarrow} (Qt_1) \overset{Y=\top}{\Longleftrightarrow} (Qt_0) \Longrightarrow (Qt'_1).$$

See [A§3.4] for these and the remaining equivalences: $(Qt)_3$ is the definition [A,Def.20] of a uniform object, $(Qt)_0$ appears in Lemma 28, $(Qt)_2$ appears in Claim 21, and $(Qt)'_1$ appears in the definition of Qt-fication in [A, Figure 34] (see also preeceding Notation) and by [A, Lemma 29 and 32] is equivalent to the rest.

2.1.4. Let $\mathfrak{R}_{\preceq} = (\mathfrak{K}, \preceq_{\mathfrak{R}})$ be a class of structures equipped with a partial order relation $\preceq_{\mathfrak{R}}$, e.g. $\mathfrak{R} = \mathfrak{R}(T)$ be the class of models of a theory T (in a fixed language $L = L(T)$), and for models $M, N \in \mathfrak{R}(T)$ $M \preceq_{\mathfrak{R}} N$ iff $M \subseteq N$ is an elementary submodel of N.

1. A set or a class X is $\mathfrak{R}$-cute iff $X \subseteq \mathfrak{R}$ and X is cute

($\longmapsto_{\preceq_{\mathfrak{R}}}$) $X \longmapsto Y$ iff $\forall M \in X\ \exists N \in Y\ (M \preceq N)$ and both $X \subseteq \mathfrak{R}$ and $Y \subseteq \mathfrak{R}$ are cute

(f) $X \overset{(f)}{\longmapsto} Y$ iff $X \overset{(f)}{\longmapsto} Y$ is an (f)-arrow and both $X \subseteq \mathfrak{R}$ and $Y \subseteq \mathfrak{R}$ are cute

(wf) $X \overset{(wf)}{\longmapsto} Y$ iff $X \overset{(wf)}{\longmapsto} Y$ is an (wf)-arrow and both $X \subseteq \mathfrak{R}$ and $Y \subseteq \mathfrak{R}$ are cute

(c) $A \overset{(c)}{\longmapsto} B$ iff $A \longmapsto B$ and $A{\longmapsto}B\ \rightthreetimes\ X \longmapsto Y$ whenever $X \overset{(wf)}{\longmapsto} Y$ for $X \subseteq \mathfrak{R}$ and $Y \subseteq \mathfrak{R}$ both cute

(wc) $A \overset{(wc)}{\longmapsto} B$ iff $A \longmapsto B$ and $A{\longmapsto}B\ \rightthreetimes\ X \longmapsto Y$ whenever $X \overset{(f)}{\longmapsto} Y$ for $X \subseteq \mathfrak{R}$ and $Y \subseteq \mathfrak{R}$ both cute

(w) $A \overset{(w)}{\longmapsto} Y$ iff $A \overset{(wc)}{\longmapsto} \bullet$ and $\bullet \overset{(wf)}{\longmapsto} Y$ for some $\bullet \subset \mathfrak{R}$ cute

Similarly to $n°$2.1,2, this definition can be interpreted as defining a c-w-f labelling on $\longmapsto$-arrows.

2.2. *Examples of c-w-f arrow notation.* [A§3.3.1,Proposition 16] spells out the set-theoretic meaning of the c-w-f arrow notation. [A, Figure 12,p.15] gives some examples. Here we give some more examples to illustrate the notation. These examples can be checked either directly by (explicit or implicit) diagram chasing or using the set-theoretic characterisation of [A§3.3.1,Proposition 16].

2.2.1. *Singleton sets.* For sets A and B, $A \subseteq B$ iff $\{A\}{\longrightarrow}\{B\}$, and $\{A\} \xrightarrow{(wc)} \{B\}$ iff $B \setminus A$ is finite and $A \subseteq B$. For sets A and B infinite $\operatorname{card} A = \operatorname{card} B$ iff $\{A\} \xrightarrow{(c)} \{B\}$, and B is countable iff $\varnothing \xrightarrow{(c)} \{B\}$ is a cofibration. Each of $\{A\} \xrightarrow{(f)} \{B\}$, $\{A\} \xrightarrow{(wf)} \{B\}$ and $\{A\} \xrightarrow{(cwf)} \{B\}$ is equivalent to $A = B$.

Also we have $x \in X$ iff $\{\{x\}\} \longrightarrow \{X\}$.

2.2.2. *Cofinal and covering families.* A class A is $\subseteq$-cofinal in B iff $A \xrightarrow{(cwf)} B$ iff $A \longrightarrow B \longrightarrow A$. For a class B we have $\varnothing \xrightarrow{(c)} B$ iff every element of B is at most countable. For a set X $\varnothing \xrightarrow{(c)} B \xrightarrow{(wf)} \{X\}$ iff every at most countable subset x of X is covered $x \subseteq b$ by a subset $b \in B$.

Let $A^{\leqslant\aleph_0}$ and $A^{<\aleph_0}$ denote the set of all at most countable, resp. finite, subsets of A. Then we have $\varnothing \xrightarrow{(wc)} A^{<\aleph_0} \xrightarrow{(f)} \{A\}$ and $\varnothing \xrightarrow{(c)} A^{\leqslant\aleph_0} \xrightarrow{(wf)} \{A\}$ (compare [A,Axiom M2] and discussion of Downward Lowenheim-Skolem theorem below). We also have $\varnothing \xrightarrow{(wc)} A^{<\aleph_0} \xrightarrow{(c)} A^{\leqslant\aleph_0} \xrightarrow{(wf)} \{A\}$.

2.2.3. *Cofinality and covering numbers.* The following observation lies at heart of this paper. Recall that the cofinality of a family of sets B is the least cardinality of a subset $B' \subseteq B$ such that every element

$b \in B$ is covered $b \subseteq b'$ by an element $b' \in B'$, and that the ω-covering number $\operatorname{cov}(\kappa, \aleph_1, \aleph_1, 2)$ of a cardinal κ is the least size of a family B' of countable sets such that every countable subset of κ is covered by an element of B'. In our notation these definitions can be expressed as follows:

$$\operatorname{cof}(B, \subseteq) = \min\{\operatorname{card} B' \,:\, B' \xrightarrow{(cwf)} B\},$$

$$\operatorname{cov}(\kappa, \aleph_1, \aleph_1, 2) = \min\{\operatorname{card} B' \,:\, \varnothing \xrightarrow{(c)} B' \xrightarrow{(wf)} \{\kappa\}\}.$$

The latter formula shall be important later: we shall explain that it means that *the covering number occurs naturally in our homotopy theory as a cofibrantly replaced derived functor of cardinality*.

2.2.4. *Ordinals.* A set or class is transitive (i.e. $x \in \alpha$ implies $x \subseteq \alpha$) iff $\alpha \longrightarrow \{\alpha\}$. The arrow $\{\alpha\} \longrightarrow \alpha$ in the opposite direction never holds: it means $\alpha \subseteq \gamma$ for $\gamma \in \alpha$. It holds $\alpha + 1 \xrightarrow{(wcf)} \{\alpha\}$ and $\{\alpha\} \xrightarrow{(wcf)} \alpha + 1$. An ordinal α is regular iff $\alpha \xrightarrow{(wf)} \alpha + 1$, or equivalently $\alpha \xrightarrow{(wf)} \{\alpha\}$. $\varnothing \xrightarrow{(wc)} \alpha$ iff $\alpha \leqslant \omega$; $\varnothing \xrightarrow{(c)} \alpha$ iff $\alpha \leqslant \aleph_1$.

2.2.5. *Large Sets. Classes.* The class $\mathcal{V}$ of all sets is a largest class in $\longrightarrow$-quasi order. The class $\mathcal{V}_{\leqslant\aleph_0}$ of all countable sets is a largest class in $\longrightarrow$-quasi order such that $\varnothing \xrightarrow{(c)} \mathcal{V}_c$. The class $\mathcal{V}_{<\aleph_0}$ of all finite sets is a largest class $\mathcal{V}_{wc}$ in $\longrightarrow$-quasi order such that $\varnothing \xrightarrow{(wc)} \mathcal{V}_{wc}$. It holds $\varnothing \xrightarrow{(wc)} \mathcal{V}_{<\aleph_0} \xrightarrow{(c)} \mathcal{V}_{\leqslant\aleph_0} \xrightarrow{(wf)} \mathcal{V}$.

2.2.6. *Transitive sets and Axiom of Regularity*. A set or class is transitive (i.e. $x \in X$ implies $x \subseteq X$) iff $X \longrightarrow \{X\}$ or equivalently $\{\cup X\} \longrightarrow \{X\}$. Alternatively, transitivity of sets is a lifting property: X is transitive iff for any set a it holds $\{\{a\}\} \longrightarrow \{a, \{a\}\} \rightthreetimes \{X\} \longrightarrow \top$.

The arrow $\{X\} \longrightarrow X$ means $X \subseteq x \in X$ for some $x \in X$ implying $x \in x$ which is forbidden by Axiom of Regularity. Recall that the Axiom of Regularity claims that for ever set X, $(X, \in)$ is a well-founded partial order. Equivalently, it says that for each non-empty set X there exists $\in$-minimal element $x \in X$ such that $x \cap X$ is empty. For X transitive, $x \cap X = x$ for $x \in X$ and thus the Axiom of Regularity claims $\varnothing \in X$ in any non-empty transitive X.

2.2.7. *Downward Lowenheim-Skolem theorem in c-w-f arrow notation.* Let M be a model of a first order theory T in a countable language. Then *there exists a collection Y of elementary submodels of such that* $\varnothing \xrightarrow{(c)} Y \xrightarrow{(wf)} \{M\}$ means that *every at most countable subset of M is contained in a countable elementary submodel of M*, a particular case of Downward Lowenheim-Skolem theorem. The full Downward Lowenheim-Skolem theorem can be stated as *every arrow $X \longrightarrow \{M\}$ decomposes as $X \xrightarrow{(c)} Y \xrightarrow{(wf)} \{M\}$ for some collection Y of elementary submodels of M*. Compare to [A, Axiom M2]. We make further remarks on Downward Lowenheim-Skolem theorem in §?? and in the Appendix.

2.3. *Classes of models. $\longmapsto$-arrows.* Fix a class $\Re$ and let $X \longmapsto Y$. By definition, notions of $X \xrightarrow{(f)} Y$ and $X \xrightarrow{(wf)} Y$ arrow are the same as $X \overset{(f)}{\longmapsto} Y$ and $X \overset{(wf)}{\longmapsto} Y$. For $\Re$ a class of models of a countable first order theory, this also holds for (c)-labels but not for (wc)-labels.

To see the former, one may use the following argument. The reader shall see that the argument is staightforward but somewhat tedious; in fact this type of argument is best represented by diagram chasing.

Notice that $A \longmapsto B$ and $A \xrightarrow{(c)} B$ then $A \longmapsto B$ right-lifts againts any weak fibration in $StNaamen$ and therefore $Qt(\Re)$, , and therefore $A \overset{(c)}{\longmapsto} B$.

The following argument proves that, conversely, if $A \overset{(c)}{\longmapsto} B$ then $A \overset{(c)}{\longrightarrow} B$. It is enough to prove that $A \longrightarrow B$ right-lifts against any weak fibration in $StNaamen$, $A \longrightarrow B \measuredangle X \overset{(wf)}{\longrightarrow} Y$. By Axiom M3(f) and Axiom M4 for $StNaamen$, $\{b \cap x \,:\, b \in B,\ x \in X\} \overset{(wf)}{\longrightarrow} \{b \cap y \,:\, b \in B,\ y \in Y\}$ as this arrow is the pull-back of $B \longrightarrow Y$ and $X \longrightarrow Y$; by [A, Proposition 15] $\{b \cap x \,:\, b \in B,\ x \in X\}$ is the inverse limit of A and B, and $\{b \cap y \,:\, b \in B,\ y \in Y\}$ is the inverse limit of B and Y. Notice that $B \longrightarrow \{b \cap y \,:\, b \in B,\ y \in Y\} \longrightarrow B$and thus $\{b \cap x \,:\, b \in B,\ x \in X\} \overset{(wf)}{\longrightarrow} B$.

First observe that the characterisation of (wf)-arrows in [A, Proposition 16(wf)] implies that the (countable) downward Lowenheim-Skolem theorem for the class $\Re$ is equivalent to the following: if $Y \longmapsto \top$ and $X \overset{(wf)}{\longrightarrow} Y$ then there exists $X' \overset{(cwf)}{\longmapsto} X$ (and therefore $X \overset{(cwf)}{\longmapsto} X'$). In plain words, if Y is a class of models in $\Re$ and $X \overset{(wf)}{\longrightarrow} Y$, then X can be replaced by an *isomorphic* class X' of models in $\Re$ such that $X' \longrightarrow X$ and $X \longrightarrow X'$.

Thus there exists a class X'_B of models in $\Re$ such that $X'_B \longrightarrow \{b \cap x \,:\, b \in B,\ x \in X\} \longrightarrow X'_B$. Finally, this implies $X'_B \overset{(wf)}{\longmapsto} B$. Now $A \overset{(c)}{\longmapsto} B \measuredangle X'_B \overset{(wf)}{\longmapsto} B$ implies $B \longrightarrow X'_B$ and $B \longrightarrow X$ as $X'_B \longrightarrow X$.

To see the latter, take $\Re_{\preceq} = (\mathfrak{K}, \preceq_{\Re})$ be the class of algebraically closed fields $\Re$ of a fixed characteristic; there elementary *embedding* $\preceq$ are *inclusions* of subfields, and *not* arbitrary injections. Then it can be easily checked, e.g. using [A§3.3.1,Proposition 16], that $\varnothing \overset{(wc)}{\longmapsto} \overline{\mathbb{Q}} \overset{(wc)}{\longmapsto} \overline{\mathbb{Q}(x_1,..,x_n)}$ but $\varnothing \overset{(wc)}{\not\longrightarrow} \overline{\mathbb{Q}} \overset{(wc)}{\not\longrightarrow} \overline{\mathbb{Q}(x_1,..,x_n)}$.

3. **Basics of Categories and Model Categories. The model category $QtNaamen$.** Here in §3 we recap the contents of Part A, and define the model category $QtNaamen$. We introduce the necessary notions and yoga of category theory and model category theories in a form convenient to us. We aim our exposition here to be brief but self-contained, and sometimes we repeat parts of Part A.

3.1. "Combinatorially[1] a category is a directed graph equipped with a collection of distinguished subgraphs called *commutative diagrammes* satisfying some properties; it is is enough to consider only triangular subgraphs Δ. We allow multiple edges between two vertices as well as loop edges leaving and entering the same vertex. An arrow/edge $h : X \longrightarrow Z$ is called the composition of arrows/edges f and g, denoted $h = fg$, if the triangle with edges f, g, and h is distinguished. It is sometimes helpful to think of categories topologically: draw the underlaying graph and glue in a cell , with boundary Δ a distinguished triangle; in this way commutative diagrams represent contractible subgraphs. [Part A, §2.1(Categories)]

3.2. Take a category and "forget" the distinguished subgraphs, loops and multiplicity of edges; you get a quasi partial order i.e. a transitive reflective binary relation. In the above situation we may say that *applying a "forgetful" functor to a category* gives a partial order

In other words, a category A *carries canonically* the structure of a quasi partially ordered set $(A, \leqslant_A)$: for $\bullet_1, \bullet_2 \in ObA$, $\bullet_1 \leqslant_A \bullet_2$ iff there is a morphism from $\bullet_1$ to $\bullet_2$. Conversely, every quasi partially ordered set $(P, \leqslant)$ can be canonically considered as a category: $ObA = P$, and there is a *unique* morphism $\bullet_1 \longrightarrow \bullet_2$ iff $\bullet_1 \leqslant_P \bullet_2$. There is no morphism $\bullet_1 \longrightarrow \bullet_2$ for $\bullet_1 \not\leqslant_P \bullet_2$; the composition of morphisms $\bullet_1 \longrightarrow \bullet_2$ and $\bullet_2 \longrightarrow \bullet_3$ is the unique morphism $\bullet_1 \longrightarrow \bullet_3$: every subgraph is necessarily a commutative diagram. An *initial* object $\perp$ of $\mathcal{A}$ is a least element of $\mathcal{A}$ (whenever such exists), i.e. there exists a *unique* morphism $\perp \longrightarrow Y$ for any object Y of $\mathcal{A}$. Dually, a *terminal* object $\top$ of $\mathcal{A}$ is a largest element of $\mathcal{A}$ (whenever such exists).

[1] We borrow from the exposition of category theory from [Gromov,2009+], pp.68-71.

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3.3. Now we are able to recast the definitions of the c-w-f-arrow notation from 2.2.1-5 into the category theoretic language of [Part A].

3.4. **Definition [Part A, Def.3(StNaamen) of §3.1, Def.25(QtNaamen) of §3.4])** *Let $StNaamen$ be the category of all classes ordered by $\longrightarrow$ equipped with the c-w-f labelling, and let $QtNaamen$ be the category of all cute classes ordered by $\longrightarrow$ equipped with the c-w-f labelling. Finally, let $QtNaamen(\Re)$ be the category of all classes of structures in a class $\Re$ ordered by $\longmapsto$ equipped with the c-w-f labelling.*

3.5. "The underlying principle of the category theory/language is that the *internal* structural properties of a mathematical object are fully reflected in the combinatorics of the graph (or rather the 2-polyhedron) of morphisms-arrows *around* it. *Amazingly*, this language, if properly (often non-obviously) developed, does allow a concise uniform description of mathematical structures in a vast variety of cases. (Some mathematicians believe that no branch of mathematics can claim maturity before it is set in a category theoretic or similar framework and some bitterly resent this idea.) [Gro09, page 70]

3.6. Accordingly, *diagram chasing* is an efficient way to reason graphically about a category: draw a commutative diagram and start adding arrows while keeping the diagram commutative. Often commonly used properties of objects and arrows are defined as rules to be used in diagram chasing: given a commutative diagram with certain properties, you can always extend it to a larger commutative diagram with certain properties. ([A§3] explains how diagram chasing leads us to define $QtNaamen$; see [A, Fig.3-5] and Appendix B for examples of simple properties defined by commutative diagrams. Lemma 35 and Lemma 36 of [A] give somewhat more involved examples of diagram chasing.)

3.7. The usual definitions of a model category (and a category) can be presented in a computational manner, as an world/environment where you can carry out certain diagram chasing constructions with labelled arrows. From this point of view, the axioms of a model category (and in fact, those of a category) are procedures one can use to add new arrows to existing diagrams and figuring out their correct labeling. Viewed this way, *a model category* by definition is a category equipped with a labelling on arrows by symbols c, w, f where you can carry out certain diagram chasing constructions. [A§2.3, Axiom M0-M5; A§3, esp. Remark 5]. For example, we interpret Axiom M1 as the following rule: Given given a commutative square of four arrows $A \xrightarrow{(wc)} B$, $B \longrightarrow Y$, $A \longrightarrow X$, $X \xrightarrow{(f)}$, add diagonal arrows $A \longrightarrow X$ and $B \longrightarrow Y$. The definition of a model category is given in [A§2.3,Axioms M0-M5].

It it this computational approach that we use in this paper. Originally model categories were introduced by Quillen [Qui67] to provide a common axiomatisation to a variety homotopy theories, e.g. homotopy of topological spaces and that of simplicial sets; arrows carrying labels (c), (w), (f), resp., are called *cofibrations, weak equivalences, fibrations*, resp.

3.8. Diagram chasing is particularly straightforward and efficient in a category that comes from a quasi partial order: all diagrams commute and the process reduces to adding labelled arrows. A quasi partial order viewed as a *category* is a directed graph where you can always add an arrow $X \longrightarrow Z$ to the diagram $X \longrightarrow Y \longrightarrow Z$ and where you do not distinguish arrows between two given objects; all diagrams commute.

3.9. In [A§3] we introduce a semi-automatic fictional homotopy theorist trying to learn (basics of naive) set theory. We claim that he might *discover* the theorem below by a fairly automatic *model category* diagram chasing greedy strategy; the proof of the Theorem is represented as his journey. See [A§3] for a description of the hero and his strategy.

The point of this paper is to convince the reader that our fictional hero may also have discovered the definition of the covering number of PCF.

3.10. **Theorem** **($QtNaamen$).** *The quasi partial order $\longrightarrow$ on cute classes, with c-w-f labels as de-*

fined, is a model category.

It is reasonable to ask whether $QtNaamen(\Re)$ is a model category for a nice class $\Re$ of models; we state a precise conjecture in ???.

4. **Functors and derived functors.** The summary of [A] is finished; now we continue and introduce the necessary language of category theory and homotopy theory adopted to our case.

4.1. A *covariant functor* between two categories $\mathcal{C}$ and $\mathcal{C}'$ is a pair of maps

$$\{\bullet\} \rightsquigarrow_\bullet \{\bullet\}' \text{ and } \{\rightarrow\} \rightsquigarrow_\rightarrow \{\rightarrow\}'$$

which sends the arrows $\bullet_1 \longrightarrow \bullet_2$ to the corresponding (by $\rightsquigarrow$) arrows $\bullet_1' \longrightarrow \bullet_2'$ that every commutative diagram in $\mathcal{C}$ goes to a commutative diagram in $\mathcal{C}'$. A contravariant functor sends the set of $\bullet_1 \longrightarrow \bullet_2$-arrows to the corresponding set of $\bullet_1' \longrightarrow \bullet_2'$-arrows, where the *functoriality* also means preservation of commutativity of all diagrams.

4.2. For quasi partially ordered sets/classes A, B considered as categories, a *functor* $F : A \longrightarrow B$ is a *monotonic function*, and a *covariant functor* is a non-decreasing function. For covariant functors $F, G : A \longrightarrow B$, there exists a natural transformation taking F into G iff $\forall X \in \mathcal{O}bA(F(X) \leqslant_B G(X))$; such a natural transformation is necessarily unique if exists. The functors $F, G : A \longrightarrow B$ are naturally equivalent iff $F(\bullet) \leqslant_B G(\bullet) \leqslant_B F(\bullet)$ for every object $\bullet \in \mathcal{O}bA$. (As any diagram is commutative in these categories, we need not state the conditions that the functors have to respect commutative diagrams.)

4.3. For quasi-partially ordered sets/classes A, A', B as categories, and covariant functors $\gamma : A \longrightarrow A'$ and $F : A \longrightarrow B$, *the left derived functor* $\mathbb{L}^\gamma F : A' \longrightarrow B$ *with respect to* $\gamma : A \longrightarrow A'$ is "the functor from A' to B such that $\mathbb{L}^\gamma F \circ \gamma$ is closest to F from the left" ([Qui67, I§4.1]) in the following precise sense. By definition $\mathbb{L}^\gamma F : A' \longrightarrow B$ is a covariant functor, i.e. an order-preserving function from A' to B, such that (i) firstly, $\forall \bullet \in ObA(\mathbb{L}^\gamma F \circ \gamma(\bullet) \leqslant_B F(\bullet))$, and (ii) secondly, for every non-decreasing function (functor) $G : A' \longrightarrow B$ such that $\forall \bullet \in ObA(G \circ \gamma(\bullet) \leqslant_B F(\bullet))$, it holds $\forall \bullet \in ObA'(G(\bullet) \leqslant_B \mathbb{L}^\gamma F(\bullet))$. Similarly we may define *the right-derived (covariant) functor* inverting the direction of all the inequalities in the above formulae. To define derived functors for arbitrary categories A, A', B we also need to state which commutative diagrams are preserved; we refer to Definition 1 in [Qui67, I§4.1].

4.4. *Example: limits of commutative diagrams as derived functors.* Consider the diagram $X_i \xleftarrow{f_i} X_k \xrightarrow{f_j} X_j$ in a category B. The diagram may be viewed as a covariant functor $X : I_X \longrightarrow B$ where I_X is the category corresponding to the partial order $i \geqslant k \leqslant j$. Take $A = I_X$ and let A' be the trivial category with a single object $\bullet$ and a single morphism $\mathrm{id}_\bullet : \bullet \longrightarrow \bullet$. Let $\gamma : A \longrightarrow A'$ be the unique covariant functor taking every object of A into the only object of A', and every morphisms of A into the only morphism of A'. Note that a functor $G : A' \longrightarrow B$ is determined by, and may be identified with, the object $G(\bullet)$. Finally, consider the left derived functor $\mathbb{L}^\gamma X : A' \longrightarrow B$. By (i) and (ii) above, $\mathbb{L}^\gamma X(\bullet)$ is the unique object such that (i') $\mathbb{L}^\gamma X(\bullet) \xrightarrow{f_i'} X_i$ and $\mathbb{L}^\gamma X(\bullet) \xrightarrow{f_j'} X_j$ and $\mathbb{L}^\gamma X(\bullet) \xrightarrow{f_k'} X_k$ making the diagram commute, and (ii') for any object Y with arrows as in (i'), there exists a unique arrow $Y \longrightarrow X$. The two properties above mean ([A§4.1,Fig.1,Rem.2]) that $\mathbb{L}^\gamma X(\bullet)$ is the *limit* of the diagram $X_i \xleftarrow{f_i} X_k \xrightarrow{f_j} X_j$.

We may take an arbitrary quasi-partial order as I_X above, and the same argument shows that *the limit of a commutative diagram D is the left derived functor of D viewed as a functor*, and dually, *the colimit of a commutative diagram D is the right derived functor of the diagram viewed as a functor*.

4.5. *For quasi-partially ordered classes as above, the left-derived functor is given by the formula*

$$\mathbb{L}^\gamma F(\bullet') = \inf\{F(\bullet) : \bullet' \leqslant_{A'} \gamma(\bullet),\ \bullet \in ObA\}$$

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In particular, the left derived functor exists iff the right hand side is well-defined.

To prove this, first check that transitivity of $\leqslant$ implies that the right hand side defines a monotone function; thus the right hand side defines a covariant functor and items (i) and (ii) hold by definition of the infimum as the greatest lower bound. Thus the right hand side defines the left derived functor whenever it is defined.

Assume now the left derived functor $\mathbb{L}^\gamma F : A' \longrightarrow B$ exists, yet the infimum $\inf\{F(\bullet) : \bullet' \leqslant_{A'} \gamma(\bullet),\ \bullet \in ObA\}$ does not exist for some $\bullet' \in \mathcal{O}bA'$. That means that there exist $\bullet'_B \in \mathcal{O}bB$ such that $\bullet'_B \not\leqslant_B \mathbb{L}^\gamma F(\bullet')$ and $\bullet'_B \leqslant_B F(\bullet)$ for any $\bullet' \leqslant_{A'} \gamma(\bullet)$. Consider now the functor $G_{\bullet'_B} : A' \longrightarrow B$ where $G_{\bullet'_B}(\bullet') = \bullet'_B$ if $\bullet \leqslant \bullet'$ (equiv., $\bullet' \longrightarrow \bullet$), and $G_{\bullet'_B}(\bullet') = \mathbb{L}^\gamma F(\bullet')$ otherwise. Then notice (ii) above fails.

Note that the formula defines, up to natural equivalence, a functor, i.e. an order-preserving function, satisfying (i) and (ii) for $F : A \longrightarrow B$ an *arbitrary* function not necessarily order-preserving (i.e. functorial). For example, the left derived functor always exists when the target B is well-ordered or $B = (\mathbb{R}_{\geqslant 0}, \leqslant)$ is the set of non-negative reals.

4.6. If A is equipped with a c-w-f labelling satisfying the axioms M0-M5 of a model category, Quillen defines the homotopy category $A' = HoA$ and the localisation functor $\gamma : A \longrightarrow HoA$. By construction of the homotopy category we outlined in [Part A, §2.3], $\mathcal{O}bA = \mathcal{O}bA' = \mathcal{O}bHoA$, and a morphism in $A' = HoA$ from X to Y is (an equivalence class represented by) a chain of morphisms in A of the following form :

$$X \longrightarrow X_1 \stackrel{(w)}{\longleftarrow} X_2 \longrightarrow X_3 \stackrel{(w)}{\longleftarrow} \dots \longrightarrow X_n \longrightarrow Y,$$

the localisation $\gamma : A \longrightarrow HoA$ is the identify on objects and (almost so) on morphisms (a morphism is taken into the equivalence class of itself). The homotopy category HoA inherits the c-w-f labelling from A: each arrow in the homotopy category is labelled by (cf), and each isomorphism is labelled by (cwf).

4.7. If both $\mathcal{A}$ and $\mathcal{B}$ are also equipped with a c-w-f labelling, we say that a functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ is homotopy-invariant iff for any arrow $X \stackrel{(w)}{\longrightarrow} Y$ (weak homotopy equivalence), it holds $F(X) \stackrel{(w)}{\longrightarrow} F(Y)$. Equivalently, F is homotopy-invariant iff $F = F' \circ \gamma$ factors via the homotopy category as $\mathcal{A} \stackrel{\gamma}{\rightarrow} Ho\mathcal{A} \stackrel{F'}{\longrightarrow} \mathcal{B}$. It is easy to see that by this definition, a left-derived functor is necessarily homotopy-invariant, and indeed, $\mathbb{L}^\gamma F(X)$ is referred to as a homotopy invariant functor "closest from the left"(Quillen, I:4.1) to the function $F : StNaamen \longrightarrow On$.

4.8. *Example: limits of commutative diagrams as derived functors with respect to a model structure.*. In 4.4 above we show that limits of commutative diagrams may be viewed as left-derived functors. Now we that that, furthermore, *limits of commutative diagrams may be viewed as left-derived functors with respect to a model structure.* Every category I has three degenerate model structures: every non-isomorphism arrow carries labels either (cf) or (wc) or (fw); by Axiom M3(c) the isomorphisms carry all the three labels (cwf). For these structures, the axioms M0-M6 hold for trivial reasons: for example, the lifting property of Axiom M1([A§2.3.1]) holds because in every lifting square of Axiom M1 one of the labelled arrows is necessary an isomorphism. For the (cf)-model structure, $Ho_{cf}A = A$, and for the (wc)- and (fw)-model structures we have $Ho_{wc}A = Ho_{wf}A$ is the trivial single-object single-morphism category. Thus 4.4 says that the limit of a commutative diagram $X : I_X \longrightarrow B$ is the left-derived homotopy-invariant functor $\mathbb{L}^\gamma X : Ho_{wc}I_X \longrightarrow B$ viewed as an object of B. Dually, colimits are right-derived functors.

4.9. *Example: complete linear orders and well-ordered classes*. In a quasi partially ordered set considered as a category, a commutative diagram may be identified with the set of its vertices; the underlying graph is not important because the morphisms are unique whenever exist [A,Remark 9]. The colimit of a diagram is its least upper bound of its vertices, and its limit is the greatest lower bound.

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A partially ordered set is *linear* iff either all finite limits or all finite colimits (exist and) are degenerate, i.e. the limit and colimit of each finite commutative diagram is degenerate. A linearly ordered set is *complete* iff every diagram has both a limit and a colimit. A partially ordered class is *well-ordered* iff limits always exist and are always degenerate.

4.10. *Functors preserve degenerate limits, and conversely, a limit preserved by all functors is degenerate.* We say that the limit or colimit of a diagram D is *degenerate* iff it is isomorphic to one of its vertices. Any functor preserves all degenerate limits and colimits. Conversely, if A is a quasi-partial order, for a commutative diagram D, if for any functor $F : A \longrightarrow A'$ it holds that $\lim F(D)$ and $F(\lim D)$ are isomorphic, then the limit of D is degenerate. A straightforward check gives the former; to check the latter, add the limit(infimum) of $\lim F(D)$ formally to the category A. That is, consider the category A', $\mathcal{O}bA' = \mathcal{O}bA \cup L$, and for $X, Y \neq L$ $\mathcal{M}or_{A'}(X,Y) = \mathcal{M}or_A(X,Y)$, $\mathcal{M}or_{A'}(X,L) = \mathcal{M}or_A(X, \lim D)$. Finally, set $\mathcal{M}or_{A'}(L,X) \neq \varnothing$ iff for some vertex D_i in D it holds $\mathcal{M}or_A(D_i, X)$. Now it is left to check that A' is indeed a category (quasi partial order) and $L, \lim D \in \mathcal{O}bA'$ are not isomorphic if the limit of D is non-degenerate.

In particular, a quasi partially ordered class A is well-ordered iff for every diagram D and every functor $F : A \longrightarrow B$, it holds $F(\lim D) = \lim F(D)$, i.e. $\lim D$ exists iff $\lim F(D)$ exists, and if they both exists, the formula holds.[2]

4.11. Let On be the category of ordinals where each arrow is labelled (cf) and each isomorphism is labelled (cwf), and let $On^\top$, $\mathcal{O}b\, On^\top = \mathcal{O}b\, On \cup \{\top\}$ be the category of ordinals with a formally added terminal object $\top$.

For a function $F : \mathcal{A} \longrightarrow On^\top$, define (minimum is taken over all finite sequences labelled as shown)

$$\mathbb{L}_c F(X) = \min \left\{ F(Y) : \; X \dashrightarrow X_1 \xleftarrow{(w)} X_2 \dashrightarrow X_3 \xleftarrow{(w)} \cdots \dashrightarrow X_n \dashrightarrow Y \xleftarrow{(c)} \bot \right\}$$

Notice that by Axiom M0 and M2 if F is a covariant functor, then $\mathbb{L}_c F = \mathbb{L}^\gamma F \circ \gamma$ is the left-derived functor of F: for any object $\bullet \in \mathcal{O}bA$ there exists a decomposition $\bot \xrightarrow{(c)} \bullet_c \xrightarrow{(wf)} \bullet$ and therefore $F(\bullet_c) \leqslant_A F(\bullet)$ implying $\mathbb{L}_c F(\bullet) \leqslant_A \mathbb{L}^\gamma F(\gamma(\bullet))$; the other inequality $\mathbb{L}_c F(\bullet) \geqslant_A \mathbb{L}^\gamma F(\gamma(\bullet))$ is by definition. However, for F a function not necessarily order-preserving, $\mathbb{L}_c F$ and $\mathbb{L}^\gamma F \circ \gamma$ may differ, and we refer to $\mathbb{L}_c F$ as the *cofibrantly replaced left-derived functor of* F.

4.12. Note that considerations above give that $\mathbb{L}_c F(X)$ *is a homotopy invariant functor "closest from the left"(Quillen, I:4.1) to the cofibrant replacement of the function* $F : StNaamen \longrightarrow On$, by which is meant: for any homotopy-invariant functor $G : StNaamen \longrightarrow On$ such that $G(X) \longrightarrow F(X)$ whenever $\bot \xrightarrow{(c)} X$, it holds that $G(Y) \longrightarrow \mathbb{L}_c F(Y)$ whenever $\bot \xrightarrow{(c)} Y$ (note then there is a natural transformation from functor G to functor $\mathbb{L}_c F$).

In particular, the function $\mathbb{L}_c F : StNaamen \longrightarrow On$ is the left derived functor of $F : StNaamen \longrightarrow On$ provided that F is a functor.

4.13. Now that we defined the necessary notions of model categories, we state a bit of model category yoga/ideology/technique we now apply: Given a functor on a model category, it is often interesting to calculate (values of) its left derived functor. It is often interesting to take a functor "forgetting" part of structure, e.g. homology/homotopy groups are derived functors of global sections functor sending a sheaf Sh_X of functions on a space X, into the ring $\Gamma(Sh(X))$ of functions defined on the whole of X, or restriction of scalars $S - Mod \longrightarrow R - mod$ for rings $R \subset S$.

[2]The authors thank Marco Porta for these observations.

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We note that we also observed that for categories quasi partially orders, the definition of a derived functors extends to that of a derived functor of an arbitrary function not necessarily preserving order.

5. **An application in a nutt-shell: the covering number of $\aleph_\omega$ as a value of a derived functor.** Now we describe the simplest application of the machinery above. To streamline the exposition we omit the proofs; the proofs can be found in the next section.

5.1. What is a simplest function that forgets part of structure of a set(class) ? Consider cardinality. By definition, this is a function $\mathrm{card}\ : QtNaamen \longrightarrow On^\top,\ \ X \longmapsto \mathrm{card}\,(X)$ that *forgets everything* about a set but the number of its elements, in a way similar to saying that the function sending a topological space into the set of its connected components forgets everything about the topology but the connected components. In ZFC the cardinality of a class is not defined: for a proper class K, we postulate $\mathrm{card}\,(K) = \top$.

Cardinality is not a functor on $QtNaamen$, as the following two counterexamples show: $\{X\} \longrightarrow \mathcal{P}(X) \longrightarrow \{X\}$ but by Cantor's diagonal argument $\mathrm{card}\,(X) < \mathrm{card}\,(\mathcal{P}(X)) > \mathrm{card}\,(X)$, and also $\{\{\bullet_1\},\{\bullet_1,\bullet_2\}\} \xrightarrow{(wcf)} \{\{\bullet_1,\bullet_2\}\}$ is an isomorphism but $2 = \mathrm{card}\,\{\{\bullet_1\},\{\bullet_1,\bullet_2\}\} > \mathrm{card}\,\{\{\bullet_1,\bullet_2\}\} = 1$ are non-isomorphic.

5.2. Take $F = \mathrm{card}$ to be the cardinality function. Arguably, the model category formalism suggests we view $\mathbb{L}_c\mathrm{card}\ : StNaamen \longrightarrow On$ as an analogue of a cofibrantly replaced left derived functor of the "forgetful functor" $\mathrm{card}\ : StNaamen \longrightarrow On$. Then homotopy yoga suggests we view values of $\mathbb{L}_c\mathrm{card}$, e.g. $\mathbb{L}_c\mathrm{card}\,(\{\aleph_\alpha\}) = \mathbb{L}_c\mathrm{card}\,(\{X : X \subseteq \aleph_\alpha\})$, as (homotopy-invariant and therefore) more robust and interesting invariants, as compared with the non-homotopy-invariant values $\mathrm{card}\,(\{X : X \subseteq \aleph_\alpha\})$.

5.3. And indeed, it is for the reasons of being more robust and less prone to change by forcing that the values of $\mathbb{L}_c\mathrm{card}\,(\{\aleph_\alpha\})$ (for limit $\aleph_\alpha$) have been introduced in set theory (Shelah, Cardinal Arithmetic); in §6 below we state these results in detail. Set-theoretically (Lemma 7.2 below) $\mathbb{L}_c\mathrm{card}\,(\{\aleph_\alpha\}) = \mathrm{cov}(\aleph_\alpha,\aleph_1,\aleph_1,2)$ is the least size of a family X of countable subsets of $\aleph_\alpha$, such that every countable subset of $\aleph_\alpha$ is a subset of a set in the family X. This may used, for example, to study the cardinality $(\aleph_\alpha)^{\aleph_0}$ of the set of countable subsets of $\aleph_\alpha$, via the bound $(\aleph_\alpha)^{\aleph_0} \leqslant \mathrm{cov}(\aleph_\alpha,\aleph_1,\aleph_1,2) + 2^{\aleph_0}$, by decomposing it into a "noise" "non-homotopy-invariant" part $2^{\aleph_0}$ whose value is known to be highly independent of ZFC (and easy to force to change), and a homotopy-invariant part $\mathrm{cov}(\aleph_\alpha,\aleph_1,\aleph_1,2)$ which admit bounds in ZFC (and is harder to force to change).

5.4. A short calculation gives $\mathbb{L}_c\mathrm{card}\,(\{X : X \subseteq \aleph_0\}) = 1$ (in ZFC) whereas it is known that there are models of ZFC where e.g. $\mathrm{card}\,(\{X : X \subseteq \aleph_0\}) = 2^{\aleph_0} > \aleph_{\omega_\omega}$. Similarly for (many) other values ZFC-independence is avoided: $\mathbb{L}_c\mathrm{card}\,(\{\aleph_\alpha\}) = \aleph_\alpha$ for $\aleph_\alpha$ regular ($\mathrm{cf}\aleph_\alpha = \aleph_\alpha$) (whereas by Silver's theorem $\mathrm{card}\,((\{X : X \subset \aleph_\alpha\})$ on *regular* cardinals $\aleph_\alpha$ is almost an arbitrary function satisfying some cardinal-arithmetic identities). Meanwhile, non-trivially, Shelah (Cardinal Arithmetic, IX:4) proves $\mathbb{L}_c\mathrm{card}\,(\{\aleph_\omega\}) < \aleph_{\omega_4}$. Similar upper bounds exist on $\mathbb{L}_c\mathrm{card}\,(\{\aleph_\alpha\})$ for (most) $\aleph_\alpha$ limit (but $\aleph_\alpha = \alpha$), and are provided by PCF theory. We list these bounds in Lemma 7.5 below.

Thus we see that $\mathbb{L}_c\mathrm{card}\,(\{\aleph_\omega\})$ is an interesting cardinal arithmetic invariant that helps to solve a classical question of finding an upper bound on the number of countable subsets of $\aleph_\omega$, and, moreover, it behaves as homotopy theory suggests it should.

5.5. How would our fictional hero of [A§3] would come to consider the covering number $\mathrm{cov}(\aleph_\alpha,\aleph_1,\aleph_1,2)$? Now we have the necessary background to describe his steps. He just defined the model category $QtNaamen$. By 5.8 he knows it is interesting to define and calculate the derived functor $\mathbb{L}_cF$ of a forgetful functor F, the homotopy invariant approximation to F. One of the basic notions he strives to understand does look somewhat like a forgetful functor : $\mathrm{card}\ : QtNaamen \longrightarrow On^\top$. Moreover, cardinality was the only notion he encountered so far that can be used as a (forgetful) functor: plug in an object (here a class) and get something simpler (here its cardinality). Thus he defines the functor

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$\mathbb{L}_c\mathrm{card}$ and then looks its values at the simplest objects — the singletons $\mathbb{L}_c\mathrm{card}\,(\{X\})$. By $n°5.7$ it is fine that he is oblivious to the fact that the cardinality is not a functor.

6. **More on PCF theory and derived functor** $\mathbb{L}_c\mathrm{card}$

In the previous section we briefly explained how can one arrive to a definition of the simplest covering number of $\aleph_\omega$. In this section we expand on this observation, give proofs and references, and use the category formalism to define more general covering numbers.

6.1. By definition the covering number[3]

$$\mathrm{cov}(\lambda, \kappa, \theta, \sigma)$$

is the *least size of a family* $X \subseteq [\lambda]^{<\kappa}$ *of subsets of* λ *of cardinality less than* κ*, such that every subset of* λ *of cardinality less than* θ*, lies in a union of less than* σ *subsets in* X*.*

6.2. **Lemma** (the covering number as a derived functor). *For* $\{\lambda\} \in QtNaamen$*,*

$$\mathbb{L}_c\mathrm{card}\,(\{\lambda\}) = cov(\lambda, \aleph_1, \aleph_1, 2)$$

Proof. Call $X \in \mathcal{O}bStNaamen$ a (wc)-*covering family* of λ iff every countable subset of λ is a subset, up to finitely many elements, of an element of X. Prove by induction on n that each X, $X_1, .., X_n, Y$ is a (wc)-covering family for λ. For $X = \{\lambda\}$ this is obvious; for X_1 this is immediate by the definition of a morphism $X \longrightarrow X_1$, for X_2 this is immediate by the definition of a weak equivalence $X_1 \xleftarrow{(w)} X_2$, etc. Thus Y is a (wc)-covering family for λ; the condition $\{\} \xrightarrow{(c)} Y$ implies that every element of Y is countable. In notation, $\varnothing \xrightarrow{(c)} Y \xrightarrow{(wc)} \{\lambda\}$; apply [A, Lemma 35 (A continuity fixed-point argument)] to find Λ such that $\varnothing \overset{(c)}{\dashrightarrow} Y' \overset{(wf)}{\dashrightarrow} \{\Lambda\} \overset{(wc)}{\dashrightarrow} \{\lambda\}, Y' \dashrightarrow Y$. This shows $\mathbb{L}_c\mathrm{card}\,(\{\lambda\}) \leqslant cov(\lambda, \aleph_1, \aleph_1, 2)$. Conversely, for Y a covering family, take $n = 2$, $X_1 = X$, $X_2 = Y$, then by the definition of a covering family $\{\lambda\} = X_1 \xleftarrow{(w)} X_2 = Y \xleftarrow{(c)} \varnothing$.

6.3. We explained that the definition of $\mathbb{L}_c\mathrm{card}\,(X)$ is natural and straightforward in homotopy theory and particularly in Quillen's formalism of model categories. The following two modifications are seemingly minor and not entirely unnatural from the homotopy point of view:

6.4. **Definition.** $\mathbb{L}_c^{Qt/A}\mathrm{card}\,(A \longrightarrow X) =$

$$= \min\{\mathrm{card}\, X' : A \xrightarrow{(c)} X' \longleftarrow X_1 \xrightarrow{(w)} X_2 \longleftarrow ... \xrightarrow{(w)} X_n \longleftarrow X, \quad A \longrightarrow X', A \longrightarrow X_1, ..., A \longrightarrow X_n \in QtNaamen\} \qquad (***)$$

and

$\mathbb{L}_c^{Qt}\mathrm{card}\,(A \longrightarrow X) =$

$$= \min\{\mathrm{card}\, X' : A \xrightarrow{(c)} X' \longleftarrow X_1 \xrightarrow{(w)} X_2 \longleftarrow ... \xrightarrow{(w)} X_n \longleftarrow X, \quad X', X_1, ..., X_n \in QtNaamen\} \qquad (****)$$

6.5. **Lemma** *For a cardinal* $\{\aleph_\alpha\} \in QtNaamen$*,*

[3] For reader's convenience here we define somewhat incoherent notation used in the references we cite. The 4-parameter notation $\mathrm{cov}(\lambda, \lambda, \kappa, 2)$ is standard and follows [Shelah, Cardinal Arithmetic], p. ?? (Appendix S). We refer to two expository papers [Shelah, Cardinal arithmetics for skeptics] and [Kojman, 2001](PCF Theory) that use slightly different 2-parameter notation for the covering number $\mathrm{cov}(\lambda, \kappa) := \mathrm{cov}(\lambda, \lambda, \kappa, 2)$ (Shelah) and $\mathrm{cov}(\lambda, \omega) := \mathrm{cov}(\lambda, \aleph_1, \aleph_1, 2)$ (Kojman). In [Shelah, Cardinal arithmetics for skeptics], Theorem 5.7 identifies $\mathrm{pp}_\kappa(\lambda)$ as $\mathrm{pp}_\kappa(\lambda) = \mathrm{cov}(\lambda, \kappa) = \mathrm{cov}(\lambda, \lambda, \kappa, 2)$ for $\mathrm{cf}\lambda \leqslant \kappa < \lambda$ and $\lambda \neq \aleph_\lambda$. It is not known whether it is consistent with ZFC that $\mathrm{pp}_\kappa(\lambda) \neq \mathrm{cov}(\lambda, \lambda, \kappa, 2)$ for some λ. Theorem 6.3 [ibid.] is what we call Homotopy Generalised Continuum Hypothesis. The relevant two pages of the paper are in the appendix; at a later stage we shall provide references to the book "Cardinal Arithmetic" which contains proofs.

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(i)

$$\mathbb{L}_c\mathrm{card}\,(\{\aleph_\alpha\}) = \mathbb{L}_c^{Qt}\mathrm{card}\,(\varnothing\longrightarrow\{\aleph_\alpha\}) = \mathbb{L}_c^{St}\mathrm{card}\,(\varnothing\longrightarrow\{\aleph_\alpha\}) = \mathrm{cov}(\aleph_\alpha,\aleph_1,\aleph_1,2).$$

In particular, (i')

$$\mathbb{L}_c\mathrm{card}\,(\{\aleph_\omega\}) = \mathbb{L}_c^{Qt}\mathrm{card}\,(\varnothing\longrightarrow\{\aleph_\omega\}) = \mathbb{L}_c^{St}\mathrm{card}\,(\varnothing\longrightarrow\{\aleph_\omega\}) = \mathrm{cov}(\aleph_\omega,\aleph_1,\aleph_1,2) = \mathrm{pp}(\aleph_\omega).$$

(ii)

$$\mathbb{L}_c^{Qt/\aleph_\alpha}(\aleph_\alpha\longrightarrow\{\aleph_\alpha\}) = \mathrm{cov}(\aleph_\alpha,\aleph_\alpha,\aleph_\alpha,2)$$

(iii)

$$\mathbb{L}_c^{Qt}(\aleph_\alpha\longrightarrow\{\aleph_\alpha\}) = \mathrm{cov}(\aleph_\alpha,\aleph_\alpha,\aleph_1,2)$$

6.6. **Proof.** (i) has the same proof as the previous Lemma. (iii) Take $n=1$, $X_1 := [\aleph_\alpha]^{\leqslant\omega}$ be the set of countable subsets, X' be a covering family as in the definition of $\mathrm{cov}(\aleph_\alpha,\aleph_\alpha,\aleph_1,2)$. (ii) As in the previous Lemma, we know that every countable subset of $\aleph_\alpha$ is a subset of an an element of every X_i, up to finitely many elements. However, we also know that $X_i \in QtNaamen$ and there is an arrow $\aleph_\alpha\longrightarrow X_i$, for each X_i. Use the lifting property of [A, Lemma 23] in the definition of QtNaamen to show that in fact X_i covers every subset of $\aleph_\alpha$ of cardinality less than $\aleph_\alpha$, e.g. by taking $A=\aleph_\mu$, $\mu<\alpha$, and $B=\aleph_\mu\cup B'$ where B' is arbitrary such that $\mathrm{card}\,B'\leqslant\aleph_\mu$.

6.7. The reader would have little trouble giving other examples, e.g. by replacing the arrow $\aleph_\alpha\longrightarrow\{\aleph_\alpha\}$ by $[\aleph_\alpha]^{<\aleph_\alpha}\longrightarrow\{\aleph_\alpha\}$ to get rid of assumption $\aleph_\alpha\in QtNaamen$.

6.8. We summarise some of what is known in our notation. As explained in the introduction, Shelah ([Shelah, Cardinal Arithmetic], [Shelah, Logical Dreams]) views these bounds as answers to the *right* questions. Note that analogously, from the homotopic point of view, these are answers to natural homotopy-invariant questions. In the introduction we say more on PCF as a homotopy-invariant theory. We note that passing to homotopy-invariant/PCF questions avoids independence of ZFC.

Theorem 1 (Shelah; bounds towards hGCH) *There are following bounds on the values of the derived functors $\mathbb{L}_c$ and $\mathbb{L}_c^{Qt},\mathbb{L}_c^{St}$.*

(i) if $\mathrm{cf}\aleph_\alpha=\aleph_\alpha$ a regular cardinal, then

$$\mathbb{L}_c(\{\aleph_\alpha\}) = \mathbb{L}_c(2^{\aleph_\alpha}) = \mathrm{cov}(\aleph_\alpha,\aleph_1,\aleph_1,2) = \aleph_\alpha$$

(ii) $\mathbb{L}_c(\{\aleph_\omega\}) = \mathbb{L}_c(2^{\aleph_\omega}) = \mathrm{cov}(\aleph_\omega,\aleph_1,\aleph_1,2) < \aleph_{\omega_4}$

(iii) if δ is a limit ordinal, $\mathrm{cf}\delta=\omega$, and $\delta<\aleph_{\alpha+\delta}$, $\alpha+\delta<\aleph_{\alpha+\delta}$, then

$$\mathbb{L}_c^{Qt}(\aleph_{\alpha+\delta}\longrightarrow\{\aleph_{\alpha+\delta}\}) = \mathrm{cov}(\aleph_{\alpha+\delta},\aleph_{\alpha+\delta},\aleph_1,2) < \aleph_{\alpha+\delta^{+4}}$$

Proof. Todo: give references to [Shelah, Cardinal Arithmetics].

Note that we do not say anything about the *fixed points* $\alpha=\aleph_\alpha$ of $\aleph_\bullet$-function. (todo: is there an explanation)

6.9. Arguably, the above justifies saying that the homotopy-invariant version of Generalised Continuum Hypothesis has less independence of ZFC, as suggested by homotopy theory.

7. **A conjectural connection to model theory of categorical classes** In this section we present a conjecture relating our approach to model theory.

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7.1. Further we conjecture that often a (not necessarily first-order) *categorical* theory, via the class of (families of) its models, defines a model subcategory of $QtNaamen$

7.2. ***The Eyjafjallajoekull conjecture*** (Bays and Gavrilovich). *Let* $\Re_{\preceq} = (\mathfrak{K}, \preceq_{\Re})$ *be an excellent abstract elementary class. Then there is a subcategory* $Qt(\Re_{\preceq})$ *of* $QtNaamen(\Re_{\preceq})$ *which is a model category, and is not degenerate, e.g. for every cardinality* λ *there is a model* $M_\lambda \in \Re$ *of cardinality* λ *such that* $\{M_\lambda\} \in \mathcal{O}bQt(\Re_{\preceq})$*, and for any* $M, N \in \Re$ *it holds that* $\{M\} \xrightarrow{(wc)} \{N\}$ *iff* $M \preceq N$ *and* N *is primary over* $M \cup \bar{b}$ *for a finite set* $\bar{b} \subset N$.

7.3. Axiom M5(2-out-of-3) is delicate to check and seem to impose structural constrains; mere diagram chasing suffices to prove that *any complete quasi-poset with fibrantly or cofibrantly generated model category labelling, and particularly the categories* $StNaamen(\Re_{\preceq})$ *and* $QtNaamen(\Re_{\preceq})$ *with labels as defined above, satisfy Axioms M1-M4 and M6 of model categories* but not necessarily M5(2-out-of-3).

7.4. Our intention is that the definition of $Qt(\Re_{\preceq})$ is to be such that model category diagram chasing corresponds to arguments in excellent classes, e.g. drawing the pushout square on the left is to correspond to a categoricity transfer argument in excellent classes (§3,p.17 of [Les04]). Take a model M of cardinality $\aleph_1$, take its cofibrant replacement $\varnothing \xrightarrow{(c)} \{M_i\}_{i\in\omega_1} \xrightarrow{(wf)} \{M\}$ by splitting $M = \cup_{i\in\omega_1} M_i$ into a continuous increasing chain of countable models M_i. Pick an element a and construct an acyclic cofibration $\{M_i\}_i \xrightarrow{(wc)} \{\overline{M_i a}\}_i$ of models $\overline{M_i a}$ countable primary over $M_i \cup \{a\}$. Finally, take the pushout of $\{M\} \xleftarrow{(wf)} \{M_i\}_{i\in\omega_1} \xrightarrow{(wc)} \{\overline{M_i a}\}_{i\in\omega_1}$. In $StNaamen(\Re_{\preceq})$ the pushout is simply $\{M\} \cup \{\overline{M_i a}\}_{i\in\omega_1}$, whereas in $QtNaamen(\Re_{\preceq})$ the pushout is a single model $\{N\}$, $N \supseteq M \cup \{a\}$. By Axioms M3 and M4 or M5 we get $\{M\} \xrightarrow{(wc)} \{N\}$, and thus N is primary over $M \cup \{a\}$.

7.5. The conjecture appears to relate model categories and such questions as *Mumford-Tate, Kummer theory, Mordell-Weil, Schanuel conjecture and dual thereto*, as these algebro-geometric properties are essentially used to prove excellency and uncountable categoricity (rather, finitely many models) of the following explicitly given abstract elementary classes of models, cf. [Bays, DPhil] and references therein.

7.6. Fix an elliptic curve $E/\overline{\mathbb{Q}}$ without complex multiplication defined over a number field. Consider the classes of all short exact sequences of the form

$$\begin{array}{ll} (\overline{K}^*) & 0 \longrightarrow \mathbb{Z} \longrightarrow V \xrightarrow{\varphi} \overline{K}^* \longrightarrow 1 \\ (\overline{K}_p^*) & 0 \longrightarrow \mathbb{Z}[\frac{1}{p}] \longrightarrow V \xrightarrow{\psi} \overline{K}_p^* \longrightarrow 1 \\ (E) & 0 \longrightarrow \mathbb{Z}^2 \longrightarrow V \xrightarrow{\varphi} E(\overline{K}) \longrightarrow 0 \\ (\mathbb{C}_{\exp}) & \text{a field } \overline{K} \text{ equipped with a homomorphism } \exp : \overline{K} \longrightarrow \overline{K}^* \end{array}$$

where, as notation suggests, V varies among $\mathbb{Q}$-vector spaces, $\overline{K}$ varies among algebraically closed fields of zero characteristic, and φ varies among group homomorphisms; and $\overline{K}_p$ varies among algebraically closed fields of prime characteristic p, and ψ varies among $\mathbb{Z}[\frac{1}{p}]$-module homomorphisms such that the isomorphism type of the restriction $\psi_{|\mathbb{Q}\mathbb{Z}[\frac{1}{p}]} : \mathbb{Q}\mathbb{Z}[\frac{1}{p}] \longrightarrow \bar{\mathbb{F}}_p$ is fixed. For the first three classes, let $\preceq_{\Re}$ be all inclusions of submodels respecting field and vector space structure; these are necessarily elementary. The definition of class $(\mathbb{C}_{\exp})$ is more complicated and may be found elsewhere [Zilber, Pseudoexponentiation]. All four are excellent abstract elementary classes [ibid.,Bays] that have finitely many, up to isomorphism, models of each uncountable cardinality, i.e. for every uncountable $\bar{K}$ there are finitely many short exact sequences up to a linear isomorphism of V inducing a field automorphism on $\bar{K}$.

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8. **Remarks.** These remarks are explained in more details in [Gavrilovich].

8.1. Gromov [Ergosystems] writes that "The category/functor modulated structures can not be directly used by ergosystems, e.g. because the morphisms sets between even moderate objects are usually unlistable. But the ideas of the category theory show that there are certain (often non-obviuos) rules for generating proper concepts." Curiously, in our categories where this obstruction does not arise, all definitions we make seem to be a result of a rather direct and automatic, straightforward repeated application of the lifting property to basic concepts of naive set theory, and the axioms of a model category admit a functional semantics whereby they are interpreted as rules to draw arrows and add labels on labelled graphs. We say more on this in [Gavrilovich], particularly §1.0.4,p.5 and §1.3,pp.12-14.

8.2. Shelah explicitly states his ideology of PCF theory in Shelah (Logical Dreams), e.g. Thesis 5.10, and we find it remarkably similar to the model category ideology as applied to StNaamen. It is unclear whether a deeper connection with PCF theory exists, e.g. whether the sequence of PCF generators is a (non-pointed, non-functorial) analogue of a (co)fibration sequence, or whether $X \longmapsto \{X\}$ and $X \longmapsto \cup_{x\in X} x$ can be usefully viewed as analogues of suspension $X \longmapsto \Sigma X$ and loop $X \longmapsto \Omega X$ spaces, cf. Kojman (A short proof of PCF theorem).

8.3. Manin (A course in logic, 2010, p.174) discusses the Continuum Hypothesis and the possibility for a need to "try to find alternative languages and semantics" for set theory. It would seem that the connection between homotopy theory (in the model category formalism) and set theory (in ZFC or NF, or similar formalisms) we suggest, may provide for such an alternative language and semantics.

8.4. Note that a topological space T determines a homotopy-invariant functor $\mathsf{acc}_T : QtNaamen \longrightarrow Naamen$, $\mathcal{X} \longmapsto \cup_{X\in\mathcal{X}} \mathsf{acc}_T(X \cap T)$ sending a family $\mathcal{X}$ into the set of accumulation points $\cup_{X\in\mathcal{X}} \mathsf{acc}_T(X \cap T)$ of a member of the family; here $Naamen$ is the poset of all sets under inclusion. It appears that the definition of a topological space may be stated purely category-theoretically in terms of this functor and the functor $\{\cdot\} : Naamen \longrightarrow QtNaamen$, $X \longmapsto \{X\}$.

8.5. Our original motivation was to associate a model category (via the class of families of models) to an uncountably categorical theory and, more generally, to an excellent abstract elementary class (Shelah, Classification theory of non-elementary classes). In particular, we wanted to use the language of homotopy theory to perform the model-theoretic analysis of complex exponentiation $(\mathbb{C}, +, *, \exp)$ (Zilber, Pseudo-exponentiation on algebraically closed fields of characteristic zero) and covers of semi-Abelian varieties ([Bays] and references therein). These results claim there exist a unique, up to an appropriate notion of isomorphism (*not* respecting topology), function $\mathrm{e}x : \mathbb{C} \longrightarrow \mathbb{C}$ satisfying $\mathrm{e}x\,(x + y) = \mathrm{e}x\,(x)\mathrm{e}x\,(y)$, the Schanuel conjecture and a dual thereto; Bays replaces $\mathbb{C}$ and $\mathrm{e}x$ by an elliptic curve and its cover $\mathrm{e}x_E : \mathbb{C} \longrightarrow \mathbb{C}/\Lambda$. Their analysis leads to a number- and geometric-theoretic conditions on semi-Abelian varieties (Mumford-Tate, Kummer theory, Mordell-Weil, Schanuel Conjecture); we wanted an analysis covering more general algebraic varieties which would to lead to geometric conditions in place of those above.

9. **Thanks.** I thank my Mother and Father for support, patience and more. I thank Boris Zilber to whose ideas this work owes a large debt. I also thank Artem Harmaty for attention to this work, and encouraging conversations, and Martin Bays for reading and discussing and point out several inaccuracies. The first author would also like to thank the St. Petersburg Steklov Mathematical Institute (PDMI RAS) and the participants of the seminar organised by N. Durov and A. Smirnov for their hospitality and insight into model categories.

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Appendix A. Some examples.

Below we use cwf-notation and the language of model categories to give some examples in our model category $QtNaamen$ and the model category Top of topological spaces. All claims we make are either standard or follow from the definitions and may be found in [Gavrilovich].

9.1. *Homotopy category. Cofibrant and fibrant objects.* Cofibrant objects, i.e. objects X such that $\varnothing \xrightarrow{(c)} X$, are families of countable sets. A family X is fibrant, i.e. $X \xrightarrow{(f)} \{x : x = x\}$, iff for every $x \in X$ and every a finite, the union $\{x \cup a\}$ is also covered by a member of X, in notation $\{x \cup a\} \longrightarrow X$. We ignore non-existence of $\{x : x = x\}$ in ZFC; note that in ZFC fibrant objects are necessarily proper classes. The homotopy category $HoQtNaamen$ is, up to equivalence of categories: (i) $HoQtNaamen$ is the full subcategory of fibrant and cofibrant objects, (ii) $HoQtNaamen$ is the category of families of countable sets, with the arrows: $X \longrightarrow Y$ iff every $x \in X$ is almost covered by an element $y \in Y$, i.e. $X \dashrightarrow \cdot \overset{(wc)}{\dashleftarrow} Y$.

9.2. $StNaamen$ *vs* $QtNaamen$. Put a label (q) on an arrow $A \longrightarrow B$ iff $A \longrightarrow B \rightthreetimes X \longrightarrow Y$ lifts with respect to any arrow $X \longrightarrow Y$ between objects of $QtNaamen$. Then for any $A \in \mathcal{O}bStNaamen$ there exists an $\tilde{A} \in \mathcal{O}bQtNaamen$, unique up to isomorphism, such that $A \xrightarrow{(q)} \tilde{A}$. Diagram chasing using (q)-labels and M6 of StNaamen shows that the category $QtNaamen$ is closed under $M2$-decomposition, i.e. if $A, Y \in \mathcal{O}bQtNaamen$ and $A \xrightarrow{(wc)} B \xrightarrow{(f)} Y$ and $A \xrightarrow{(c)} X \xrightarrow{(wf)} Y$, then $B, X \in \mathcal{O}bQtNaamen$.

9.3. *Singletons.* For sets A and B, $A \subseteq B$ iff there is a (necessarily unique) arrow/morphism $\{A\} \longrightarrow \{B\}$, and $\{A\} \xrightarrow{(wc)} \{B\}$ is an acyclic cofibration iff $B \setminus A$ is finite (and $A \subseteq B$). For sets A and B infinite $\operatorname{card} A = \operatorname{card} B$ iff $\{A\} \xrightarrow{(c)} \{B\}$, and B is countable iff $\varnothing \xrightarrow{(c)} \{B\}$ is a cofibration.

9.4. *Ordinals.* For a ordinal it holds $\alpha \longrightarrow \{\alpha\}$ and $\cup\alpha \longrightarrow \alpha$.
$\{\alpha\} \longrightarrow \alpha + 1 \longrightarrow \{\alpha\}$, i.e. $\{\alpha\}$ and $\alpha + 1$ are isomorphic
$\alpha \xrightarrow{(f)} \alpha + 1$ iff $\alpha = \cup_{\beta<\alpha}\beta$ is limit.
$\alpha \xrightarrow{(wf)} \alpha + 1$ iff α is a regular cardinal, i.e. $\operatorname{cf}\alpha = \alpha$
$\alpha \xrightarrow{(c)} \alpha + 1$ iff $\alpha = \omega$ or α is not a cardinal
$\alpha \xrightarrow{(wc)} \alpha + 1$ iff α is not a limit ordinal, i.e. $\alpha \neq \cup_{\beta<\alpha}\beta$
$\alpha \xrightarrow{(c)} \beta$ iff $\alpha = \beta$ or α is not a cardinal and
either $\operatorname{card}\beta \leqslant \operatorname{card}\alpha + \aleph_0$ or β is a cardinal and $\operatorname{card}\beta \leqslant (\operatorname{card}\alpha + \aleph_0)^+$.
$\alpha \xrightarrow{(wc)} \beta$ iff $\beta < \alpha + \omega$ and α not a limit ordinal
$\alpha \in ObQtNaamen$ iff $\operatorname{cf}\alpha = \omega$ or $\operatorname{cf}\alpha = \alpha$

9.5. *Fibrations. Increasing chains. Paths.* Take a set M and represent M as a union of a continuous increasing chain $M = \cup_{i<\lambda} M_i$; then $\{M_i\}_{i<\lambda} \xrightarrow{(f)} \{M\}$ is a fibration. Let $M^{<\lambda}$ be the set of all subsets of M of cardinality strictly less than λ, then $M^{<\lambda} \xrightarrow{(f)} \{M\}$. If $\lambda > \omega$, then $M^{<\lambda} \xrightarrow{(wf)} \{M\}$ and $\{M_i\}_{i<\lambda} \xrightarrow{(wf)} \{M\}$ provided $\operatorname{card} M_i < \operatorname{card} M$ and $\operatorname{card} M = \operatorname{cf} \operatorname{card} M$ is regular.

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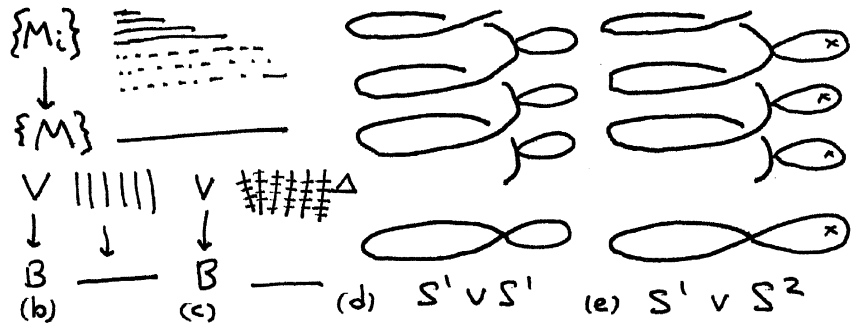

Fig. 1. Fibrations in $StNaamen$ and Top. (a) a union of increasing chain $\{M_i\}_{i<\lambda} \xrightarrow{(f)} \{M\}$ (b) a schematic picture of a fibration $V \xrightarrow{(f)} B$ of topological spaces; the vertical lines denote fibres $f^{-1}(b)$. (c) a schematic picture of a fibration $V \xrightarrow{(f)} B$ of topological spaces with a homotopy connexion Δ, a rule for lifting uniquely paths $\gamma : [0,1] \longrightarrow B$ in B to $\tilde{\gamma} : [0,1] \longrightarrow V$, $p(\tilde{\gamma}(t) = \gamma(t)$ from an arbitrary point y_0, $\tilde{\gamma}(0) = y_0$. the horizontal lines represent paths so obtained. (d) a fibration whose base is a bouquet $S^1 \vee S^1$ of two circles (e) a fibration whose base is a bouquet $S^1 \vee S^2$ of a circle and a sphere.

9.6. *Cofibrations. Countable sets. Closed inclusions. Simplices.* We have $\varnothing \xrightarrow{(wc)} X$ is an acyclic cofibration iff X is a family of finite sets, and $\varnothing \xrightarrow{(c)} X$ is a cofibration iff X is a family of countable sets. An arrow $A \xrightarrow{(c)} B$ is a cofibration iff for every $b \in B$ there exists finitely many $b_0, ..., b_n = b \in B, n \in \omega$ and $a_0 \in A$ such that $\operatorname{card} b \leqslant \operatorname{card}(a_0 \cap b_0)$ and $\operatorname{card} b \leqslant \operatorname{card}(b_i \cap b_{i+1})$ for every $0 \leqslant i \leqslant n-1$. An arrow $A \xrightarrow{(wc)} B$ is an acyclic cofibration iff every $b \in B$ is almost a subset of an $a \in A$ (and of course $A \longrightarrow B$).

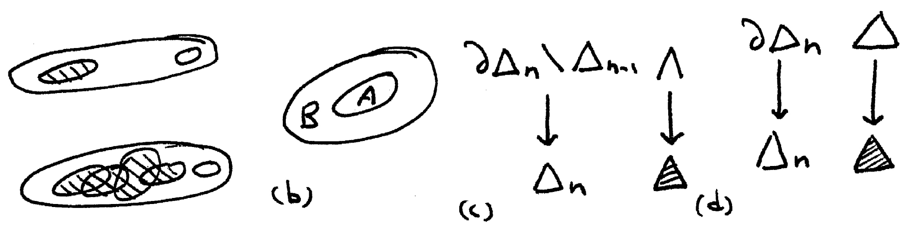

Fig. 2. Cofibrations in $StNaamen$ and Top. (a) a cofibration in $StNaamen$ (b) a picture of a cofibration $V \xrightarrow{(c)} B$ of topological spaces in Hurewicz model structure; this is just a closed inclusion. (c) a generating acyclic cofibration in Quillen model structure on Top: inclusion of the boundary without a face of a simplex into the simplex (d) a generating cofibration in Quillen model structure on Top: inclusion of the boundary of a simplex into the simplex

9.7. *Axiom M2. Path and cylinder spaces.* Let $A \longrightarrow Y$ be a morphism in $StNaamen$. The M2-decomposition can be explicitly given as follows:

$$A \xrightarrow{(wc)} \{\, (a \cup y_{fini}) \cap y \ :\ a \in A,\ y \in Y,\ y_{fini} \text{ finite} \,\} \xrightarrow{(f)} Y$$

and

$$A \xrightarrow{(c)} \{\, y :\ a_0 \in A,\ n \in \omega,\ y \subseteq y_n,\ y_0, ..., y_n \in Y,\ \operatorname{card} y \leqslant \operatorname{card}(a_0 \cap y_0) + \aleph_0 \text{ and } \operatorname{card} y \leqslant \operatorname{card}(y_i \cap y_{i+1}) + \aleph_0 \text{ for every } 0 \leqslant i \leqslant n-1 \,\} \xrightarrow{(wf)} Y$$

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In Top, for sufficiently nice topological spaces A and Y, there are the following decompositions of a map $g : A \longrightarrow Y$:

$$A \xrightarrow{(wc)} \{ (a, \gamma) \; : \; a \in A, \; \gamma : [t_0, t_1] \longrightarrow Y, \; \gamma(t_0) = g(a) \} \xrightarrow{(f)} Y$$

$$A \xrightarrow{(c)} A \times [0,1] \cup Y/(a,0) \approx g(a) \xrightarrow{(wf)} Y$$

The maps involved are: in the wc-f-decomposition, a point $a \in A$ goes into the pair $(a, \gamma_{g(a)})$ where γ is the constant path at point $g(a)$, $t_0 = t_1 = 0$; a pair (a, γ) goes into $\gamma(t_2)$; and in the c-wf-decomposition, a point $a \in A$ goes into $(a, 1)$; and a point $(a, t) \in A \times [0, 1]$ goes into $g(a)$, and $y \in Y$ goes into itself.

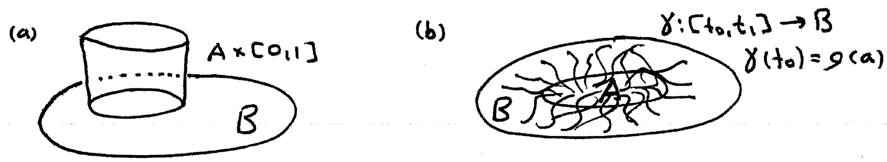

Fig. 3. M2-decompositions. Cones, cylinders and paths. (a) A c-wf-decomposition in Top using a cone object. (b) A wc-f-decomposition in Top using a cocone object of paths.

9.8. *Axiom M2. Downward Lowenheim-Skolem theorem as an instance of M2(c-wf).* Downward Lowenheim-Skolem theorem, e.g. for a first-order theory in a countable language, claims that every infinite subset of a model is contained in an elementary submodel of the same cardinality; equivalently, every subset $A \subseteq M$ is contained in an elementary submodel $A \subseteq M' \preceq M$ of M of cardinality card $M' =$ card $A + \aleph_0$. In our notation this is

$$\{A\} \xrightarrow{(c)} \{M' \; : \; M' \preceq M\} \xrightarrow{(wf)} \{M\}$$

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Appendix B. Examples of lifting properties.

We give examples of some widely used notions that can be defined by a lifting property. Arguably, it is useful to think of these definitions as follows: we take a counterexample and "forbid" it by requiring the lifting property with respect to to it. The following example may make this more clear. Assume we are interested in counting something, and we realise that to hope to preserve the count we need to avoid the two simplest possible(?) operations: adding a point $\{\} \longrightarrow \{\cdot\}$ to nothing or gluing two points into one $\{\cdot,\cdot\} \longrightarrow \{\cdot\}$. However, avoiding just these two is not enough: what we want is a class of operations(morphisms) which have nothing to do with these two bad ones. And we define such a class by requiring the left lifting property (Fig.4(b-c)). This gives us the class of bijections, i.e. exactly the operations that preserve the count.

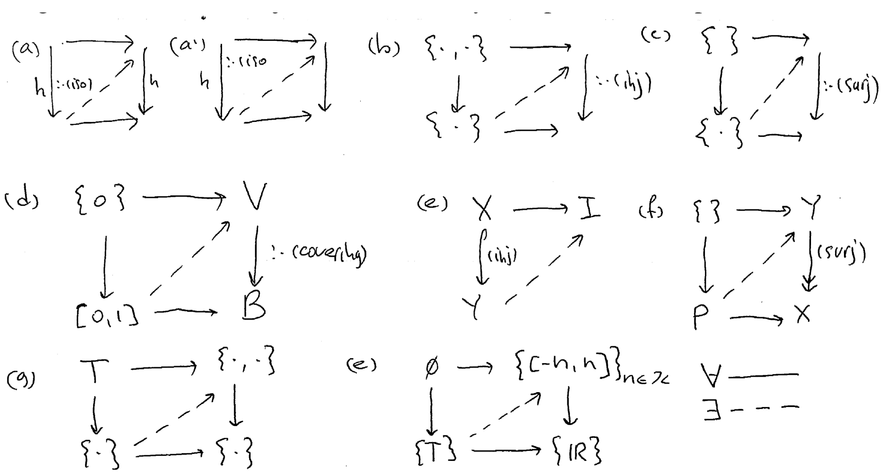

Fig.4. Read a diagramme as an $\forall\exists$-formula with parameters: for the arrows labelled ► or ◄, the following property holds: "for each commutative diagramme of solid arrows carrying labels as shown, there exists dashed arrows carrying labels as shown, making the diagramme of all the arrows commutative" (a) Isomorphism. In a category an arrow is an isomorphism iff it has (either left or right) lifting property with respect to itself (and consequently (a') any other arrow). (b) an arrow is injective iff it has the right lifting property with respect to $\{\cdot,\cdot\}\longrightarrow\{\cdot\}$ whenever(=in most categories where) the latter notation/arrow makes sense. (c) an arrow is surjective iff it has left lifting property with respect to $\{\}\longrightarrow\{\cdot\}$ whenever(=in most categories where) the latter notation/arrow makes sense. (d) Let $I = [0,1]$ be the unit interval of the real line, and let $0 \in [0,1]$ be its end point; the morphism $V\longrightarrow B$ is a *covering* of topological spaces iff there is always exists a unique lifting arrow $I\longrightarrow V$ making the diagramme commute. (e) an object I is injective iff for each injective arrow $X\longrightarrow Y$ and any arrow $X\longrightarrow I$, there exists an arrow $Y\longrightarrow I$. (f) dually, an object P is a projective object, e.g. a free module, iff for each surjective arrow $X \longleftarrow Y$ and an arrow $X \longleftarrow P$, there exists an arrow $Y \longleftarrow P$. (g) a topological space T is connected iff $T\longrightarrow\{\cdot\}$ has the right lifting property with respect to to $\{\cdot,\cdot\}\longrightarrow\{\cdot\}$ in the category of topological spaces) (e) a topological space T is compact iff every continuous map $T\longrightarrow\mathbb{R}$ factors via an interval $[-n,n]$ for some $n \in \mathbb{Z}$.

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Appendix C. Some Open Questions

A number of questions appeared in the course of our work, and list them below mostly for our own record. One would not expect such a list to appear in a published paper, and it may well be best for the reader to ignore it.

9.9. **Variations. Similar model categories**. Find a model category interpretaiton for other covering numbers.

Call a set A is *closed under homotopy countable unions* iff for every countable family of sets $a_1, a_2, ... \in A$ there exists $a \in A$ such that each a_i is contained in *a up to finitely many elements*.

(0) Check that this notion is homotopy-invariant, i.e. if (X and Y are objects of QtNaamen and) $X \xrightarrow{(w)} Y$ and either of X or Y is closed under homotopy countable unions, then both X and Y are closed under homotopy countable unions.

(1) prove that objects of QtNaamen closed under homotopy countable unions, form a model category.

(2) Calculate, in the subcategory,

$$\mathbb{L}_c\text{card}\,(\{\aleph_1\}\longrightarrow\{\kappa\}) := \min\{\text{card}\,X' : \{\aleph_1\} \xrightarrow{(c)} X' \xleftarrow{(w)} X_1 \xrightarrow{(w)} X_2 \xleftarrow{(w)} ... \xrightarrow{(w)} X_n \xleftarrow{(w)} X\} \qquad (***)$$

(Conjecture: this is $cov(\kappa, \aleph_1, \aleph_1, \omega)$)

9.10. **Question 1(i)**. Find a natural (e.g. in homotopy theory) characterisation/axiomatisation of QtNaamen (or StNaamen), possibly adding more structure, e.g. that of infinity-category. For example, characterise/axiomatise QtNaamen as (a) a labelled category up to isomor- phism, or (b) as a model category, e.g. up to Quillens equivalence of model categories, or perhaps (c) offer an interesting and relevant notion of equivalence.

9.11. **Question 1(ii).** Rewrite first-order axioms of (a large fragment) of ZFC in terms of arrows, lifting properties, commutative diagrammes in StNaamen, or, better, QtNaamen, preferably in the spirit of homotopy theory, e.g. Quillens model category book. If necessary, find and add more structure to QtNaamen/StNaamen to axiomatise the whole of ZFC, e.g. something of higher category structure. Does this clarify any issues in ZFC ? Does this reformulation makes ZFC easier to appreciate or use by a non-specialist mathematician? How much is lost?

Question 1(iii). Use methods of set theory, possibly also model theory, to suggest a natural notion extending in some way the notion of a model category. Does it make sense in the context of homotopy theory?

Question 2(i). Use methods of stability/classification/model theory (of mathemat- ical logic) to study the structure on the homotopy category induced by the model category, even if in our rather degenerate setting. We already saw that there is some- what of a non-trivial connection to set theory. As the main topic of interest of both authors is model theory, we cannot resist asking whether methods of model theory can contribute to the study of model categories, e.g. the category QtNaamen. For exam- ple, in the explanatory exposition we said ”axioms of a model category require that the labelling induces no further structure on the homotopy category”. Do this words admit an interpretation that certain structure in stably/conservatively embedded in the labelled category as a structure, here all structures as in the sense of logic?

Question 3. Observe that most of common computational tools of algebraic topol- ogy, e.g. fibration and cofibrations sequences, loop or suspension objects, path spaces, maps spaces degenerate in our setting. Try to enlarge the category by adding either new morphisms or objects. One way to start is to add all injective maps as morphisms, “quantise” to add formal limits, new path and map objects as necessary. And iterate this step countably many times.

Question 4. Elaborate explicitly our hero’s strategy in the context of Gromov’s Ergosystems. Is it really as simple and automatic as our exposition seem to suggest?

Question 4’. Develop better notation so that everything our hero does, becomes a calculation on rather simple marked graphs or surfaces. E.g. use N.Durov’s idea to consider the dual of the commutative diagramme and his observation that (todo: state the observation)

Question 5. Construct a model category whose objects are (some) families of models of an excellent abstract elementary class, e.g. an uncountably categorical first-order theory in a countable language a quasi-minimal excellent class of Zilber. Is the expressive power of the ”homotopy” language of category theories, sufficient to develop the theory or at least state its main results and lemmas ? If not, is it possible to enrich it while keeping the "homotopic" and category-theoretic character of the exposition? Does this allow a an exposition of the theory of AEC or its results easier to a non-specialist?

One way to start is to consider the full subcategory of StNaamen

$$StNaamen(\mathcal{M}) := \{\mathbb{M} : \forall M(M \in \mathbb{M} \implies M \prec \mathcal{M})\}$$

consisting only of families of elementary submodels of a fixed monster model $\mathcal{M}$ of the class we are interested in. Label an arrow (c) or (wf) iff it carries the same label in $StNaamen$. Rest of labelling is already not entirely clear: for most AEC, no arrow but identity in $StNaamen(\mathcal{M})$ may inherit (wc)-label.

Intuition may suggest the following conditions to place on families.

A topologist may imagine every model $M \in \mathbb{M}$ as a *simplex* in a *simplicial set* $\mathbb{M}$ and $\{K\} \prec \{M\}$ as being *faces*, *subsimplices* of simplex $M \in \mathbb{M}$. It may be reasonable to place finiteness restrictions on families $\mathbb{M}$, e.g. requiring $\mathbb{M}$ to be (w-o) *well-founded* there is no strictly decreasing infinite chain $... \prec M_{n+1} \prec M_n \prec ... \prec M_0$ in $\mathbb{M}$ or (Δ-fini)("*that a simplex has finitely many faces*") for every $M \in \mathbb{M}$ there exists finitely many *faces* $M_1, ..., M_n \in \mathbb{M}$, $M_1, ..., M_n \prec M$, $M_1, ..., M_n \neq M$, such that for every $M' \in \mathbb{M}$ either $M \prec M'$ or $M' \cap M \prec M_1$ or ... or $M' \cap M \prec M_n$ (every $M' \cap M$ either is the whole *simplex* M or lies in one of its finitely many *faces* $M_1, ..., M_n$). This condition appears in the definition of a *good system* of the first page of [Shelah, 1973], and in fact was a starting point of this research.

A logician may imagine an inductive construction (or proof of something about) a large model U, and that a family $\mathbb{M}$ is *a stage of induction*, a collection of models already constructed at an (infinite) inductive step, or perhaps the models the inductive hypothesis says something about at a step.

Question 6. A 'functorial' definition of a topological space. Given a compact nice (sequential, Hausdorff, etc) topological space T, consider a pair of functors:

$$\mathsf{acc}_T : StNaam(T) \longrightarrow T$$

$$\{\cdot\}_T : T \longrightarrow StNaamen(T)$$

where $StNaamen(T)$ is the full subcategory of StNaamen, $\mathcal{O}bStNaamen(T) := 2^{2^T}$ with induced structure, for a subset $Z \subseteq T$ $\{U\}_T = \{U\}$, and

$$\mathsf{acc}_T(\mathcal{X}) := \cup_{X \in \mathcal{X}} \{t : t \text{ is an accumulation point of } X \subseteq T\}.$$

Observe that for T compact, a subset $U \subseteq T$ is *open* iff for any $\mathcal{X} \in StNaamen(T)$, any arrow $\mathsf{acc}_T(\mathcal{X}) \longrightarrow U$ "lifts" to an arrow $\mathcal{X} \overset{Ho}{\dashrightarrow} \{U\}_T$ in the homotopy category (equivalently, to $\mathcal{X} \overset{(wc)}{\dashleftarrow} X' \dashrightarrow \{U\}_T$).

Furthermore, it seems that such pairs of functors could be characterised in a reasonable functorial manner by properties like

(i) $\mathcal{X} \xrightarrow{(w)} \mathcal{Y}$ implies $\mathsf{acc}_T(\mathcal{X}) = \mathsf{acc}_T(\mathcal{Y})$

(ii) $\mathsf{acc}_T \mathcal{X} \xrightarrow{(iso)} \varnothing$ implies $\mathcal{X} \xleftarrow{(w)} \varnothing$

(iii) if $\mathsf{acc}_T(\mathcal{X}) \longrightarrow U \longleftarrow V$, then there exists $\mathcal{Y} \in \mathcal{O}bStNaamen(T)$, $\mathcal{Y} \dashrightarrow \mathcal{X}$, $\mathsf{acc}_T(\mathcal{Y}) \dashrightarrow V$

Is this characterisation useful for anything ? Does it give rise to a nice category of topological spaces ? Does it generalise, e.g. if we take $StNaamen(\mathcal{M})$ instead of a topological space T ? Can one nicely define the unit interval $[0, 1]$ in this way ?

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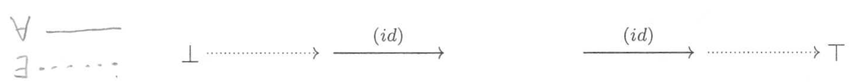
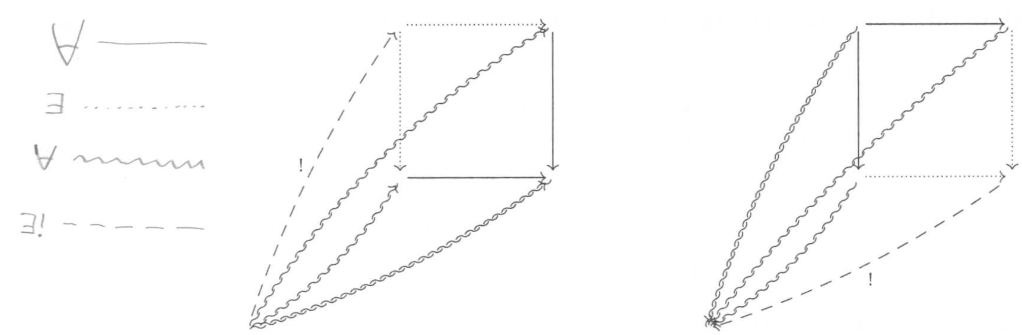
Axioms of a model category in labelled commutative diagrammes notation.

We state the axioms of Quillen of a model category in their original form. In particular, we follow the axiom numeration of Quillen(Homotopical Algebra).

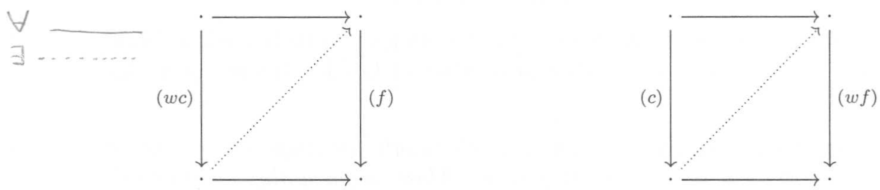
Notation (Commutative diagrammes). *Commutative diagrammes will be used systematically throughout this note. Most importantly, diagrammes will be used to introduce new definitions. We introduce our notation for commutative diagrams. The properties defined are always properties of arrows. To distingish the arrows in the diagrammes which are the object of the definition we will denote them by ◄ or ►. We will mostly use commutative diagrammes to introduce ∀∃-definitions. In such cases solid arrows will be universally quantified and dashed arrows will be existentially quantified. Whenever definitions involving higher quantifier depth (such as in Figure) a legend will be provided. As in Figure 1, we will use the notation* $X \xrightarrow{\therefore(\cdot)} Y$ *to mean "if the commutative diagram is true, then* $X \longrightarrow Y$ *is labeled* $(\cdot)$*". Notation* $X \stackrel{!}{\to} Y$ *indicates uniqueness. A legend on the right might be used to indicate the quantifiers and their order (from top to bottom). Unless stated otherwise, solid arrows are quantified universally, and dotted arrows are quantified existentially.*

Axiom $(M0)$. *The category* $\mathcal{C}$ *is closed under finite projective and injective limits. It is known that it is enough to require existence of initial objects, terminal objects and pullbacks and pushouts.*

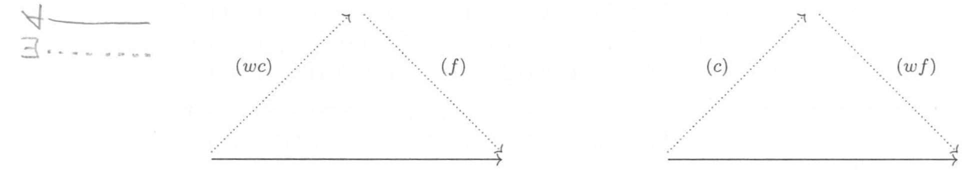




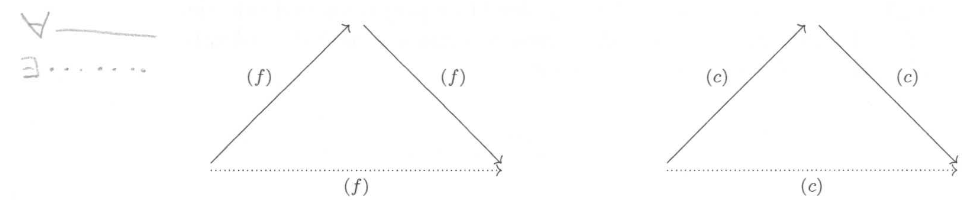
Axiom (M1). *The two following lifting properties for labeled arrows hold:*



Axiom (M2). *The following two ∀∃-diagrams hold:*



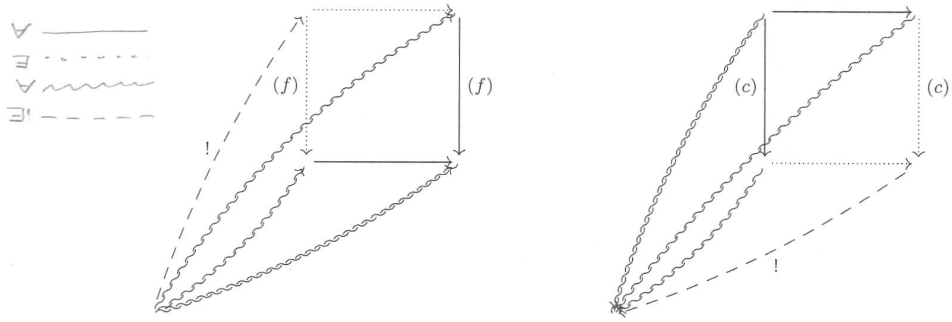
Axiom (M3(ccc,fff)). *Fibrations and cofibrations are stable under compositions. Namely, the following two ∀∃-diagrams hold:*



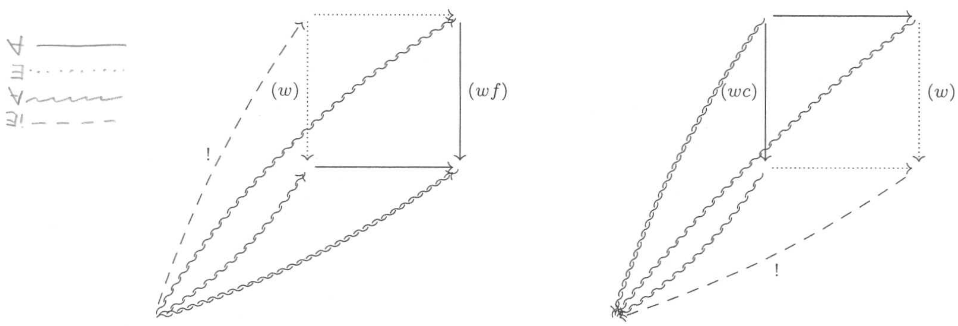
Axiom (M2(cwf)). *Isomorphisms are fibrations, co-fibrations and weak equivalences:*

Figure 1: The figure reads: if the commutative ∀∃-diagramme is true then *the left arrow is labeled (wcf).*

Axiom (M3($f \leftarrow f, c \rightarrow c$)). *Fibrations and cofibrations are stable under base change and co-base change respectively. I.e. the following diagrammes are true:*



Axiom (M4($wf \leftarrow w, wc \rightarrow w$)). *The base extension of an arrow labeled (wc) and the co-base extension of an arrow labeled (wf) are both labeled (w):*



The last axiom assures that weak equivalence is close enough to being transitive:

Axiom (M5, Two out of three). *In a triangluar diagram, if any two of the arrows are labeled (w) so is the third*