Point-set topology as diagram chasing computations

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Question: define a proof system formalising diagram chasing arguments (computations with commutative diagrams) in category theory, a common method of "computational" proof using category theory. (Did not find in literature).

Observation: some standard easy proofs in point-set topology are computations with commutative diagrams of finite preorders (which happen to be degenerate finite categories) in disguise, e.g.

implications between separation axioms T_0, T_1, T_2

$f(x) = g(x)$ defines a closed subset of a Hausdorff space

However, there are other examples of (parts of) category theory arguments disguised in a similar way, say in the theory of metric spaces.

We explain how an example from (Ganesalingam, Gowers, *A fully automatic problem solver with human-style output* is a sequence of applications of a single diagram chasing rule, the lifting property.

Relation to (Ganesalingam, Gowers):

- "our programs really are thinking in a human way"

- "that in the long term, paying close attention to human methods will pay dividends"

- "*we do not allow our programs to do anything that a good human mathematician wouldn't do*", in particular no backtracking for routine problems

- (difference) BUT no human-readable output (important for [GG]); possibly may be added later

- Arguably: [GG]'s automatic prover sometimes does diagram chasing, or computation with commutative diagrams, *in disguise*.

A proof as presented in (Ganesalingam, Gowers):

Problem. Let X be a complete metric space and let A be a closed subset of X. Prove that A is complete.

The proof discovery process would usually be something like this.

1. *[Clarify what needs to be proved.]* We must show that every Cauchy sequence in A converges in A.

2. *[We must show something about* every *Cauchy sequence, so pick an arbitrary one.]* Let (a_n) be a Cauchy sequence in A.

3. *[Clarify what now needs to be proved.]* We are trying to show that (a_n) converges in A.

4. *[See what we can say about the sequence* (a_n)*.]* The sequence (a_n) is a Cauchy sequence in the space X, and X is complete; therefore (a_n) converges in X.

5. *[Give a name to the object that we have just implicitly been presented with.]* Let x be the limit of the sequence (a_n).

6. *[See what we can say about* x*.]* But A is closed under taking limits, so $x \in A$.

7. *[Recognise that the problem is solved.]* Thus, (a_n) converges in A, as we wanted.

Our program is designed to imitate these typical human moves as closely as possible.

- High level statements “out of nowhere” as if they come all by themselves

- No explicit “combinatorial” pattern; implicit semantics

- What does “We must”, “Clarify”, “we have just implicitly been presented with” mean to a computer?

- Each step (application of a heuristic) hard-coded into the prover ?

In our exposition/translation:

- Explicit “combinatorial” patterns; no words but in the definition of the semantics

- Standard derivation rules from category theory

- Most creative part is the definition of the underlying category and thereby semantics

- “Reading off” from the text of the definitions used

Our interpretation: the argument above is a diagram chasing computation consisting only of application of lifting properties, once the right notation has been set up.

Let us translate this argument step-by-step to the language of category theory of diagram chasing.

Problem. Let X be a complete metric space and let A be a closed subset of X. Prove that A is complete.

(0) Translate the statement to the language of arrows.

Fix the category of metric spaces with continuous distance-non-increasing maps. (Why? Arguably, the most creative step.)

Translate the notions used in the theorem:

a Cauchy sequence, a convergent sequence, a complete metric space, a closed subspace of a metric space.

(0′) A *Cauchy sequence* (a_n) *in metric space* X is a sequence of points $a_n \in X$,$n \in \mathbb{N}$ such that

$$dist_A(a_n, a_m) \leq \frac{1}{\min(m,n)}.$$

This implicitly defines a (non-complete) metric space (a_n) whose points are $\{a_n : n \in \mathbb{N}\}$ and distance

$$dist(a_n, a_m) := \frac{1}{\min(m,n)}.$$

Rewrite: A *Cauchy sequence* (a_n) *in metric space* X is a continuous distance-non-increasing map

$$(a_n) \longrightarrow A$$

(0$''$) *the Cauchy sequence* (a_n) *in* A *converges in* A iff there is a limit point a_∞ in A such that

$$dist_A(a_\infty, a_n) \leq \frac{1}{n}.$$

This implicitly defines a (complete) metric space (a_n, a_∞) whose points are $\{a_n : n \in \mathbb{N}\} \cup \{a_\infty\}$ and distance

$$dist(a_n, a_m) := \frac{1}{\min(m, n)} \quad \text{(know already)}$$

$$dist(a_\infty, a_n) := \frac{1}{n}$$

Rewrite: *the Cauchy sequence* $(a_n) \longrightarrow A$ *converges in* A iff the map $(a_n) \longrightarrow A$ factors as

$$(a_n) \longrightarrow (a_n, a_\infty) \longrightarrow A$$

in the category of metric spaces with distance-non-increasing maps.

(0''') X *is complete*: each arrow $(a_n) \longrightarrow X$ factors as

$$(a_n) \longrightarrow (a_n, a_\infty) \longrightarrow X$$

in the category of metric spaces with distance-non-increasing maps.

$$\begin{tikzcd}
(a_n) \arrow[r] \arrow[d] & X \arrow[d] \\
(a_n, a_\infty) \arrow[r] \arrow[ur, dashed] & \{\bullet\}
\end{tikzcd}$$

(0'''') A *is closed under taking limits*: for each sequence (a_n) in A, if the sequence (a_n) in A has a limit a_∞ in X, then $a_\infty \in A$.

the sequence (a_n) *in* $A \subseteq X$ *has a limit* a_∞ *in* X*:* the composition

$$(a_n) \longrightarrow A \longrightarrow X$$

factors as

$$(a_n) \longrightarrow (a_n, a_\infty) \longrightarrow X$$

then $a_\infty \in A$*:*

$$\begin{tikzcd}
(a_n) \arrow[r] \arrow[d] & A \arrow[d] \\
(a_n, a_\infty) \arrow[r] \arrow[ur, dashed] & X
\end{tikzcd}$$

(1) *[Clarify what needs to be proved.]* We must show that every Cauchy sequence in A converges in A.

(2) *[We must show something about* every *Cauchy sequence, so pick an arbitrary one.]* Let (a_n) be a Cauchy sequence in A.

(2') Draw arrow

$$(a_n) \longrightarrow A$$

(3) *[Clarify what now needs to be proved.]* We are trying to show that (a_n) converges in A.

(3') Draw arrows

$$(a_n) \longrightarrow (a_n, a_\infty) \xrightarrow{(to\ construct)} A$$

(4) *[See what we can say about the sequence (a_n).]* The sequence (a_n) is a Cauchy sequence in the space X, and X is complete; therefore (a_n) converges in X.

(4') We have Cauchy sequence

$$(a_n) \longrightarrow A, \qquad A \longrightarrow X,$$

and *therefore* their composition *Cauchy sequence* $(a_n) \longrightarrow X$ in X.

As X *is complete*, each arrow $(a_n) \longrightarrow X$ factors as $(a_n) \longrightarrow (a_n, a_\infty) \longrightarrow X$.

Therefore we construct

$$(a_n) \longrightarrow (a_n, a_\infty) \longrightarrow X$$

$$\begin{tikzcd}
(a_n) \arrow[r] \arrow[d] & X \arrow[d] \\
(a_n, a_\infty) \arrow[r] \arrow[ur, dashed] & \{\bullet\}
\end{tikzcd}$$

(5) *[Give a name to the object that we have just implicitly been presented with.]* Let x be the limit of the sequence (a_n).

(5') done already: $x = a_\infty$

(6) *[See what we can say about x.]* But A is closed under taking limits, so $x \in A$.

(6') *A is closed under taking limits*:

$$\begin{tikzcd}
(a_n) \arrow[r] \arrow[d] & A \arrow[d] \\
(a_n, a_\infty) \arrow[r] \arrow[ur, dashed] & X
\end{tikzcd}$$

(6'') *so $x \in A$*: apply the lifting property above to

$$(a_n) \longrightarrow X \text{ and } (a_n, a_\infty) \longrightarrow X$$

and construct the diagonal arrow

$$(a_n, a_\infty) \longrightarrow A$$

(7) *[Recognise that the problem is solved.]* Thus, (a_n) converges in A, as we wanted.

(7') We have constructed a factorisation

$$(a_n) \longrightarrow (a_n, a_\infty) \longrightarrow A$$

for the arrow

$$(a_n) \longrightarrow A$$