Topological and metric spaces as full subcategories of the category of simplicial objects of the category of filters.

an early draft of a research proposal
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Abstract

We rewrite several notions of elementary topology in the first two chapters (Bourbaki, General Topology), notably compactness, completeness, equicontinuity, in the language of category theory using embeddings of topological and metric spaces into the category of simplicial objects of the category of filters. This approach suggests a number of open questions, which we formulate, largely for the author's own use.

This is an early draft of a research proposal.

1 Introduction.

It appears that a number of notions as defined in [Bourbaki, General Topology] may conveniently and concisely be expressed in the language of category theory. It may be worthwhile to express them this way, for two reasons: it may provide a fresh point of view on foundations of topology (tame topoology) and it may lead to a development of the language of category theory. It is possible that this may be of use in formalisation of foundations of topology.

In this proposal we define two fully faithful embeddings of the category of topological spaces and that of uniform metric spaces into the category of simplicial objects of the category of filters, and, based on this, use these two functors to reformulate several elementary notions including that of being compact, complete, a Cauchy sequence, and equicontinuity.

We formulate a number of open questions, largely for the author's own use: this proposal is at a very early stage and it is possible that some of the questions are easy or indeed well-known.

It is quite possible we are unaware of some relevant literature.

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2 Main constructions and open questions.

2.1 The category of simplicial filters.

We say a topological space is *filtered* iff

(F_I) any superset of a non-empty open set is open.

We say a subset of a topological space is *big* iff it is non-empty and open.

A *filter* is a filtered topological space X which is not discrete. This implies that the intersection of two big, i.e. non-empty open, subsets is big, i.e. non-empty.

Let *Filt* be the full subcategory of the category of topological spaces whose objects are filtered topological spaces. This category has all small limits and colimits and a non-commutative tensor product [Blass,Thm.7]. Limits and colmits are set-wise the same as in $Sets$ and the topology is defined as the finest/coarsest filtered topology such that the necessary maps are continuous. Let *sFilt* be the category of simplicial objects in the category *Filt* of filtered topological spaces, i.e. $sFilt = Func(\mathrm{Ord}^{op}_{<\omega}, Filt)$ where $\mathrm{Ord}_{<\omega}$ denotes the category of categories corresponding to finite linear orders

$$\bullet_1 \longrightarrow .. \longrightarrow \bullet_n,\ 0 \leq n < \omega.$$

There are two natural functors $Filt \longrightarrow sFilt$:

$\text{ኔ} : F \longmapsto (F, F, F, ...),$ identity maps

$E : F \longmapsto (F, F \times F, F \times F \times F, ...)$, face and degeneracy maps are coordinate maps $F^n \longrightarrow F^m, (x_1, ..., x_n) \mapsto (x_{i_1}, ..., x_{i_m})$ where $1 \leq i_1 \leq i_2 \leq ... \leq i_m \leq n$.

2.2 Topological and metric spaces as simplicial filters

Topological and uniform spaces are defined [Bourbaki, Ch1., Ch.2] as systems of neighbourhood filters satisfying certain compatibility conditions, and lead us to define two fully faithful functors $\text{ቲ} : Top \longrightarrow sFilt$, $\text{ኂ} : \mathcal{MU} \longrightarrow sFilt$.

In Appendix B we show how to "read off" the latter embedding from the definition of uniform structures in [Bourbaki, Chapter 2].

For a topological space X, let $\text{ቲ}(X)$ denote the following object in *sFilt*.

$$\text{ቲ} : X \longmapsto (|X|, |X| \times |X|, |X| \times |X| \times |X|, ...)$$

where a subset of $|X|^n$ is *big* iff the following formula holds:

$\forall x_1 \in X \exists U_{x_1} \ni x_1$ a neighbourhood of x_1
$\forall x_2 \in U_{x_1} \exists U_{x_2} \ni x_2$ a neighbourhood of x_2
....
$\forall x_n \in U_{x_{n-1}} \exists U_{x_n} \ni x_n$ a neighbourhood of x_n
$(x_1, x_2, ..., x_n) \in U$

Face and degeneracy maps are coordinate maps

$$|X|^n \longrightarrow |X|^m, (x_1, ..., x_n) \mapsto (x_{i_1}, ..., x_{i_m})$$

where $1 \leq i_1 \leq i_2 \leq ... \leq i_m \leq n$.

Note that any big subset of $|X|^n$ contains the diagonal, in particular the topology on $|X|$ is always antidiscrete. Topology on X is discrete iff the diagonal in $|X|^n, n \geq 2$ is open, equivalently a subset of $|X|^n$ is big iff it contains the diagonal.

For X finite, a subset of $|X|^n, n \geq 2$ is big iff it contains all the "non-strictly decreasing" sequences $(x_1, x_2, ..., x_n)$ such that $x_i \in cl(x_{i+1})$, $1 \leq i < n$.

Note that the topology on $|X|^n$ is the coarsest filter such that the maps $|X|^n \longrightarrow |X|^2, (x_1, ..., x_n) \mapsto (x_i, x_{i+1})$, $1 \leq i < n$ are continuous.

For a metric space M, let ሚ(M) denote the following object in *sFilt*.

$$\text{ሚ} : M \longmapsto (|M|, |M| \times |M|, |M| \times |M| \times |M|, ...)$$

where a subset of $|M|^n$ is *big* iff it contains an ε-neighbourhood of the diagonal $\{(x, .., x) : x \in M\}$. Face and degeneracy maps are coordinate maps

$$|X|^n \longrightarrow |X|^m, (x_1, ..., x_n) \mapsto (x_{i_1}, ..., x_{i_m})$$

where $1 \leq i_1 \leq i_2 \leq ... \leq i_m \leq n$.

Note that, as before, any big subset of $|M|^n$ contains the diagonal, in particular the topology on $|M|$ is always antidiscrete. The diagonal in $|M|^n, n \geq 2$ is open, iff the metric space M is discrete.

Note that the topology on $|M|^n$ is the coarsest filter such that the maps $|X|^n \longrightarrow |X|^2, (x_1, ..., x_n) \mapsto (x_i, x_j)$, $1 \leq i < j \leq n$ are continuous.

Let $\mathcal{MU}$ denote the category of uniform spaces [Bourbaki, II§1.1]. In a similar way ሚ(M) is defined also for M a uniform space.

Note that permutations of coordinates act on ሚ(M).

Claim 1. *ቲ $: Top \longrightarrow sFilt$ and ሚ $: \mathcal{MU} \longrightarrow sFilt$ are fully faithful functors.*

Proof. The verification is straightforward and we only consider ቲ. The formula is positive and therefore a superset of a big subset is big. The intersection of two neighbourhoods is a neighbourhood [Bourbaki,I§1.2,Ax.(V_{II})] and this carries though the quantifiers. Finally, each neighbourhood of a point contains the point, and this implies that big subsets necessarily contain the diagonal and thus form a filter. To see continuity of a degeneracy map $(x_1, ..., x_n) \longmapsto (x_1, .., x_i, x_{i+1}, ..., x_n)$, pick the same neighbourhood twice, $U_{x_i} = U_{x_{i-1}}$ or $U_{x_1} = X$ if $i-1 < 1$; this uses that an open subset is a neighbourhood of each of its points [Bourbaki,I§1.2,Ax.(V_{IV})] and that X itself is open [Bourbaki,I§1.2,Ax.(V_I)]. To check continuity of a face map $(x_1, ..., x_n) \longmapsto (x_1, .., x_i, x_i, x_{i+1}, ..., x_n)$, pick $x_{i+1} = x_i$; this uses that a point is contained in any neighbourhood of itself [Bourbaki,I§1.2,Ax.(V_{III})]. The functor is faithful because the morphism on X^1 uniquely determines morphisms on all the other Cartesian powers. A function $f : X \longrightarrow Y$ is continuous iff for each point $x \in X$ and each neighbourhood

V_y of $y = f(x)$ there is a neighbourhood U_x such that $V_y \subset f(U_x)$. This is implied by the fact that the preimage of a big set $\{y\} \times V_y \cup \cup_{y' \neq y} \{y'\} \times Y$ contains $\{x\} \times U_x$ for some U_x, as it is big. $\square$

These two embeddings immediately give rise the following questions.

Question 1. (Category theory and homotopy)

1. Do functors $\mathfrak{t}, \mathfrak{m}$ have adjoints? Do they preserve limits and colimits?

2. Characterise the full subcategories $\mathfrak{t}(Top)$ and $\mathfrak{m}(\mathcal{MU})$ of *sFilt* in terms of the ambient category *sFilt*.

3. Characterise systems of neighbourhoods such that the construction of $\mathfrak{t}$ gives rise to a simplicial object.

4. Does a model structure on Top extend to a model structure on *sFilt*? Does *sFilt* have an interesting model structure?

The following questions are vague.

Question 2. (Naive homotopy theory)

1. What is the "right" notion in *sFilt* of a real line interval $[0, 1]$, a fibration, and path, loop, and suspension objects?

2. Is there an interesting object of *sFilt* which corresponds to the path space of a topological space and which is more "finitary"? Note that in *sFilt* topological spaces have dimension 2 and that path space is thought of a space "shifted". Can this "shift" be realised in *sFilt* somehow, e.g. so that the *sFilt*-path space of a topological space have dimension 3 ?

3. Is there an interesting object of *sFilt* which corresponds to a foliation, particularly an irrational foliation?

Question 3. ("Combinatorial" definition of compactness and completeness)

1. A number of elementary topological properties can be defined by, in a sense, combinatorial expressions, by taking iterated orthogonals in Top of a single morphism between finite topological spaces [Gavrilovich, Lifting Property]. Calculate these expressions in *sFilt* using the embedding $\mathfrak{t} : Top \longrightarrow sFilt$. Note that this would give properties of both topological spaces and metric spaces. Do they define the same properties of topological spaces? Do they provide an interesting analogy between topological spaces and metric spaces, e.g. compactness [Bourbaki, I§10.2, Thm.1(d), p.101] and completeness [Bourbaki, II§3.6, Prop.11]?

2. The following is an example of a precise conjecture. Evaluated in Top, the following expression defines the class of almost(?) all proper maps [Gavrilovich, Lifting Property, Claim 1]:

$$((\{\{o\} \longrightarrow \{o \to c\}\}^r)_{<5})^{lr}$$

If evaluated in *sFilt*, does the same expression define both the class of compact space and complete metric space, i.e. is the following true:

(a) A Hausdorff space X is compact iff

$$𝔗\,(X \longrightarrow \{pt\}) \in ((𝔗\,(\{\{o\} \longrightarrow \{o \to c\}\})\}^r)_{<5})^{lr}$$

(b) A metric space M is complete iff

$$𝔐\,(M \longrightarrow \{pt\}) \in ((𝔗\,(\{\{o\} \longrightarrow \{o \to c\}\})\}^r)_{<5})^{lr}$$

Question 4. (Tame topology of Grothendieck)

1. Does this point of view shed light on tame topology of Grothendieck, k, i.e. a foundation of topology "without false problems" and "wild phenomena" "at the very beginning"? Is there a better construction of "the" tubular neighbourhood of closed tame subspace in a tame space? I quote a specific suggestion by Grothendieck [Esquisse d'un Programme, translation,§5,p.33]:

> Among the first theorems one expects in a framework of tame topology as I perceive it, aside from the comparison theorems, are the statements which establish, in a suitable sense, the existence and uniqueness of the tubular neighbourhood of closed tame subspace in a tame space (say compact to make things simpler), together with concrete ways of building it (starting for instance from any tame map $X \longrightarrow \mathbb{R}^+$ having Y as its zero set), the description of its "boundary" (although generally it is in no way a manifold with boundary!) ∂T , which has in T a neighbourhood which is isomorphic to the product of T with a segment, etc. Granted some suitable equisingularity hypotheses, one expects that T will be endowed, in an essentially unique way, with the structure of a locally trivial fibration over Y, with ∂T as a subfibration.

Question 5. (History and formalisation of mathematics)

Early works on topology talk about topological spaces in terms of neighbourhood systems. Could it be that they are implicitly trying to express (say, functorial) constructions in *sFilt*, are implicitly using category theoretic language but describing it in words? Can this question be made precise?

For example, it was a convention to always mean by $U(x)$ a neighbourhood of a point x, and that's somewhat natural from the point of view of our definition of $𝔗 : Top \longrightarrow sFilt$.

In Appendix B we show how to "read off" our construction of $𝔐 : \mathcal{MU} \longrightarrow$ *sFilt* from Bourbaki. Arguably, $𝔗 : Top \longrightarrow sFilt$ and the combinatorial definitions of elementary topological properties [Gavrilovich, Lifting Property] are “contained” in Bourbaki. In what sense are these reformulations “contained implicitly” there? Can this sense be made explicit?

Could these reformulations be of use in formalisation of topology and analysis, e.g. Chapter 1 and 2 of Bourbaki?

2.3 Elementary theory of topological and metric spaces

We reformulate several notions from [Bourbaki, Chapter 1, 2] in terms of functors ቲ $: Top \longrightarrow sFilt$ and ሚ $: \mathcal{MU} \longrightarrow sFilt$.

2.3.1 Compact and complete metric spaces

An *ultrafilter* is a filter such that if the union of finitely many sets is open, then one of them is; equivalently, each subset A is either open or closed.

With a filter $\mathcal{F}$ on the set of points of a topological space X associate [Bourbaki, II§5,Example] a topological space $X \cup_{\mathcal{F}} \{\infty\}$ such that $\mathcal{F}$ is the neighbourhood filter of ∞: $|X \cup_{\mathcal{F}} \{\infty\}| = |X| \cup \{\infty\}$, and a subset is open iff it is either an open subset of X or contains ∞ and is a union of $\{\infty\}$ and an open subset of X which is also $\mathcal{F}$-open.

Let X be a topological space such that $|X| = |\mathcal{F}|$. An filter $\mathcal{F}$ *converges* on a topological space X iff one of the two equivalent conditions holds [Bourbaki,II§7, Def.1]:

- there is a point $\infty \in X$ such that each X-open neighbourhood of X is also $\mathcal{F}$-open
- the obvious map $|F| \longrightarrow X$ extends to a map $|F| \cup_{\mathcal{F}} \{\infty\} \longrightarrow X$

Question 6.

1. Find a category theoretic way to work with ultrafilters, possibly using that in Top, for an ultrafilter $\mathcal{F}$, in Top $\mathcal{F} \longrightarrow \mathcal{F} \cup_{\mathcal{F}} \{\infty\} \rightthreetimes g$ for each closed map of finite topological spaces, and, more generally, $\mathcal{F} \longrightarrow \mathcal{F} \cup_{\mathcal{F}} \{\infty\} \in (\{\{0\} \longrightarrow \{0 \to 1\}\}^{l}{}_{finite})^{r}$ (see [Gavrilovich, Lifting property, Claim 1].
2. Find a category theoretic way or convenient notation to work with cluster points of filters rather than limit point of ultrafilters.

A topological space X is *quasi-compact* iff one of the two equivalent conditions holds [Bourbaki, I§10.2, Thm.1(d), p.101]:

- each ultrafilter on X converges
- for each ultrafilter $\mathcal{F}$ it holds in Top $\mathcal{F} \longrightarrow \mathcal{F} \cup_{\mathcal{F}} \{\infty\} \rightthreetimes X \longrightarrow \{\bullet\}$
- for each ultrafilter $\mathcal{F}$ it holds in *sFilt* ቲ$(\mathcal{F}) \longrightarrow$ ቲ$(\mathcal{F} \cup_{\mathcal{F}} \{\infty\}) \rightthreetimes$ ቲ$(X) \longrightarrow$ ቲ$(\{\bullet\})$

More generally, a map $X \longrightarrow Y$ is proper iff for each ultrafilter $\mathcal{F}$ either of the following equivalent conditions holds [Bourbaki, I§10.2, Thm.1(d), p.101]:

- in *Top*, $\mathcal{F} \longrightarrow \mathcal{F} \cup_{\mathcal{F}} \{\infty\} \;\rightthreetimes\; X \longrightarrow Y$
- in *sFilt*, $\mathfrak{t}(\mathcal{F}) \longrightarrow \mathfrak{t}(\mathcal{F} \cup_{\mathcal{F}} \{\infty\}) \rightthreetimes \mathfrak{t}(X) \longrightarrow \mathfrak{t}(Y)$

A *Cauchy filter* $\mathcal{F}$ on a metric space M ([Bourbaki,II§3.1,Def.2]) is a filter on $|M|$ such that one of the two equivalent conditions holds:

- for each $\epsilon > 0$ there is a $\mathcal{F}$-open non-empty subset $V \subset |M|$ of diameter at most ε
- the map $\mathcal{F} \times \mathcal{F} \longrightarrow |M| \times |M|$ is continuous where $|M| \times |M|$ is equipped with the topology coming from $\mathfrak{m}(M)$
- the obvious map $\mathrm{E}(\mathcal{F}) \longrightarrow \mathfrak{m}(M)$ is well-defined

A *Cauchy sequence* in M is a map $\mathrm{E}(\omega_{cofinite}) \longrightarrow \mathfrak{m}(M)$ where $\omega_{cofinite}$ is the set of natural numbers equipped with cofinite topology (i.e. a subset is closed iff it is finite).

A metric space is *precompact* iff one of the two equivalent conditions holds :

- for each $\varepsilon > 0$ there is a finite covering of M by subsets of diameter at most ε [Bourbaki, II§4,Thm.3]
- each ultrafilter on M is a Cauchy ultrafilter [Bourbaki, II§4,Exer.5]
- for each ultrafilter it holds in *sFilt* $\mathfrak{l}(\mathcal{F}) \longrightarrow \mathrm{E}(\mathcal{F}) \rightthreetimes \mathfrak{m}(M) \longrightarrow \mathfrak{m}(\{\bullet\})$

A metric space M is *complete* iff one of the two equivalent conditions holds [Bourbaki,II§3.3,Def.3]:

- each Cauchy filter on M converges
- in *sFilt*, $\mathrm{E}(\mathcal{F}) \longrightarrow \mathrm{E}(X \cup_{\mathcal{F}} \{\infty\}) \rightthreetimes \mathfrak{m}(M) \longrightarrow \mathfrak{m}(\{\bullet\})$
- in *sFilt*, $\mathfrak{t}(\mathcal{F}) \longrightarrow \mathfrak{t}(X \cup_{\mathcal{F}} \{\infty\}) \rightthreetimes \mathfrak{m}(M) \longrightarrow \mathfrak{m}(\{\bullet\})$

Question 7. Define the completion of a uniform space [Bourbaki, II§3.7] as something like inner hom $\underline{Hom}(\mathrm{E}(\omega_{cofinite}), \mathfrak{m}(M))$. Develop the theory [Bourbaki, II§3,4] of complete and precompact uniform spaces in terms of *sFilt* and the lifting properties.

Let M be a metric space. The following are equivalent [Bourbaki,II§1.2,Def.3]:

- topological space M_{top} is homeomorphic to $|M|$ with the topology induced from the metric on M
- there is an arrow $\mathfrak{t}(M_{top}) \xrightarrow{\gamma} \mathfrak{m}(M)$ and for each topological space X, in *sFilt* any map $\mathfrak{t}(X) \longrightarrow \mathfrak{m}(M)$ factors as

$$\mathfrak{t}(X) \longrightarrow \mathfrak{t}(M_{top}) \xrightarrow{\gamma} \mathfrak{m}(M)$$

For a compact space K, there exists a unique uniform space K_{uni} which induces on K its topology. In other words, there is a unique map ⵜ$(K) \xrightarrow{\gamma}$ ⵎ(K_{uni}) such that each map ⵜ$(K) \longrightarrow$ ⵎ(M) factors as

$$\text{ⵜ}(K) \xrightarrow{\gamma} \text{ⵎ}(K_{uni}) \longrightarrow \text{ⵎ}(M)$$

[Bourbaki, II§4.1,Thm.1].

Remark 1. The notion of the topology induced by a metric is reminiscent of an adjoint functor to ⵜ. Does either ⵎ or ⵜ have adjoints?

2.3.2 Equicontinuous functions and Arzela-Ascoli theorem

Let X be a topological space, let M be a metric space, and let $(f_i)_{i\in\mathbb{N}}$ be a family of functions $f_i : X \longrightarrow M$.

The family f_i is *equicontinuous* if either of the following equivalent conditions holds:

- for every $x \in X$ and $\epsilon > 0$, there exists a neighbourhood U of x such that $d_Y(f_i(x'), f_i(x)) \leq \epsilon$ for all $i \in \mathbb{N}$ and $x' \in U$
- the map ⵜ$(X) \times$ ƚ$(\{\mathbb{N}\}) \longrightarrow$ ⵎ(M), $(x, i) \longmapsto f_i(x)$ is well-defined
- the map ⵜ$(X) \times$ ƚ$(\mathbb{N}_{cofinite}) \longrightarrow$ ⵎ(M), $(x, i) \longmapsto f_i(x)$ is well-defined

If $X = (X, d_X)$ is also a metric space, we say that the family f_i is *uniformly equicontinuous* iff either of the following equivalent conditions holds:

- for every $\epsilon > 0$ there exists a $\delta > 0$ such that $d_Y(f_i(x'), f_i(x)) \leq \epsilon$ for all $i \in \mathbb{N}$ and $x', x \in x$ with $d_X(x, x') \leq \delta$
- the map ⵎ$(X) \times$ ƚ$(\{\mathbb{N}\}) \longrightarrow$ ⵎ(M), $(x, i) \longmapsto f_i(x)$ is well-defined
- the map ⵎ$(X) \times$ ƚ$(\mathbb{N}_{cofinite}) \longrightarrow$ ⵎ(M), $(x, i) \longmapsto f_i(x)$ is well-defined

The family is *uniformly Cauchy* iff either of the following equivalent conditions holds:

- for every $\epsilon > 0$ there exists a $\delta > 0$ and $N > 0$ such that $d_Y(f_i(x'), f_j(x)) \leq \epsilon$ for all $i, j > N$ and $x', x \in x$ with $d_X(x, x') \leq \delta$.
- the map ⵎ$(X) \times E(\mathbb{N}_{cofinite}) \longrightarrow$ ⵎ(M), $(x, i) \longmapsto f_i(x)$ is well-defined

Here $\{\mathbb{N}\}$ denotes the trivial filter on $\mathbb{N}$ with a unique big subset $\mathbb{N}$ itself, and $\mathbb{N}_{cofinite}$ denotes the filter of cofinite subsets of $\mathbb{N}$.

Question 8. (Azrela-Ascoli)

1. Reformulate various notions of equicontinuity and convergence of a family of functions $f_i : X \longrightarrow M$ in terms of maps in *sFilt* using e.g. ƚ$(\mathbb{N}_{cofinite})$, E$(\mathbb{N}_{cofinite})$, ƚ$(\mathbb{N}_{cofinite} \cup_{\mathbb{N}_{cofinite}} \{\infty\})$, ⵜ$(\mathbb{N}_{cofinite} \cup_{\mathbb{N}_{cofinite}} \{\infty\})$, $E(\mathbb{N}_{cofinite} \cup_{\mathbb{N}_{cofinite}} \{\infty\})$, ⵜ$(\mathbb{N}_{cofinite})$, ⵜ$(X)$, ⵎ$(X)$, and ⵎ$(M)$.

2. Reformulate and prove Arzela-Ascoli theorem in terms something like inner Hom in $sFilt$ and the lifting properties definiting precompactness, compactness etc.

3. Define function spaces in terms of something like inner Hom in $sFilt$.

3 Appendix A.

Embeddings of geometric categories

Here we define several embeddings of geometric categories of metric spaces into the category of "infinitary" simplicial objects of the category of filters, notably the categroy of metric spaces up to quasi-isometry. To make the exposition self-contained, we repeat here some of the notation introduced above. In part this is motivated by a remark in [Gromov, Hyperbolic dynamics, §2.7,p.54, footnote 90].

Let $\mathrm{Ord}_{<\alpha}$ denote the category of finite ordinals less than α and non-decreasing maps; equivalently but more conceptually, this is the full subcategory of the category of categories consisting of the categories $\bullet_0 \longrightarrow \bullet_1 \longrightarrow ...$ corresponding to well-ordered sets of size less than α. When $\alpha = \omega + 1$, the category $\mathrm{Ord}_{<\omega}$ is the category of finite ordinals usually denoted Δ.

For a category C and ordinal α, *$<\alpha$-simplicial objects in C* is a functor $F : \mathrm{Ord}^{op}_{<\alpha} \longrightarrow C$. These objects naturally form a category which we denote $s_{<\alpha}C = Func(\mathrm{Ord}^{op}_{<\alpha}, C)$ of functors from $\mathrm{Ord}^{op}_{<\alpha}$ to C. When $\alpha = \omega + 1$, this is the usual category of simplicial objects of C.

With an object X we can associate two $<\alpha$-simplicial objects in C as follows. $\text{よ}(X)$ sends each ordinal to X itself and each morphism to the identity The functor $E_\alpha(X)$ sends an ordinal $\beta < \alpha$ to the Cartesian power X^β, and morphisms are sent to the coordinate maps.

These two functors define two fully faithful embeddings of C into $s_{<\alpha}C$. $\text{よ} : C \longrightarrow s_{<\alpha}C$ and $E : C \longrightarrow s_{<\alpha}C$.

Let *Filt* be the category of filters, i.e. the full subcategory of the category of topological spaces consisting of spaces such that any superset of a non-empty open set is open.

3.1 Metric spaces as "infinitary" simplicial filters

We define several embeddings of categories of metric spaces with various kinds of geometric maps, e.g. uniformly continuous maps, Lipschitz maps on large scale. We do so by definition various filters on (possibly infinite) Cartesian powers of a metric space which preserve certain geometric information about the metric space. In the usual way these collections of filters give rise to simplicial objects of $s_{\leq\omega}Filt$.

Let M be a metric space. Let us now define a number of topologies on Cartesian powers of $|M|$.

- A non-empty subset of M^n is τ-open (big) iff the following formula holds:
  $\forall x_1 \in M \exists U_{x_1} \ni x_1$ a neighbourhood of x_1
  $\forall x_2 \in U_{x_1} \exists U_{x_2} \ni x_2$ a neighbourhood of x_2
  ....
  $\forall x_n \in U_{x_{n-1}} \exists U_{x_n} \ni x_n$ a neighbourhood of x_n
  $(x_1, x_2, ..., x_n) \in U$

- A non-empty subset of M^n is τ_U-open iff it contains an ε-neighbourhood of the diagonal $\{ (x, x, ..., x) : x \in |M| \}$ for some $\varepsilon > 0$, i.e. $U \subseteq |M|^n$ is open iff there is $\varepsilon > 0$ such that for each $x_1, ..., x_n \in M$, it holds $(x_1, ..., x_n) \in U$ provided there is $x \in M$ such that $dist(x, x_i) < \varepsilon$, $i = 1, .., n$.

  Note that no proper subset $\emptyset \subsetneq U \subsetneq |M|$ is open as you may take $x = x_1$.

- Fix a real number $D > 0$. A non-empty subset of M^n is τ_D-open iff it contains an D-neighbourhood of the diagonal $\{ (x, x, ..., x) : x \in |M| \}$ for some $\varepsilon > 0$, i.e. $U \subseteq |M|^n$ is open iff there for each $x_1, ..., x_n \in M$, it holds $(x_1, ..., x_n) \in U$ provided there is $x \in M$ such that $dist(x, x_i) \leq D$, $i = 1, .., n$.

  Note that no proper subset $\emptyset \subsetneq U \subsetneq |M|$ is open as you may take $x = x_1$.

- A non-empty subset of M^ω is $\tau_{\mathcal{L}}$-open iff there is $\lambda > 0$, $N > 0$, $D < \lambda N$ such that for each $x_1, ..., x_n, ... \in M$, it holds $(x_1, ..., x_n, ...) \in U$ provided there is $x \in M$ such that $x = x_1 = ... = x_N$ and $dist(x, x_i) \leq \lambda i - D$ for each $i > N$.

  A non-empty subset U of M^n is $\tau_{\mathcal{L}}$-open iff it contains the diagonal $\{(x, .., x) : x \in M\}$.

- A non-empty subset of M^ω is $\tau_{\mathcal{L}_1}$-open iff there is $N > 0$, $D < N$ such that for each $x_1, ..., x_n, ... \in M$, it holds $(x_1, ..., x_n, ...) \in U$ provided there is $x \in M$ such that $x = x_1 = ... = x_N$ and $dist(x, x_i) \leq i - D$ for each $i > N$.

  A non-empty subset U of M^n is τ_1-open iff it contains the diagonal $\{(x, .., x) : x \in M\}$.

A map $f : |M| \longrightarrow |N|$ induces a map $f_n : |M|^n \longrightarrow |N|^n$. The following is easy to check:

- For $n > 1$, f_n is is τ-continuous iff it is continuous.

- For $n > 1$, f_n is is τ_U-continuous iff it is uniformly continuous.

- For $n > 1$, f_n is τ_D-continuous iff for each $x, y \in M$ $dist(x, y) \leq D$ implies $dist(f(x), f(y)) \leq D$

- f_ω is $\tau_{\mathcal{L}}$-continuous iff it is λ-Lipschitz on large scale for some $\lambda, D > 0$, i.e. $dist(f(x), f(y)) \leq \lambda dist(x, y)$ whenever $dist(x, y) \geq D$, $x, y \in M$.

- f_ω is $\tau_{\mathcal{L}_1}$-continuous iff it is 1-Lipschitz on large scale, i.e.for some D for each $x, y \in M$ $dist(f(x), f(y)) \leq dist(x, y) + D$

A map $f : M \longrightarrow M$ is *an almost isometry* iff either of the following equivalent conditions holds:

- $dist(f(x), f(y)) \leq dist(x, y) + D$ for some D

- $f_\omega : M^\omega \longrightarrow M^\omega$ is $\tau_{\mathcal{L}_1}$-continuous

A map $f : M \longrightarrow M$ is *a quasi-isometry* iff either of the following equivalent conditions holds:

- $dist(f(x), f(y)) \leq \lambda dist(x, y) + D$ for some $\lambda, D > 0$ for each $x, y \in M$
- $f_\omega : M^\omega \longrightarrow M^\omega$ is $\tau_{\mathcal{L}}$-continuous

A verification shows that these topologies define fully faithful functors

$$m_{\mathcal{U}} : \mathcal{MU} \longrightarrow Func(\mathrm{Ord}^{\mathrm{op}}_{<\omega}, Filt),$$

$$m_D : \mathcal{M}_D \longrightarrow Func(\mathrm{Ord}^{\mathrm{op}}_{<\omega}, Filt),$$

$$m_{\mathcal{L}} : \mathcal{ML} \longrightarrow Func(\mathrm{Ord}^{op}_{<\omega+1}, Filt),$$

$$m_{\mathcal{L}_1} : \mathcal{ML}_1 \longrightarrow Func(\mathrm{Ord}^{op}_{<\omega+1}, Filt)$$

from the relevant geometric categories of metric spaces.

4 Appendix B.

Reading Bourbaki definition of the uniform spaces

(Bourbaki, II§1.1.1) treats metric spaces as uniform spaces; we observe that the uniform space is a simplicial object.

We quote (Bourbaki, I§6.1.1) and (Bourbaki, II§1.1.1):

> DEFINITION I. A filter on a set X is a set $\mathcal{F}$ of subsets of X which has the following properties:
> (F_I) Every subset of X which contains a set of F belongs to $\mathcal{F}$.
> (F_{II}) Every finite intersection of sets of $\mathcal{F}$ belongs to $\mathcal{F}$.
> (F_{III}) The empty set is not in $\mathcal{F}$.
>
> 1. DEFINITION OF A UNIFORM STRUCTURE
>
> DEFINITION I. *A uniform structure (or uniformity) on a set X is a structure given by a set $\mathfrak{U}$ of subsets of $X \times X$ which satisfies axioms (F_I) and (F_{II}) of Chapter I, 6, no. I and also satisfies the following axioms:*
> *(U_I) Every set belonging to $\mathfrak{U}$ contains the diagonal Δ.*
> *(U_{II}) If $V \in \mathfrak{U}$ then $V^{-1} \in \mathfrak{U}$.*
> *(U_{III}) For each $V \in \mathfrak{U}$ there exists $W \in \mathfrak{U}$ such that $W \circ W \subset V$.*
> *The sets of $\mathcal{U}$ are called entourages of the uniformity defined on X by $\mathcal{U}$. A set endowed with a uniformity is called* a uniform space. *If V is an entourage of a uniformity on X, we may express the relation $(x, x') \in V$ by saying that "x and x' are V-close".*

The set of points of a metric space X carries a canonical uniform space: $V \in \mathfrak{U}$ iff $\{(x, x') : dist(x, x') < \varepsilon\} \subset V$ for some $\varepsilon > 0$.

Let us translate the definitions above to the language of arrows: we shall see that a uniform space may be viewed as a simplicial object of the category of topological spaces.

First notice that a filter can equivalently be defined as a non-discrete topology such that a superset of a non-empty open set is necessarily open: a filter $\mathcal{F}$ on a set X defines a topology on X where a subset is open iff it is either $\mathcal{F}$-big or empty. Indeed, Axioms (F_I) and (F_{II}) of a filter imply that the family of subsets $\mathfrak{U} \cup \{\emptyset\}$ is a topology on a set X.

In this way a uniform structure on a set X defines a topology on $X \times X$.

Axiom (U_I) implies that the diagonal map $X \xrightarrow{(x,x)} X \times X$ is continuous as a map from the set X equipped with antidiscrete topology to the set $X \times X$ equipped with the topology above, and is almost equivalent to this. Indeed, the latter says that an $\mathfrak{U}$-big subset of $X \times X$ either contains the diagonal or does not intersect it.

Axiom (U_{II}) says that permuting the coordinates $X \times X \longrightarrow X \times X, (x_1, x_2) \mapsto (x_2, x_1)$ is continuous in this topology.

Define topology on the set $X \times X \times X$ via the pullback square in the category of filter topological spaces *sFilt*:

$$\begin{tikzcd}
X \times X \times X \arrow[r, "(p_1 \times p_2)"] \arrow[d, "(p_2 \times p_3)"'] & X \times X \arrow[d, "p_2"] \\
X \times X \arrow[r, "p_2"] & X
\end{tikzcd}$$

Axiom (U_{III}) says that the map $X \times X \times X \xrightarrow{(p_1,p_3)} X \times X, (x_1, x_2, x_3) \mapsto (x_1, x_3)$ is continuous in this topology. Indeed, by definition

$$W_1 \circ W_2 = \{(x_1, x_3) \,:\, (x_1, x_2) \in W_1,\ (x_2, x_3) \in W_2\}$$

and the sets of form

$$\{(x_1, x_2, x_3) \,:\, (x_1, x_2) \in W_1,\ (x_2, x_3) \in W_2\} = W_1 \times X \cap X \times W_2,$$

$W_1, W_2 \in \mathfrak{U}$, form a base of the pullback topology on $X \times X \times X$. Hence, (U_{III}) says that the preimage of an open subset of $X \times X$ under (p_1, p_3) contains an open subset of $X \times X \times X$, i.e. is open (as pullback is taken among filter topologies).

Axiom (U_I) implies that the diagonal map $X \xrightarrow{(x,x)} X \times X$ is continuous as a map from the set X equipped with antidiscrete topology to the set $X \times X$ equipped with the topology above.

Note that $W \circ W$ intersects the diagonal and the continuity of the diagonal map $X \xrightarrow{(x,x)} X \times X$ implies $W \circ W$ contains the diagonal. Thus, in presence of (U_{III}), (U_I) is equivalent to the continuity of the diagonal map $X \xrightarrow{(x,x)} X \times X$ in the topologies indicated.

Let X_1 denote the set X equipped with the antidiscrete topology. Let X_2 and X_3 denote the sets $X \times X$ and $X \times X \times X$ equipped with the topologies above. For $n > 3$, let X_n be the pullback in *sFilt*

$$\begin{tikzcd}
X_n \arrow[r, "(p_2 \times \ldots \times p_n)"] \arrow[d, "(p_1 \times p_2)"'] & X_{n-1} \arrow[d, "p_2"] \\
X_2 \arrow[r, "p_2"] & X
\end{tikzcd}$$

The axioms above ensure that the "set-theoretic" face and degeneracy maps

$$(p_{i_1}, ..., p_{i_k}) : X \times ... \times X \longrightarrow X \times ... \times X$$

are continuous. Thus we see that a uniform structure on a set X defines a simplicial complex X_n in *sFilt*,

$$(p_{i_1}, ..., p_{i_k}) : X_n \longrightarrow X_m$$

Claim 2. *A uniform structure on a set X is a simplicial object $X.$ in the subcategory sFilt of filter topological spaces equipped with an involution $i : X. \longrightarrow X.$ such that*

- *X_1 is the set X equipped with antidiscrete topology*
- *the underlying set of X_2 is $X \times X$*
- *$i : X. \longrightarrow X.$ is the involution permuting the coordinates on $X \times X$*
- *for $n > 2$, X_n is the pullback as described above*

Question 9. Find a categorical description of the simplicial objects obtained from uniform spaces.

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This note is elementary, and it was embarrassing and boring, and embarrassingly boring, to think or talk about matters so trivial, but luckily I had no obligations for a time.

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