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POINT-SET TOPOLOGY AS DIAGRAM CHASING COMPUTATIONS

TO GRIGORI MINTS Z"L IN MEMORIAM

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> 2. Hawk/Goose effect. A baby chick does not have any built-in image of "deadly hawk" in its head but distinguishes frequent, hence, harmless shapes, sliding overhead from potentially dangerous ones that appear rarely. Similarly to "first", "frequent" and "rare" are universal concepts that were not specifically designed by evolution for distinguishing hawks from geese. This kind of universality is what, we believe, turns the hidden wheels of the human thinking machinery.
>
> Misha Gromov, *Math Currents in the Brain.*

Abstract

We observe that some natural mathematical definitions are lifting properties relative to simplest counterexamples, namely the definitions of surjectivity and injectivity of maps, as well as of being connected, separation axioms T_0 and T_1 in topology, having dense image, induced (pullback) topology, and every real-valued function being bounded (on a connected domain); abelian groups, perfect groups, and finite groups of order prime to p.

We also offer a couple of brief speculations on cognitive and AI aspects of this observation, particularly that in point-set topology some arguments read as diagram chasing computations with finite preorders.

1. *Introduction. Structure of the Paper*

An earlier version of this note was written for *The De Morgan Gazette* to demonstrate that some natural definitions are lifting properties relative to the simplest counterexample, and to suggest a way to "extract" these lifting properties from the text of the usual definitions and proofs. The exposition is in the form of a story and aims to be self-contained and accessible to a first year student who has taken some first lectures in naive set theory, topology, and who has heard a definition of a category. A more sophisticated reader may find it more illuminating to recover our formulations herself from reading either the abstract, or the abstract and the opening sentence of the next

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two sections. The displayed formulae and Figure 1(a) defining the lifting property provide complete formulations of our theorems to such a reader.

Let us stress an observation that rules to *erase arrows* from a diagram chasing computation might also be useful: they let us to more faithfully transcribe words "without loss of generality assume ..." and thus avoid unnecessary "doubling" of vertices and arrows. In particular, sequential compactness can be viewed as a lifting property followed by a rule to erase arrows.

Our approach naturally leads to a more general observation that in basic point-set topology, a number of arguments are computations based on symbolic diagram chasing with finite preorders; because of lack of space, we discuss it in a separate note [G0].

2. *Surjection and injection*

We try to find some "algebraic" notation to (re)write the *text* of the definitions of surjectivity and injectivity of a function, as found in any standard textbook. We want something very straightforward and syntactic—notation for what we (actually) say, for the text we write, and not for its meaning, for who knows what meaning is anyway?

(*)$_{\textbf{words}}$ "A function f from X to Y is *surjective* iff for every element y of Y there is an element x of X such that $f(x) = y$."

A function from X to Y is an arrow $X \longrightarrow Y$. Grothendieck taught us that a point, say "x of X", is (better viewed as) as $\{\bullet\}$-valued point, that is an arrow

$$\{\bullet\} \longrightarrow X$$

from a (the?) set with a unique element; similarly "y of Y" we denote by an arrow

$$\{\bullet\} \longrightarrow Y.$$

Finally, make dashed the arrows required to "exist". We get the diagram Fig. 1(b) without the upper left corner; there "$\{\}$" denotes the empty set with no elements listed inside of the brackets.

(**)$_{\textbf{words}}$ "A function f from X to Y is *injective* iff no pair of different points of X is sent to the same point of Y."

"A function f from X to Y" is an arrow $X \longrightarrow Y$. "A pair of points" is a $\{\bullet, \bullet\}$-valued point, that is an arrow

$$\{\bullet, \bullet\} \longrightarrow X$$

from a two element set; we ignore "different" for now. "the same point of Y" is an arrow $\{\bullet\} \longrightarrow Y$. Represent "sent to" by an arrow

$$\{\bullet, \bullet\} \longrightarrow \{\bullet\}.$$

What about "different"? If the points are not "different", then they are "the same" point of X, and thus we need to add an arrow representing a single point of X, that is an arrow

$$\{\bullet\} \longrightarrow X.$$

Now all these arrows combine nicely into diagram Figure 1(c); however, our analysis does not necessarily makes it clear that the diagonal arrow needs to be denoted differently. How do we read it? We want this diagram to have the meaning of the sentence $(**)_{\text{words}}$ above, so we interpret such diagrams as follows:

($\rightthreetimes$) "for every commutative square (of solid arrows) as shown there is a diagonal (dashed) arrow making the total diagram commutative" (see Fig. 1(a)).

(recall that "commutative" in category theory means that the composition of the arrows along a directed path depends only on the endpoints of the path)

Property ($\rightthreetimes$) has a name and is in fact quite well-known [Qui]. It is called *the lifting property*, or sometimes *orthogonality of morphisms*, and is viewed as the property of the two downward arrows; we denote it by $\rightthreetimes$.

Now we rewrite $(*)_{\text{words}}$ and $(**)_{\text{words}}$ as:

$$(*)_{\rightthreetimes} \qquad \{\} \longrightarrow \{\bullet\} \rightthreetimes X \longrightarrow Y$$

$$(**)_{\rightthreetimes} \qquad \{\bullet, \bullet\} \longrightarrow \{\bullet\} \rightthreetimes X \longrightarrow Y$$

So we rewrote these definitions without any words at all. Our benefits? The usual little miracles happen:

The notation makes apparent a similarity of $(*)_{\text{words}}$ and $(**)_{\text{words}}$: they are obtained, in the same purely formal way, from the two of the simplest arrows (maps, morphisms) in the category of Sets. More is true: it is also apparent that these two arrows are the simplest

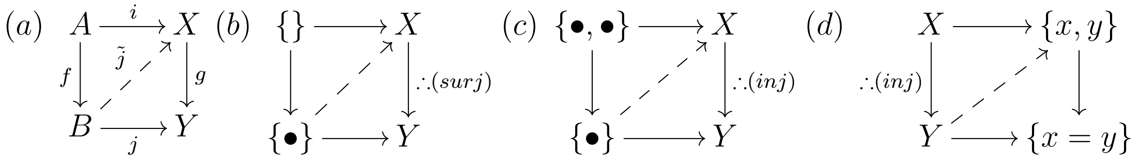

FIGURE 1: *Lifting properties. Dots* $\therefore$ *indicate free variables and what property of these variables is being defined; in a diagram chasing calculation, "*$\therefore$ $(surj)$*" reads as: given a (valid) diagram, add label* $(surj)$ *to the corresponding arrow.*
(a) The definition of a lifting property $f \rightthreetimes g$*: for each* $i : A \longrightarrow X$ *and* $j : B \longrightarrow Y$ *making the square commutative, i.e.* $f \circ j = i \circ g$*, there is a diagonal arrow* $\tilde{j} : B \longrightarrow X$ *making the total diagram* $A \xrightarrow{f} B \xrightarrow{\tilde{j}} X \xrightarrow{g} Y, A \xrightarrow{i} X, B \xrightarrow{j} Y$ *commutative, i.e.* $f \circ \tilde{j} = i$ *and* $\tilde{j} \circ g = j$*. (b)* $X \longrightarrow Y$ *is surjective*
(c) $X \longrightarrow Y$ *is injective;* $X \longrightarrow Y$ *is an epicmorphism if we forget that* $\{\bullet\}$ *denotes a singleton (rather than an arbitrary object and thus* $\{\bullet, \bullet\} \longrightarrow \{\bullet\}$ *denotes an arbitrary morphism* $Z \sqcup Z \xrightarrow{(id,id)} Z$*)*
(d) $X \longrightarrow Y$ *is injective, in the category of Sets;* $\pi_0(X) \longrightarrow \pi_0(Y)$ *is injective, when the diagram is interpreted in the category of topological spaces.*

counterexamples to the properties, and this suggests that we think of the lifting property as a category-theoretic (substitute for) negation. Note also that a non-trivial (one which is not an non-isomorphism) morphism never has the lifting property relative to itself, which fits with this interpretation.

Now that we have a formal notation and the little observation above, we start to play around looking at simple arrows in various categories, and also at not-so-simple arrows representing standard counterexamples.

You notice a few words from your first course on topology: (i) *connected,* (ii) *the separation axioms* T_0 *and* T_1*,* (iii) *dense,* (iv) *induced (pullback) topology*, and (v) *Hausdorff* are, respectively,

(i):

$$X \longrightarrow \{\bullet\} \rightthreetimes \{\bullet, \bullet\} \longrightarrow \{\bullet\}$$

(ii):

$$\{\bullet \gtreqless \star\} \longrightarrow \{\bullet\} \rightthreetimes X \longrightarrow \{\bullet\}$$

and

$$\{\bullet < \star\} \longrightarrow \{\bullet\} \rightthreetimes X \longrightarrow \{\bullet\}$$

(iii):

$$X \longrightarrow Y \rightthreetimes \{\bullet\} \longrightarrow \{\bullet \rightarrow \star\}$$

(iv):

$$X \longrightarrow Y \rightthreetimes \{\bullet < \star\} \longrightarrow \{\bullet\}$$

(v):

$$\{\bullet, \bullet'\} \hookrightarrow X \rightthreetimes \{\bullet > \star < \bullet'\} \longrightarrow \{\bullet\}$$

See the last two pages for illustrations how to read and draw on the blackboard these lifting properties in topology; here

$$\{\bullet < \star\},\ \{\bullet \gtrless \star\},\ \ldots$$

denote finite preorders, or, equivalently, finite categories with at most one arrow between any two objects, or finite topological spaces on their elements or objects, where a subset is closed iff it is downward closed (that is, together with each element, it contains all the smaller elements). Thus

$$\{\bullet < \star\},\ \{\bullet \gtrless \star\} \text{ and } \{\bullet > \star < \bullet'\} \longrightarrow \{\bullet\}$$

denote the connected spaces with only one open point $\bullet$, with no open points, and with two open points $\bullet, \bullet'$ and a closed point $\star$. Line (v) is to be interpreted somewhat differently: we consider *all* the injective arrows of form

$$\{\bullet, \bullet'\} \hookrightarrow X.$$

We mentioned that the lifting property can be seen as a kind of negation. Confusingly, there are *two* negations, depending on whether the morphism appears on the left or right side of the square, that are quite different: for example, both the pullback topology and the separation axiom T_1 are negations of the same morphism, and the same goes for injectivity and injectivity on π_0 (see Figure 1(c,d)).

Now consider the standard example of something non-compact: the open covering

$$\mathbb{R} = \bigcup_{n\in\mathbb{N}} \{\, x \,:\, -n < x < n \,\}$$

of the real line by infinitely many increasing intervals. A related arrow in the category of topological spaces is

$$\bigsqcup_{n\in\mathbb{N}} \{\, x \,:\, -n < x < n\} \longrightarrow \mathbb{R}.$$

Does the lifting property relative to that arrow define compactness? Not quite, but almost:

$$\{\} \longrightarrow X \rightthreetimes \bigsqcup_{n\in\mathbb{N}} \{x \,:\, -n < x < n\,\} \longrightarrow \mathbb{R}$$

reads, for X connected, as "Every continuous real-valued function on X is bounded, i.e. for each continuous $f : X \longrightarrow \mathbb{R}$ there is a natural number $n \in \mathbb{N}$ such that $-n < f(x) < n$ for each $x \in X$", which is an early characterisation of compactness taught in a first course on analysis. Notice that this characterisation mentions explicitly the arrow $X \longrightarrow \mathbb{R}$ and the bounded intervals of the real line, i.e. arrows $\{x : -n < x < n\} \xrightarrow{\subseteq} \mathbb{R}$, $n \in \mathbb{N}$ constituting the arrow-counterexample on the right hand side.

In a category of metric spaces with say distance non-increasing maps, a metric space X is *complete*, i.e. each Cauchy sequence $x_n \in X$, $n \in \mathbb{N}$, say $dist(x_n, x_m) \leq 1/n$, converges to some point $x_\infty \in X$ such that $dist(x_\infty, x_n) \leq 1/n$, iff

$$\{\text{“}x_n\text{”} : n \in \mathbb{N}\} \longrightarrow \{\text{“}x_n\text{”} : n \in \mathbb{N}\} \cup \{\text{“}x_\infty\text{”}\} \rightthreetimes X \longrightarrow \{\bullet\}$$

(where $dist(\text{“}x_n\text{”}, \text{“}x_m\text{”}) = \frac{1}{n}$ for $m > n$, $dist(\text{“}x_\infty\text{”}, \text{“}x_n\text{”}) = \frac{1}{n}$, as defined above.)

In functional analysis, a (partially defined!) linear operator $f : X \longrightarrow Y$ between Banach spaces X and Y is *closed* iff for every convergent sequence $x_n \in X$, if $f(x_n) \xrightarrow[n\longrightarrow\infty]{} y$ in Y, then there is a $x \in X$ such that $f(x) = y$ and $x_n \xrightarrow[n\longrightarrow\infty]{} x$, i.e.

$$\{\text{“}x_n\text{”} : n \in \mathbb{N}\} \longrightarrow \{\text{“}x_n\text{”} : n \in \mathbb{N}\} \cup \{\text{“}x_\infty\text{”}\} \rightthreetimes \text{Domain}(f) \longrightarrow Y$$

A module P over a commutative ring R is *projective* iff for an arbitrary arrow $N \longrightarrow M$ in the category of R-modules it holds

$$0 \longrightarrow R \rightthreetimes N \longrightarrow M \implies 0 \longrightarrow P \rightthreetimes N \longrightarrow M.$$

Dually, a module I over a ring R is *injective* iff for an arbitrary arrow $N \longrightarrow M$ in the category of R-modules it holds

$$R \longrightarrow 0 \rightthreetimes N \longrightarrow M \implies N \longrightarrow M \rightthreetimes I \longrightarrow 0.$$

Finite groups. There are examples outside of topology; let us give some examples in group theory, cf. Figure 2. There is no non-trivial homomorphism from a group F to G, write $F \not\twoheadrightarrow G$, iff

$$0 \longrightarrow F \rightthreetimes 0 \longrightarrow G \text{ or equivalently } F \longrightarrow 0 \rightthreetimes G \longrightarrow 0.$$

A group A is *Abelian* iff

$$\langle a, b\rangle \longrightarrow \langle a, b : ab = ba\rangle \rightthreetimes A \longrightarrow 0$$

where $\langle a, b\rangle \longrightarrow \langle a, b : ab = ba\rangle$ is the abelianisation morphism sending the free group into the Abelian free group on two generators; a group G is *perfect*, $G = [G, G]$, iff $G \not\twoheadrightarrow A$ for any Abelian group A, i.e.

$$\langle a, b\rangle \longrightarrow \langle a, b : ab = ba\rangle \rightthreetimes A \longrightarrow 0 \implies G \longrightarrow 0 \rightthreetimes A \longrightarrow 0$$

in the category of finite or algebraic groups, a group H is *soluble* iff $G \not\twoheadrightarrow H$ for each perfect group G, i.e.

$$0 \longrightarrow G \ \rightthreetimes\ 0 \longrightarrow H \text{ or equivalently } C \longrightarrow 0 \rightthreetimes H \longrightarrow 0.$$

A prime number p does not divide the number elements of a finite group G iff G has no element of order p, i.e. no element $x \in G$ such that $x^p = 1_G$ yet $x^1 \neq 1_G, ..., x^{p-1} \neq 1_G$, equivalently $\mathbb{Z}/p\mathbb{Z} \not\twoheadrightarrow G$, i.e.

$$0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \ \rightthreetimes\ 0 \longrightarrow G \text{ or equivalently } \mathbb{Z}/p\mathbb{Z} \longrightarrow 0 \rightthreetimes G \longrightarrow 0.$$

A finite group G is a p-group, i.e. the number of its elements is a power of a prime number p, iff in the category of finite groups

$$0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \rightthreetimes 0 \longrightarrow H \implies 0 \longrightarrow H \rightthreetimes 0 \longrightarrow G.$$

Sylow theorem implies in a finite group, each p-group is contained in a maximal one, and the maximal p-subgroups are isomorphic.

This can reformulated as: (in the category of finite groups) each arrow $0 \longrightarrow G$ decomposes as $0 \longrightarrow Syl_p(G) \longrightarrow G$ where

$$0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \ \rightthreetimes\ 0 \longrightarrow Syl_p(G)$$

$$0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \ \rightthreetimes\ 0 \longrightarrow H \implies H \longrightarrow Syl_p(G) \rightthreetimes G \longrightarrow 0,$$

Sylow theorem says more: the maximal p-subgroups are in fact conjugated. We only remark that the notion of an inner automorphism can be reformulated in a diagram chasing manner. An inner automorphism $g \longmapsto aga^{-1}$ of a group G extends to an automorphism $h \longmapsto \iota(a)h\iota(a)^{-1}$ of a group H for any embedding $\iota : G \longrightarrow H$. [Inn, Sch] show this is a characterisation: an automorphism $\sigma : G \longrightarrow G$ is inner iff it extends to an automorphism of H for any embedding $\iota : G \longrightarrow H$. See [Inn] and references therein for several more similar reformulations.

Feit-Thomson theorem can be expressed as a combination of lifting properties: the theorem says says that each (finite) group of odd order is soluble, i.e. for each perfect finite group G and each finite group H,

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \rightthreetimes 0 \longrightarrow H \implies 0 \longrightarrow G \rightthreetimes 0 \longrightarrow H.$$

Note that all these examples but the last one have a flavour of negation—a notion being defined by the lifting property with respect to the simplest counterexample.

Monomorphism and epimorphism. A category theorist would rewrite (**)$_\rightthreetimes$ as

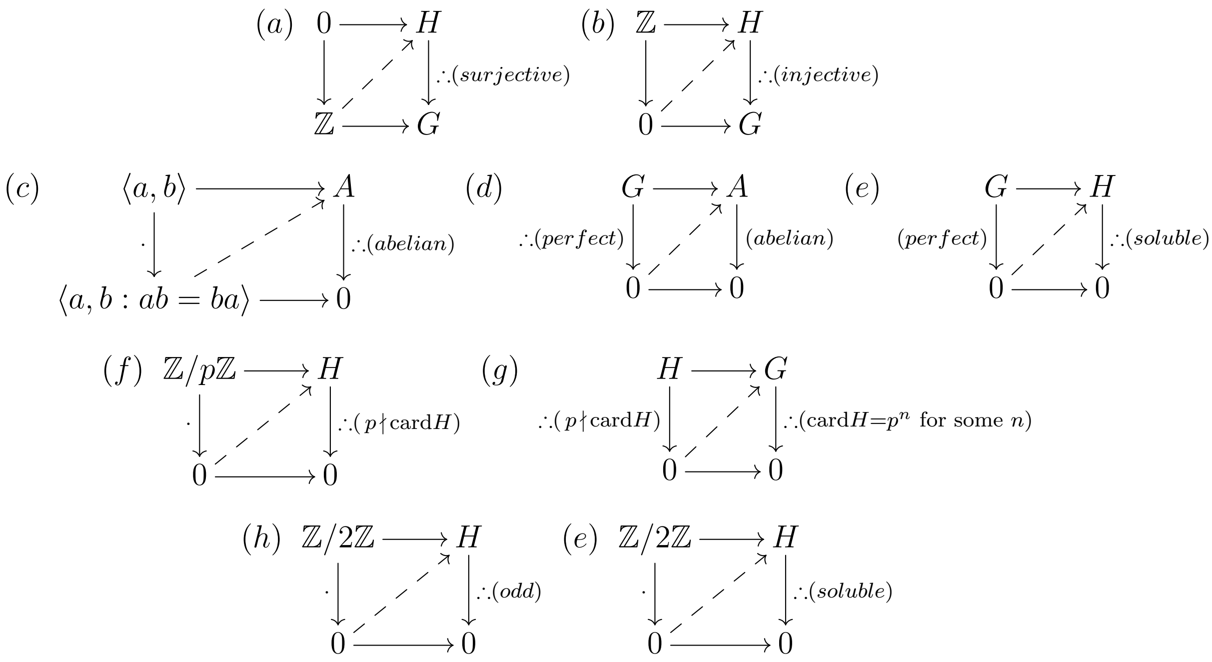

Figure 2: *Lifting properties. Dots $\therefore$ indicate free variables. Recall these diagrams represent rules in a diagram chasing calculation and "$\therefore$ (label)" reads as: given a (valid) diagram, add label (label) to the corresponding arrow. A diagram is valid iff for every commutative square of solid arrows with properties indicated by labels, there is a diagonal (dashed) arrow making the total diagram commutative. A single dot indicates that the morphism is a constant.*

(a) a homomorphism $H \longrightarrow G$ is surjective, i.e. for each $g \in G$ there is $h \in H$ sent to g

(b) a homomorphism $H \longrightarrow G$ is injective, i.e. the kernel of $H \longrightarrow G$ is the trivial group

(c) a group is abelian iff each morphism from the free group of two generators factors through its abelianisation $\mathbb{Z} \times \mathbb{Z}$.

(d) a group G is perfect, $G = [G, G]$, iff it admits no non-trivial homomorphism to an abelian group

(e) a finite group is soluble iff it admits no non-trivial homomorphism from a perfect group; more generally, this is true in any category of groups with a good enough dimension theory.

(f) by Cauchy theorem, a prime p divides the number of elements of a finite group G iff the group contains an element $e, e^p = 1, e \neq 1$ of order p

(f) a group has order p^n for some n iff iff the group contains no element $e, e^l = 1, e \neq 1$ of order l prime to p

(h) by Cauchy theorem, a finite group has an odd number of elements iff it contains no involution $e, e^2 = 1, e \neq 1$

(e) Feit-Thompson theorem says that each group of odd order is soluble, i.e. it says that this diagram chasing rule is valid in the category of finite groups. Note that it is not a definition of the label unlike the other lifting properties.

$$(**)_{\mathbf{mono}} \qquad \bullet\vee\bullet \longrightarrow \bullet \ \rightthreetimes\ X \longrightarrow Y$$

denoting by $\vee$ and $\bullet\vee\bullet \longrightarrow \bullet$ the coproduct and the codiagonal morphism, respectively, and then read it as follows: given two morphisms

$$\bullet \xrightarrow{\text{left}} X \ \text{ and } \ \bullet \xrightarrow{\text{right}} X,$$

if the compositions

$$\bullet \xrightarrow{\text{left}} X \longrightarrow Y \ = \ \bullet \xrightarrow{\text{right}} X \longrightarrow Y$$

are equal (both to $\bullet \xrightarrow{\text{down}} Y$), then

$$\bullet \xrightarrow{\text{left}} X \ = \ \bullet \xrightarrow{\text{right}} X$$

are equal (both to $\bullet \xrightarrow{\text{down}} X$). Naturally her first assumption would be that $\bullet$ denotes an *arbitrary* object, as that spares the extra effort needed to invent the axioms particular to the category of sets (or topological spaces) that capture that $\bullet$ denotes a single element, i.e. allow one to treat $\bullet$ as a single element. (A logician understands "arbitrary" as "we do not know", "make no assumptions", and that is how formal derivation systems treat "arbitrary" objects.) Thus she would read $(**)_{\rightthreetimes}$ as the usual category theoretic definition of a monomorphism. Note this reading doesn't need that the underlying category has coproducts: a category theorist would think of working inside a larger category with formally added coproducts $\bullet \vee \bullet$, and a logician would think of working inside a formal derivation system where "$\bullet$" is either a built-in or "a new variable" symbol, and "$\bullet \vee \bullet \longrightarrow \bullet$" (or "$\{\bullet,\bullet\} \longrightarrow \{\bullet\}$") is (part of) a well-formed term or formula.

And of course, nothing prevents a category theorist to make a dual diagram

$$(**)_{\mathbf{epi}} \qquad X \longrightarrow Y \ \rightthreetimes\ \bullet \longrightarrow \bullet\times\bullet, \qquad \bullet \text{ runs through all the objects}$$

and read it as:

$$X \longrightarrow Y \xrightarrow{\text{left}} \bullet \ = \ X \longrightarrow Y \xrightarrow{\text{right}} \bullet \text{ implies } Y \xrightarrow{\text{left}} \bullet \ = \ Y \xrightarrow{\text{right}} \bullet$$

which is the definition of an epimorphism.

3. *Sequential compactness. Erasing arrows as a diagram chasing rule.*

Here we argue that rules to *erase arrows* may be useful in diagram chasing on the example of sequential compactness, and that "an arrow

$X \longrightarrow Y$ factors as $X \dashrightarrow X' \dashrightarrow Y$ via an arrow $X \dashrightarrow X'$ with a certain property" can be expressed as a lifting property combined with a rule to erase arrows from the diagram chasing computation. We end with a syntactic analysis of a definition of sequential compactness.

Sequential compactness. Now let us translate to diagram chasing rules the definition of sequential compactness.

One-point compactification $\mathbb{N} \longrightarrow \mathbb{N} \cup \{\infty\}$ of the natural numbers is a standard example of a map from a non-compact space to a compact space; it is therefore tempting to check whether the lifting property

$$(***)_{\rightthreetimes} \qquad \mathbb{N} \longrightarrow \mathbb{N} \cup \{\infty\} \rightthreetimes X \longrightarrow \{\bullet\}$$

defines something like (sequential) compactness. It does not and is in fact trivial — it says that in X each sequence $x_n \in X$, n a natural number, converges. Figure 3(a) shows a related diagram which does define sequential compactness. However, in Figure 3(a) the morphism $\mathbb{N} \longrightarrow \mathbb{N} \cup \{\infty\}$ appears twice in the diagram and this duplication is often unnecessary: often we have little reason to talk about the sequence x_n's once we have chosen a subsequence of it. That is, having applied the diagram chasing rule Figure 3(a), it is often desirable to remove unnecessary arrows, perhaps after some immediate simplifications. This is represented by Figure 3(b). To derive Figure 3(a) from Figure 3(b), first apply Figure 3(b) to the inner arrow in the diagram Figure 3(c). This shows that rules represented by Figure 3(a) and Figure 3(b) are equivalent.

Finally, Figure 3(e-f) suggests a notation for the diagram chasing rule represented by Figure 3(b). This notation makes connection to the lifting property more apparent and empahasises deleting arrows as a diagram chasing rule.

Further reducing to diagram chasing with finite spaces We may reduce the definition of sequential compactness to diagram chasing with finite topological spaces. It is based on the following lifting properties:

$\mathbb{N} \longrightarrow \mathbb{N} \cup \{\infty\}$ is injective:

$$\{0,1\} \longrightarrow \{0=1\} \rightthreetimes \mathbb{N} \longrightarrow \mathbb{N} \cup \{\infty\}$$

$\mathbb{N}$ is dense in $\mathbb{N} \cup \{\infty\}$:

$$\{1\} \longrightarrow \{0<1\} \rightthreetimes \mathbb{N} \longrightarrow \mathbb{N} \cup \{\infty\}$$

each point of $\mathbb{N} \cup \{\infty\}$ is closed:

$$\{0<1\} \longrightarrow \{0=1\} \rightthreetimes \mathbb{N} \cup \{\infty\} \longrightarrow \{\bullet\}$$

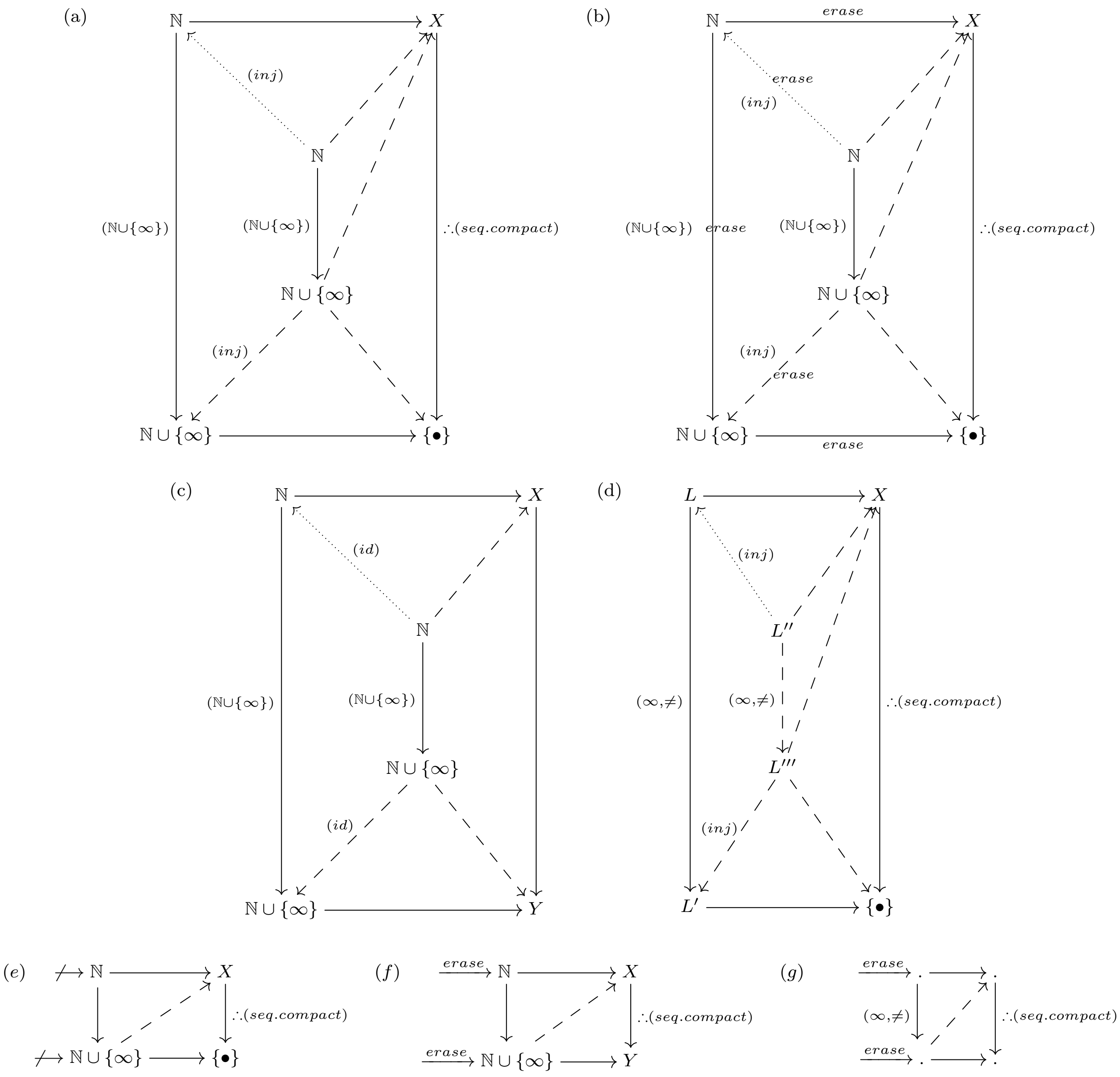

FIGURE 3: *Sequential compactness as a lifting property. Label* $(\mathbb{N}\cup\{\infty\})$ *says that the arrow* $\mathbb{N} \longrightarrow \mathbb{N}\cup\{\infty\}$ *denote the one-point topological compactification of the set of natural numbers with discrete topology. That is, each subset of* $\mathbb{N}$ *is both closed and open, and a subset of* $\mathbb{N}\cup\{\infty\}$ *is closed iff it is either finite or contains* ∞. *In (c) and (d),* $L \xrightarrow{(\infty,\neq)} L'$ *denotes that* L *is dense in* L', *the map is injective and each point of* L' *and* L *is closed, and the map is not an isomorphism. See Figure 3 below for the corresponding lifting properties. Note commutativity of the left rectangle implies the map* $\mathbb{N} \longrightarrow \mathbb{N}\cup\{\infty\}$ *has infinite image.*

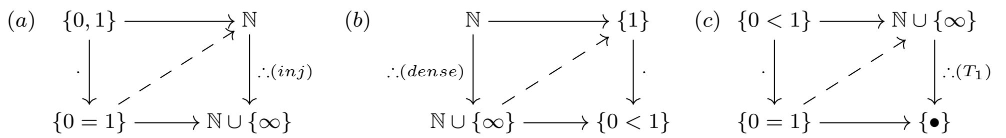

FIGURE 4: *Lifting Properties defining the label* (∞) *above: injecivity, dense image, the target is* T_1*-separated, i.e. each point of the target is closed.*

Write $X \xrightarrow{(\infty)} Y$ to mean these lifting properties hold with respect to $X \longrightarrow Y$, and write $X \xrightarrow{(\infty,\neq)} Y$ to mean $X \xrightarrow{(\infty)} Y$ and the arrow is not an isomorphism. It is easy to see that $X \xrightarrow{(\infty,\neq)} Y$ implies both spaces X and Y are infinite. Figure 4(d) replacing $\mathbb{N} \longrightarrow \mathbb{N} \cup \{\infty\}$ by $X \xrightarrow{(\infty,\neq)} Y$ in Figure 4(b) gives a definition of compactness.

Syntactical analysis of a definition of sequential compactness. Now let us translate to diagram chasing rules the definition of sequential compactness.

(***)$_{\textbf{words}}$ "A space X is *sequentially compact* iff each sequence $x_n \in X$, n a natural number, has an accumulation point, i.e. a point x_∞ such that each open neighbourhood of x_∞ in X contains infinitely many points from the sequence.

"A sequence $x_n \in X$, n a natural number" is an arrow $\mathbb{N} \xrightarrow{x_*} X$; words "each open neighbourhood of x_∞ [..] contains infinitely many points from the sequence" define a property of topology (on anything containing) $\{$"x_n" $: n \in \mathbb{N}\} \cup \{$"$x_\infty$"$\}$ or in another notation $\mathbb{N} \cup \{\infty\}$. Thus the definition above translates to:

(***)$_{\mathbb{N} \longrightarrow \mathbb{N} \cup \{\infty\}}$ each arrow $\mathbb{N} \xrightarrow{x.} X$ factors as

$$\mathbb{N} \longrightarrow \mathbb{N} \cup \{\infty\} \xrightarrow{x.,x_\infty} X$$

via a topological space such that its set of points is $\mathbb{N} \cup \{\infty\}$ and such that the topology on $\mathbb{N} \cup \{\infty\}$ is such that each open neighbourhood of x_∞ contains infinitely many points of $\mathbb{N}$.

Note there is no fixed topology on $\mathbb{N} \cup \{\infty\}$; rather, we only know (care) that each open neighbourhood of ∞ is infinite and what restriction it places on continuous maps $\mathbb{N} \cup \{\infty\} \longrightarrow X$ from $\mathbb{N} \cup \{\infty\}$; this condition says little if anything about arrows to $\mathbb{N} \cup \{\infty\}$.

Note $\mathbb{N} \xrightarrow{(\infty)} \mathbb{N} \cup \{\infty\}$ carries label (∞) implies the topology on $\mathbb{N} \cup \{\infty\}$ has the properties described above.

Above we argued that the lifting property is a meaningful negation; does this apply to this example as well? Arguably yes: $L \xrightarrow{(\infty,\neq)} L'$ is arguably the simplest lifting properties defining a class consisting only of infinite spaces. Arguably, this supports the well-know intuition that being compact is an analogue of being finite for topological spaces.

4. *Speculations.*

Does your brain (or your kitten's) have the lifting property built-in? Note [G0] suggests a broader and more flexible context making contemplating an experiment possible. Namely, some standard arguments in point-set topology are computations with category-theoretic (not always) commutative diagrams of preorders, in the same way that lifting properties define injection and surjection. In that approach, the lifting property is viewed as a rule to add a new arrow, a computational recipe to modify diagrams.

Can one find an experiment to check whether humans *subconsciously* use diagram chasing to reason about topology?

Does it appear implicitly in old original papers and books on point-set topology?

Is diagram chasing with preorders too complex to have evolved? Perhaps; but note the self-similarity: preorders are categories as well, with the property that there is at most one arrow between any two objects; in fact sometimes these categories are thought of as 0-categories. So essentially your computations are in the category of (finite 0-) categories.

Is it universal enough? Diagram chasing and point-set topology, arguably a formalisation of "nearness", is used as a matter of course in many arguments in mathematics.

Finally, isn't it all a bit too obvious? Curiously, in my experience it's a party topic people often get stuck on. If asked, few if any can define a surjective or an injective map without words, by a diagram, or as a lifting property, even if given the opening sentence of the previous section as a hint. No textbooks seem to bother to mention these reformulations (why?). An early version of [GH-I] states $(*)_{\swarrow}$ and $(**)_{\swarrow}$ as the simplest examples of lifting properties we were able to think up; these examples were removed while preparing for publication.

No effort has been made to provide a complete bibliography; the author shall happily include any references suggested by readers in the next version [G].

Acknowledgments and historical remarks

It seems embarrassing to thank anyone for ideas so trivial, and we do that in the form of historical remarks. Ideas here have greatly influenced by extensive discussions with Grigori Mints, Martin Bays, and, later, with Alexander Luzgarev and Vladimir Sosnilo. At an early stage Ksenia Kuznetsova helped to realise an earlier reformulation of compactness was inadequate and that labels on arrows are necessary to formalise topological arguments. "A category theorist [that rewrote] (**) as" the usual category theoretic definition of a monomorphism, is Vladimir Sosnilo. Exposition has been polished in the numerous conversations with students at St. Peterburg and Yaroslavl'2014 summer school.

The discussion of sequential compactness and removing arrows in a diagram chasing computation was added post publication in Dec 2014. I thank Fedor Petrov who asked me to formulate sequential compactness as a lifting property; I thank M.Bays, M.Dubashinsky, A.Luzgarev, S.Kryzhevich, K.Pimenov, S.Sinchuk and V.Sosnilo for subsequent discussions. A discussion with S.Kryzhevich motivated examples in finite group theory. M.Dubashinski suggested the notion of a closed operator in Banach spaces. I would like to add that initially I used an unconventional definition "each infinite set has an accumulation point", as was stressed several times by Fedor Petrov, and which lead to a more cumbersome lifting property. I thank Paul Schupp for pointing out the result of [Sch].

Reformulations $(*)_{\rightthreetimes}$ and $(**)_{\rightthreetimes}$ of surjectivity and injectivity, as well as connectedness and (not quite) compactness, appeared in early drafts of a paper [GH-I] with Assaf Hasson as trivial and somewhat curious examples of a lifting property but were removed during preparation for publication. After $(**)_{\rightthreetimes}$ came up in a conversation with Misha Gromov the author decided to try to think seriously about such lifting properties, and in fact gave talks at logic seminars in 2012 at Lviv and in 2013 at Munster and Freiburg, and 2014 at St. Petersburg. At a certain point the author realised that possibly a number of simple arguments in point-set topology may become diagram chasing computations with finite topological spaces, and Grigori Mints insisted these observations be written. Ideas of [ErgB] influenced this paper (and [GH-I] as well), and particularly our computational approach to category theory. Alexandre Borovik suggested to write a note for *The De Morgan Gazette* explaining the observation that 'some of human's "natural proofs" are expressions of lifting properties as applied to "simplest counterexample"'.

I thank Yuri Manin for several discussions motivated by [GH-I].

I thank Kurt Goedel Research Center, Vienna, and particulary Sy David Friedman, Jakob Kellner, Lyubomyr Zdomskyy, and Chebyshev Laboratory, St.Petersburg, for hospitality.

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I wish to express my deep thanks to Grigori Mints, to whose memory this paper is dedicated . . .

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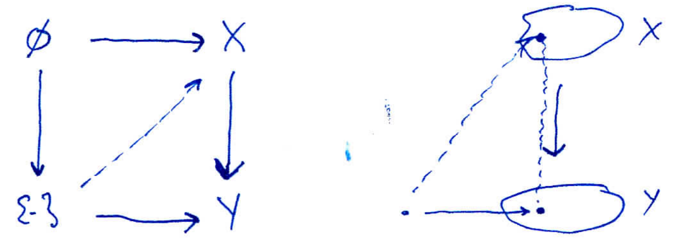
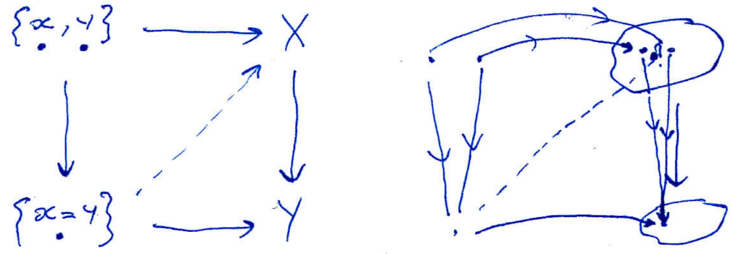
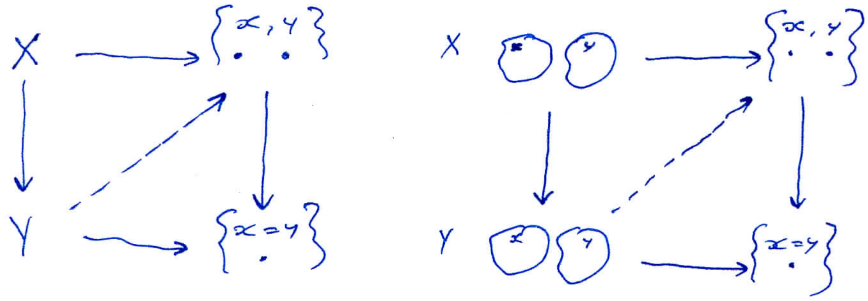


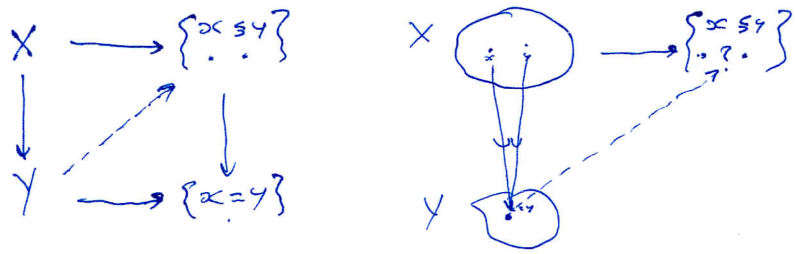
$X \twoheadrightarrow Y$



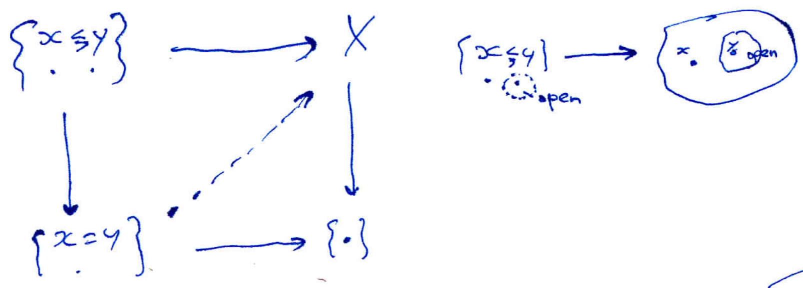
$X \hookrightarrow Y$



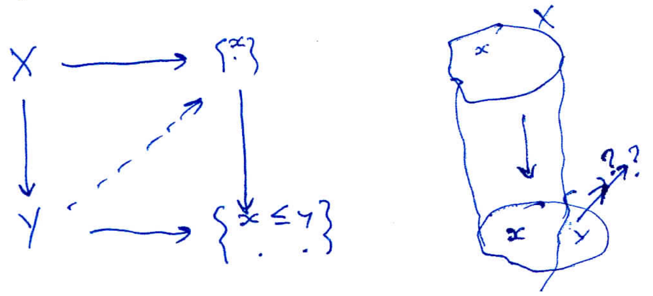
$\pi_0(X) \hookrightarrow \pi_0(Y)$



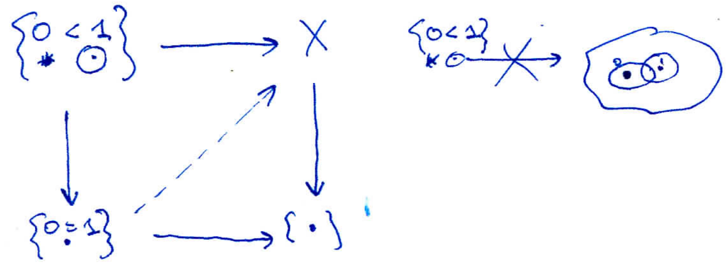
$X \hookrightarrow Y$



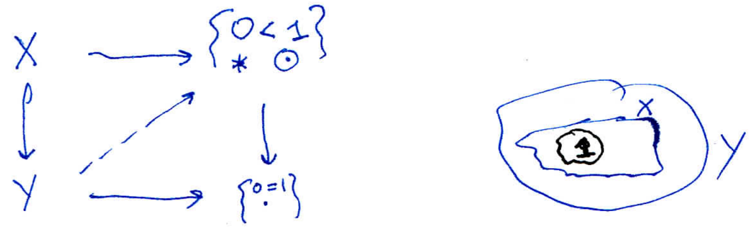
X is T_0



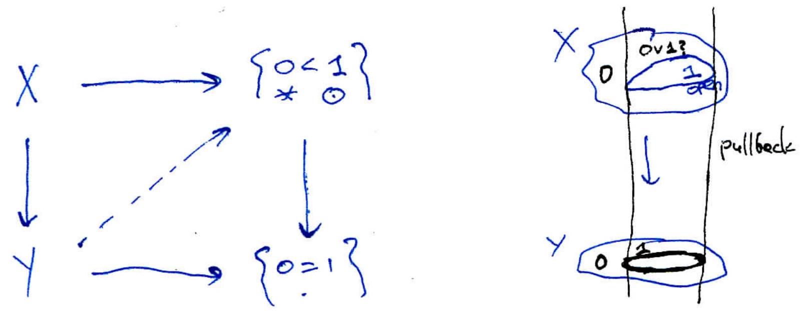
$X \twoheadrightarrow Y$



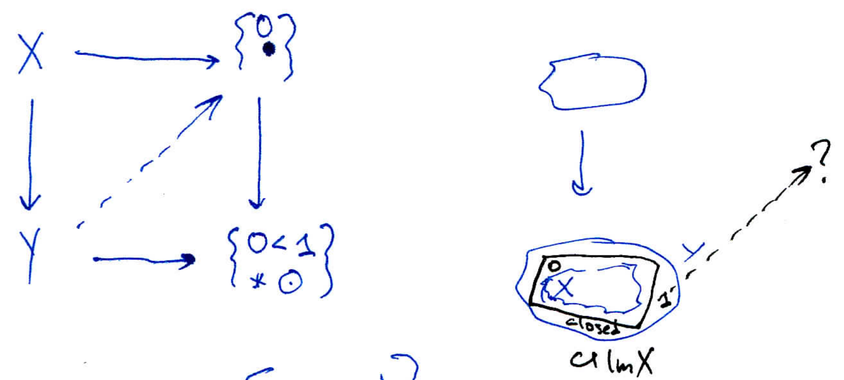
X is T_1



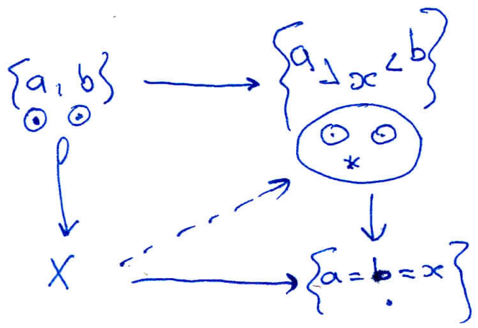
induced topology



the coarsest topology



Im X is dense



X is T_2 (Hausdorff)