A diagram chasing formalisation of elementary topological properties

misha gavrilovich*

in memoriam: evgenii shurygin

instances of human and animal behavior [...] miraculously complicated, [of] little, if any, pragmatic (survival/reproduction) value, are due to internal constraints on possible architectures of unknown to us functional "mental structures".

. Misha Gromov, Structures, Learning and Ergosystems.

Abstract

We introduce a simple formal syntax and use it to rewrite in a concise, uniform and intuitive way several standard definitions in topology which are usually expressed in words. The definitions include compact, discrete, connected, and totally disconnected spaces, dense image, induced topology, closed subsets, and some of the separation axioms. The syntax is of category-theoretic flavour and is based on the observation that these properties can all be defined category-theoretically by repeated application of a standard category theory trick, the Quillen lifting property (orthogonality of morphisms), starting from a single (counter)example.

We hope our reformulations may be use in formalisation of elementary topology and in teaching.

1 Introduction.

Generalities. Motivation. We introduce a simple formal syntax and use it to rewrite in a concise, uniform and intuitive way several standard definitions in topology which are usually expressed in words. The definitions include compact, discrete, connected, and totally disconnected spaces, dense image, induced topology, closed subsets, and some of the separation axioms.

The formal expressions are uniform and intuitive in the sense that each is based on the simplest (counter)example and treats orthogonality of morphisms as negation, i.e. as a way to avoid counterexamples. Arguably, they are short enough (several bytes) that a brute force algorithm can produce in practice a short list containing them all.

We hope this syntax may be of use in automatic theorem proving and in teaching elementary topology, and, if developed further, may allow to rewrite

*A draft; comments welcome. **mishap@sdf.org**. This draft is essentially a part of a more verbose draft on the expressive power of the lifting property available at http://mishap.sdf.org/expressive-power-of-the-lifting-property.pdf.

the traditional definitions in elementary topology into a new presentation based on diagram chasing that is easier for both humans and machines to understand. In a somewhat similar way the diagram chasing presentation[1] based on the axioms and formalism of a model category allows to rewrite some definitions of elementary homotopy theory. One cannot but wonder whether these elementary observations may provide a useful point of view on tame topology of Grothendieck, a foundation of topology "without false problems" and "wild phenomena" "at the very beginning". We also note that in a sense, our expressions are implicitly present in [Bourbaki, General Topology] but are described there in words.

The syntax is a very simple combinatorial formalism of category theoretic flavour based on finite preorders, or, equivalently, finite categories of certain kind. The semantics is based on viewing finite topological spaces as preorders or categories, and the Quillen lifting property (orthogonality of morphisms) in the category of topological spaces. The intuition (cf. [Gavrilovich, DMG]) is based on the following: (a) the expressions are based on simple counterexamples, often a map between a point and two points (a) treating the lifting property as negation (c) correlations between the diagrams and the text of the definitions.

Illustrative examples. As an illustration, let us list formal expressions for: compactness; surjective, subset, discrete, split; connected, injective; induced topology, separation axiom T_1; dense image, closed subset

```
 { {a}-->{a->b} }^r_{<5}^lr           compactness

   { {}-->{a} }^r                     surjective
   { {}-->{a} }^rr                    subset
   { {}-->{a} }^rl                    discrete
   { {}-->{a} }^lrl                   split morphisms

  { {a,b}-->{a=b} }^l                 connected
  { {a,b}-->{a=b} }^r                 injective

  { {a->b}-->{a=b} }^r                induced topology
  { {a->b}-->{a=b} }^l                separation axiom T1

  { {b}-->{a->b} }^l                  dense image
  { {b}-->{a->b} }^lr                 closed subset
```

Moreover, these expressions are straightforward to find once you note that they are based on the simplest counterexamples and treat orthogonality/lifting property as negation, e.g. `{a}-->{a->b}` denotes the simplest non-proper map sending a point into the open point of the space `{a->b}` with open point `a` and

[1] Someone interested in implementation may want to a look at a sample Python code [BaysQuilder, NOTES] and the appendix to [Gavrilovich, A homotopy approach to set theory, pp.13-16] to get some idea on how to view the axioms of a model category as diagram chasing rules.

closed point `b`; "negated" three times, this counterexample gives the notion of a proper map.

Now let us explain how to evaluate the expression for compactness in the category of topological spaces: first take (a class consisting of) the single non-proper map `{a}-->{a->b}` , the inclusion of the open point into the Sierpinski space, then take the subclass of its right orthogonal consisting of the maps between spaces each with less than 5 points (the subclass happens to be the class of all proper, equiv. closed, maps between such spaces), and then finally take its left, then right orthogonal.

An easy argument (cf. the proof of Claim 2) shows you thus obtain a subclass of the class of proper maps which contains all proper maps between finite spaces and all proper maps between normal (T4) spaces. We do not know if it is the class of proper maps (cf. Conjecture 1 below).

Our observations indicate that certain classical definitions and theorems of topology, such as Tychonoff theorem and implications between separation axioms, can be understood in a purely finitary way, as combinatorial rules of diagram chasing flavour.

Key observation The syntax is based on the following observation, which applies not only to topology:

> a number of elementary properties from a first-year course can be defined category-theoretically by repeated application of a standard category theory trick, the Quillen lifting property, starting from a class of explicitly given morphisms, often consisting of a single (counter)example

In this short note we discuss only examples of such properties in topology; they include the notions of: compact, discrete, connected, and totally disconnected spaces, dense image, induced topology, closed subsets, and separation axioms.

See [Gavrilovich, Lifting Property] for other examples. They include complete metric spaces, a subset of a metric space being closed. Examples in algebra include: finite groups being nilpotent, solvable, torsion-free, p-groups, and prime-to-p groups; injective and projective modules; injective, surjective, and split homomorphisms.

These examples and the observation above are elementary, yet we were unable to find them in literature.

Structure of the paper. The goal of this short note is a concise exposition of the examples in topology and directly related open questions. A reader interested in a broader context or a more verbose exposition might want to look at a draft [Gavrilovich, Lifting Property].

Further examples and speculations may be found in drafts [Gavrilovich, Lifting Property], [Gavrilovich, Tame topology]. There we offer detailed speculations on how to rewrite elementary topology in terms of diagram chasing rules and labelled finite categories.

History. These observations arose in an attempt to understand ideas of Misha Gromov [Memorandum Ergo] about ergologic/ergostructure/ergosystems.

Oversimplifying, ergologic is a kind of reasoning which helps to understand how to generate proper concepts, ask interesting questions, and, more generally, produce interesting rather than useful or correct behaviour. He conjectures there is a related class of mathematical, essentially combinatorial, structures, called *ergostructures* or *ergosystems*, and that this concept might eventually help to understand complex biological behaviour including learning and create mathematically interesting models of these processes.

We hope our observations may eventually help to uncover an essentially combinatorial reasoning behind elementary topology, and thereby suggest an example of an ergostructure.

From this point of view, our examples suggest a short algorithm to generate a number of interesting notions in topology.

Related works. This paper continues work started in [DMG], a rather leisurely introduction to some of the ideas presented here here which aims to express the intuition behind our syntax. Draft [Gavrilovich, Elementary Topology] shows how to view several topology notions and arguments in [Bourbaki, General Topology] as diagram chasing calculations with finite categories. Draft [Gavrilovich, Tame Topology] is more speculative but less verbose; it has several more examples dealing with compactness, in particular it shows that a number of consequences of compactness can be expressed as a change of order of quantifiers in a formula. Notably, these drafts show how to "read off" a simplicial topological space from the definition of a uniform space, see also Remark 7 and Conjecture 2 in [Gavrilovich, Lifting property].

2 Expressions for topological properties

2.1 The lifting property: the key observation

For a property C of arrows (morphisms) in a category, define

$$C^l := \{f : \text{ for each } g \in C \;\; f \rightthreetimes g\}$$
$$C^r := \{g : \text{ for each } f \in C \;\; f \rightthreetimes g\}$$
$$C^{lr} := (C^l)^r, \ldots$$

here $f \rightthreetimes g$ reads " f has the left lifting property wrt g ", " f is (left) orthogonal to g ", i.e. for $f : A \longrightarrow B$, $g : X \longrightarrow Y$, $f \rightthreetimes g$ iff for each $i : A \longrightarrow X$, $j : B \longrightarrow Y$ such that $ig = fj$ ("the square commutes"), there is $j' : B \longrightarrow X$ such that $fj' = i$ and $j'g = j$ ("there is a diagonal making the diagram commute").

The following observation is enough to reconstruct all the examples in this paper, with a bit of search and computation.

Observation.
A number of elementary properties can be obtained by repeatedly passing to the left or right orthogonal $C^l, C^r, C^{lr}, C^{ll}, C^{rl}, C^{rr}, \ldots$ starting from a simple class of morphisms, often a single (counter)example to the property you define.

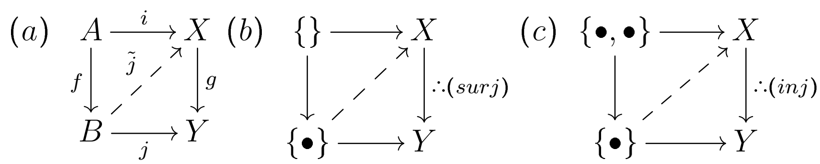

Figure 1: Lifting properties. (a) The definition of a lifting property $f \rightthreetimes g$. (b) $X \longrightarrow Y$ is surjective (c) $X \longrightarrow Y$ is injective

A useful intuition is to think that the property of left-lifting against a class C is a kind of negation of the property of being in C, and that right-lifting is another kind of negation. Hence the classes obtained from C by taking orthogonals an odd number of times, such as $C^l, C^r, C^{lrl}, C^{lll}$ etc., represent various kinds of negation of C, so $C^l, C^r, C^{lrl}, C^{lll}$ each consists of morphisms which are far from having property C.

Taking the orthogonal of a class C is a simple way to define a class of morphisms excluding non-isomorphisms from C, in a way which is useful in a diagram chasing computation.

The class C^l is always closed under retracts, pullbacks, (small) products (whenever they exist in the category) and composition of morphisms, and contains all isomorphisms of C. Meanwhile, C^r is closed under retracts, pushouts, (small) coproducts and transfinite composition (filtered colimits) of morphisms (whenever they exist in the category), and also contains all isomorphisms.

For example, the notion of isomorphism can be obtained starting from the class of all morphisms, or any single example of an isomorphism:

$$(Isomorphisms) = (all\ morphisms)^l = (all\ morphisms)^r = (h)^{lr} = (h)^{rl}$$

where h is an arbitrary isomorphism.

Example. Take $C = \{\varnothing \longrightarrow \{*\}\}$ in Sets and Top. Let us show that C^l is the class of surjections, C^{rr} is the class of subsets, C^l consists of maps $f : A \longrightarrow B$ such that either $A = B = \varnothing$ or $A \neq \varnothing$, B arbitrary. Further, in Sets, C^{rl} is the class of injections, and in Top, C^{rl} is the class of maps of form $A \longrightarrow A \cup D$, D is discrete; both in Sets and Top, C^{lr} is the class of maps $A \longrightarrow B$ such that either $A = \varnothing$ or the map is an isomorphism.

In both Sets and Top, $(\varnothing \longrightarrow \{*\})^{rr}$ is the class of subsets, i.e. injective maps $A \hookrightarrow B$ where the topology on A is induced from B.

2.2 Notation for maps of finite topological spaces

Now we introduce notation for maps of finite topological spaces we use. A topological space comes with a *specialisation preorder* on its points: for points $x, y \in X$, $x \leq y$ iff $y \in clx$, or equivalently, a category whose objects are points of X and there is a unique morphism $x \searrow y$ iff $y \in clx$.

For a finite topological space X, the specialisation preorder or equivalently the category uniquely determines the space: a subset of X is closed iff it is downward closed, or equivalently, there are no morphisms going outside the

The monotone maps (i.e. functors) are the continuous maps for this topology.

We denote a finite topological space by a list of the arrows (morphisms) in the corresponding category; '$\leftrightarrow$' denotes an isomorphism and '=' denotes the identity morphism; the empty list is denoted by either $\varnothing$ or $\{\}$. An arrow between two such lists denotes a continuous map (a functor) which sends each point to the correspondingly labelled point, but possibly turning some morphisms into identity morphisms, thus gluing together some points.

Thus, each point goes to "itself" and

$$\{a,b\} \longrightarrow \{a\searrow b\} \longrightarrow \{a \leftrightarrow b\} \longrightarrow \{a = b\}$$

denotes

$$(\textit{discrete space on two points}) \longrightarrow (\textit{Sierpinski space}) \longrightarrow (\textit{antidiscrete space}) \longrightarrow (\textit{single point})$$

In $A \longrightarrow B$, each object and each morphism in A necessarily appears in B as well; we avoid listing the same object or morphism twice. Thus both

$$\{a\} \longrightarrow \{a,b\} \text{ and } \{a\} \longrightarrow \{b\}$$

denote the same map from a single point to the discrete space with two points. Both

$$\{a\swarrow U\searrow x\swarrow V\searrow b\} \longrightarrow \{a\swarrow U = x = V\searrow b\} \text{ and } \{a\swarrow U\searrow x\swarrow V\searrow b\} \longrightarrow \{U = x = V\}$$

denote the morphism gluing together points U, x, V.

In $\{a\searrow b\}$, the point a is open and point b is closed.

Finally, for a class C of morphisms in Top, let

$$C_{<n} := \{f : f \in C,\ \text{both the domain and range of } f \text{ are finite of size less than } n\}.$$

$$C_{\text{fini}} := \{f : f \in C,\ \text{both the domain and range of } f \text{ are finite }\}.$$

2.3 Expressions for elementary properties of topological spaces

Toying with the observation leads to the examples in the claim below which is trivial to verify, an exercise in deciphering the notation in all cases but (i) compactness.

Claim 1. *(i) In the category of topological spaces, the following holds:*

a Hausdorff space K is compact iff $K \longrightarrow \{*\}$ is in $((\{a\} \longrightarrow \{a\searrow b\})^r_{<5})^{lr}$

a Hausdorff space K is compact iff $K \longrightarrow \{*\}$ is in

$$\{\ \{a \leftrightarrow b\} \longrightarrow \{a = b\},\ \{a\searrow b\} \longrightarrow \{a = b\},\ \{b\} \longrightarrow \{a\searrow b\},\ \{a\swarrow o\searrow b\} \longrightarrow \{a = o = b\}\ \}^{lr}$$

a space D is discrete iff $\varnothing \longrightarrow D$ is in $(\varnothing \longrightarrow \{*\})^{rl}$

a space D is antidiscrete iff $D \longrightarrow \{*\}$ is in $(\{a,b\} \longrightarrow \{a=b\})^{rr} = (\{a \leftrightarrow b\} \longrightarrow \{a=b\})^{lr}$

a space K is connected or empty iff $K \longrightarrow \{*\}$ is in $(\{a,b\} \longrightarrow \{a=b\})^{l}$

a space K is totally disconnected and non-empty iff $K \longrightarrow \{*\}$ is in $(\{a,b\} \longrightarrow \{a=b\})^{lr}$

a space K is connected and non-empty iff for some arrow $\{*\} \longrightarrow K$
$\{*\} \longrightarrow K$ is in $(\varnothing \longrightarrow \{*\})^{rll} = (\{a\} \longrightarrow \{a,b\})^{l}$

a space K is non-empty iff $K \longrightarrow \{*\}$ is in $(\varnothing \longrightarrow \{*\})^{l}$

a space K is empty iff $K \longrightarrow \{*\}$ is in $(\varnothing \longrightarrow \{*\})^{ll}$

a space K is T_0 iff $K \longrightarrow \{*\}$ is in $(\{a \leftrightarrow b\} \longrightarrow \{a=b\})^{r}$

a space K is T_1 iff $K \longrightarrow \{*\}$ is in $(\{a \searrow b\} \longrightarrow \{a=b\})^{r}$

a space X is Hausdorff iff for each injective map $\{x,y\} \hookrightarrow X$ it holds $\{x,y\} \hookrightarrow X \rightthreetimes \{x \searrow o \swarrow y\} \longrightarrow \{x=o=y\}$

a non-empty space X is regular (T3) iff for each arrow $\{x\} \longrightarrow X$ it holds $\{x\} \longrightarrow X \rightthreetimes \{x \searrow X \swarrow U \searrow F\} \longrightarrow \{x = X = U \searrow F\}$

a space X is normal (T4) iff $\varnothing \longrightarrow X \rightthreetimes \{a \swarrow U \searrow x \swarrow V \searrow b\} \longrightarrow \{a \swarrow U = x = V \searrow b\}$

a space X is completely normal iff $\varnothing \longrightarrow X \rightthreetimes [0,1] \longrightarrow \{0 \swarrow x \searrow 1\}$ where the map $[0,1] \longrightarrow \{0 \swarrow x \searrow 1\}$ sends 0 to 0, 1 to 1, and the rest $(0,1)$ to x

a space X is path-connected iff $\{0,1\} \longrightarrow [0,1] \rightthreetimes X \longrightarrow \{*\}$

a space X is path-connected iff for each Hausdorff compact space K and each injective map $\{x,y\} \hookrightarrow K$ it holds $\{x,y\} \hookrightarrow K \rightthreetimes X \longrightarrow \{*\}$

$(\varnothing \longrightarrow \{*\})^{r}$ is the class of surjections

$(\varnothing \longrightarrow \{*\})^{rr}$ is the class of subsets, i.e. injective maps $A \hookrightarrow B$ where the topology on A is induced from B

$(\varnothing \longrightarrow \{*\})^{lrl}$ is the class of maps $A \longrightarrow B$ which split

$(\{b\} \longrightarrow \{a \searrow b\})^{l}$ is the class of maps with dense image

$(\{b\} \longrightarrow \{a \searrow b\})^{lr}$ is the class of closed subsets $A \subset X$, A a closed subset of X

(ii) in the category of topological spaces,
for a connected topological space X, each function on X is bounded iff

$$\varnothing \longrightarrow X \rightthreetimes \cup_n(-n,n) \longrightarrow \mathbb{R}$$

(iii) *in the category of metric spaces and uniformly continuous maps, a metric space X is complete iff* $\{1/n\}_n \longrightarrow \{1/n\}_n \cup \{0\} \rightthreetimes X \longrightarrow \{0\}$ *where the metric on* $\{1/n\}_n$ *and* $\{1/n\}_n \cup \{0\}$ *is induced from the real line a subset* $A \subset X$ *is closed iff* $\{1/n\}_n \longrightarrow \{1/n\}_n \cup \{0\} \rightthreetimes A \longrightarrow X$

Proof. We defer the proof of the first two items to the next claim. The rest is straightforward to verify using the definitions. □

Many of the separation axioms can be expressed as lifting properties with respect to maps involving up to 4 points and the real line, see [Appendix A].

The following is a list of properties which are defined using the lifting property starting from a single morphism between spaces of at most two points and whose meaning is easy to describe in words.

Claim 2. *In the category of topological spaces, it holds:*

$((\{a\} \longrightarrow \{a\searrow b\})^r_{<5})^{lr}$ *is almost the class of proper maps, namely a map of T4 spaces is in the class iff it is proper*

$(\{b\} \longrightarrow \{a\searrow b\})^l$ *is the class of maps with dense image*

$(\{b\} \longrightarrow \{a\searrow b\})^{lr}$ *is the class of maps of closed inclusions* $A \subset X$, A *is closed*

$(\varnothing \longrightarrow \{*\})^r = (\{0\} \longrightarrow \{0 \leftrightarrow 1\})^l$ *is the class of surjections*

$(\varnothing \longrightarrow \{*\})^{rl}$ *is the class of maps of form* $A \longrightarrow A \cup D$, D *is discrete*

$(\varnothing \longrightarrow \{*\})^{rll} = (\{a\} \longrightarrow \{a, b\})^l$ *is the class of maps* $A \longrightarrow B$ *such that each open closed non-empty subset of* B *intersects* $Im A$.

$(\varnothing \longrightarrow \{*\})^l$ *is the class of maps* $A \longrightarrow B$ *such that* $A = B = \varnothing$ *or* $A \neq \varnothing$, B *arbitrary*

$(\varnothing \longrightarrow \{*\})^{lr}$ *is the class of maps* $A \longrightarrow B$ *such that either* $A = \varnothing$ *or the map is an isomorphism*

$(\varnothing \longrightarrow \{*\})^{lrl}$ *is the class of maps* $A \longrightarrow B$ *which split*

$(\varnothing \longrightarrow \{*\})^{rr}$ *is the class of subsets, i.e. injective maps* $A \hookrightarrow B$ *where the topology on* A *is induced from* B.

$(\{a \leftrightarrow b\} \longrightarrow \{a = b\})^l$ *is the class of injections*

$(\{a\searrow b\} \longrightarrow \{a = b\})^l$ *is the class of maps* $f : X \longrightarrow Y$ *such that the topology on* X *is induced from* Y

$(\{a, b\} \longrightarrow \{a = b\})^l$ *describes being connected, and is the class of maps* $f : X \longrightarrow Y$ *such that* $f(U) \cap f(V) = \varnothing$ *for each two open closed subsets* $U \neq V$ *of* X; *if both* X *and* Y *are unions of open closed connected subsets, this means that the map* $\pi_0(X) \hookrightarrow \pi_0(Y)$ *is injective*

$(\{a \leftrightarrow b\} \longrightarrow \{a = b\})^r$ *fibres are T0 spaces*

$(\{a\searrow b\} \longrightarrow \{a=b\})^r$ *fibres are T1 spaces*

$(\{a,b\} \longrightarrow \{a=b\})^r$ *is the class of injections*

$(\{a\} \longrightarrow \{a\leftrightarrow b\})^l$ *is the class of surjections*

$(\{a\} \longrightarrow \{a\leftrightarrow b\})^r$ *is the class of surjections*

$(\{b\} \longrightarrow \{a\searrow b\})^l$ *something T1-related but not particularly nice*

$(\{a\} \longrightarrow \{a\searrow b\})^l$ *something T0-related*

$(\{a\} \longrightarrow \{a,b\})^l$ *is the class of maps* $f : X \longrightarrow Y$ *such that either* X *is empty or* f *is surjective*

Proof. All items are trivial to verify, with the possible exception of the first item. [Bourbaki, General Topology, I10.2, Thm.1(d), p.101], quoted in Appendix B, gives a characterisation of proper maps by a lifting property with respect to maps associated to ultrafilters. Using this it is easy to check that each map in $(\{a\} \longrightarrow \{a\searrow b\})^r_{<5}$ being closed, hence proper, implies that each map in $((\{a\} \longrightarrow \{a\searrow b\})^r_{<5})^{lr}$ is proper. A theorem of [Taimanov], cf. [Engelking, 3.2.1,p.136], quoted in Appendix B, states that for a compact Hausdorff space K, a Hausdorff space K is compact iff the map $K \longrightarrow \{*\}$ is in C_T^{lr} where

$$C_T := \{\, \{a\leftrightarrow b\} \longrightarrow \{a=b\},\, \{a\searrow b\} \longrightarrow \{a=b\},$$

$$\{b\} \longrightarrow \{a\searrow b\},\, \{a\swarrow o\searrow b\} \longrightarrow \{a=o=b\} \,\}$$

It is easy to check that all the maps listed in the formula above are closed, hence proper, and therefore

$$C_T^{lr} \subseteq ((\{a\} \longrightarrow \{a\searrow b\})^r_{<5})^{lr}$$

Finally, note that the proof of Taimanov theorem generalises to give that a proper map between normal Hausdorff (T4) spaces is in the larger class. □

Remark 1. Note that the proof of the Tychonoff theorem via ultrafilters is viewed as a formal property of class of morphisms defined by Quillen lifting properties. The standard proof of Urysohn lemma is viewed as an infinite application of the lifting property characterising axiom T4 and passing to the limit.

Conjecture 1. *In the category of topological spaces,*

$$((\{a\} \longrightarrow \{a\searrow b\})^r_{<5})^{lr}$$

is the class of proper maps.

Remark 2. It is easy to see that $((\{a\} \longrightarrow \{a \searrow b\})^r_{<m})^{lr} \subset ((\{a\} \longrightarrow \{a \searrow b\})^r_{<n})^{lr}$ for any $m < n$. However, I do not know whether there is $n > m > 3$ such that the inclusion is strict. An example using cofinite topology (suggested by Sergei Kryzhevich) shows that C_T^{lr} does not define the class of compact spaces: indeed, consider infinite sets $A \subset B$, $\omega \leq \operatorname{card} A < \operatorname{card} B$, equipped with cofinite topology (i.e. a subset is closed iff it is finite). Then $A \subseteq B \in C_T^l$ yet $A \subseteq B \not\rightthreetimes A \longrightarrow \{*\}$ fails: for a map $f : B \longrightarrow A$ the preimage of some (necessarily closed) point is infinite as $\operatorname{card} B > \operatorname{card} A$, hence not closed, and the map is not continious. Hence, $A \longrightarrow \{*\} \notin C_T$ yet A is compact (non-Hausdorff). This example could probably be generalised to show that that $((\{a\} \longrightarrow \{a \searrow b\})^r_{<4})^{lr} \subsetneq ((\{a\} \longrightarrow \{a \searrow b\})^r_{<5})^{lr}$.

Question 1. (a) Calculate

$$((\{b\} \longrightarrow \{a \searrow b\})^r_{<5})^{lr},\ ((\{b\} \longrightarrow \{a \searrow b\})^{lrr}, \text{ and}$$

$$(\{a \swarrow U \searrow x \swarrow V \searrow b\} \longrightarrow \{a \swarrow U = x = V \searrow b\})^{lr}$$

Could either be viewed as a "definition" of the real line?

(b) Characterise the interval $[0,1]$, a circle $\mathbb{S}^1$ and, more generally, spheres $\mathbb{S}^n$ using their topological characterisations provided by the Kline sphere charterisation theorem and its analogues. An example of such a characterisation is that a topological space X is homomorphic to the circle $\mathbb{S}^1$ iff X is a connected Hausdorff metrizable space such that $X \setminus \{x,y\}$ is not connected for any two points $x \neq y \in X$ ([Hocking,Young, Topology, Thm.2-28,p.55]); another example is that a topological space X is homomorphic to the closed interval $[0,1]$ iff X is a connected Hausdorff metrizable space such that $X \setminus \{x\}$ is not connected for exactly two points $x \neq y \in X$ ([Hocking,Young, Topology, Thm.2-27,p.54]).

Remark 3. Is there a model category or a factorisation system of interest associated with any of these lifting properties, for example compactness/properness?

Many of the separation axioms can be expressed as lifting properties with respect to maps involving up to 4 points and the real line, see [Appendix A].

2.4 Hausdorff axioms of topology as diagram chasing computations with finite categories

Above we reformulated a number of elementary notions in topology in terms of preorders. Now we observe that the original axioms of topology as formulated by Hausdorff can also be viewed as rules for manipulating finite preorders.

Early works talk of topology in terms of *neighbourhood* systems U_x where U_x varies though *open neighbourhoods of points* of a topological space; this is how the notion of topology was defined by Hausdorff. In the notation of arrows, *a neighbourhood system* U_x, $x \in X$ would correspond to a system of arrows

$$\{x\} \longrightarrow X \xrightarrow{U} \{x \searrow x'\}$$

and Hausdorff's axioms (A),(B),(C) (see Appendix B) would correspond to diagram chasing rules.

Here we show the axioms of topology stated in the more modern language of open subsets can be seen as diagram chasing rules for manipulating diagrams involving notation such as

$$\{x\} \longrightarrow X,\ X \longrightarrow \{x \searrow y\},\ X \longrightarrow \{x \leftrightarrow y\}$$

in the following straightforward way; cf. [Gavrilovich, Elementary Topology,.2.1] for more details.

As is standard in category theory, identify a point x of a topological space X with the arrow $\{x\} \longrightarrow X$, a subset Z of X with the arrow $X \longrightarrow \{z \leftrightarrow z'\}$, and an open subset U of X with the arrow $X \longrightarrow \{u \searrow u'\}$. With these identifications, the Hausdorff axioms of a topological space become rules for manipulating such arrows, as follows.

Both the empty set and the whole of X are open says that the compositions

$$X \longrightarrow \{c\} \longrightarrow \{o \searrow c\} \text{ and } X \longrightarrow \{o\} \longrightarrow \{o \searrow c\}$$

behave as expected (the preimage of {o} is empty under the first map, and is the whole of X under the second map).

The intersection of two open subsets is open means the arrow

$$X \longrightarrow \{o \searrow c\} \times \{o' \searrow c'\}$$

behaves as expected (the "two open subsets" are the preimages of points $o \in \{o \searrow c\}$ and $o' \in \{o' \searrow c'\}$; "the intersection" is the preimage of (o, o') in $\{o \searrow c\} \times \{o' \searrow c'\}$).

Finally, *a subset U of X is open iff each point u of U has an open neighbourhood inside of U* corresponds to the following diagram chasing rule:

for each arrow $X \underset{\xi_U}{\longrightarrow} \{U \longleftrightarrow \bar{U}\}$ it holds

$$\begin{array}{ccc} & & \{U \to \bar{U}\} \\ & \nearrow & \downarrow \\ X & \xrightarrow[\xi_U]{} & \{U \longleftrightarrow \bar{U}\} \end{array} \quad \text{iff for each } \{u\} \longrightarrow X, \quad \begin{array}{ccc} \{u\} & \longrightarrow & \{u \to U \longleftrightarrow \bar{U}\} \\ \downarrow & \nearrow & \downarrow \\ X & \xrightarrow[\xi_U]{} & \{u = U \longleftrightarrow \bar{U}\} \end{array}$$

The preimage of an open set is open corresponds to the composition

$$X \longrightarrow Y \longrightarrow \{u \searrow u'\} \longrightarrow \{u \leftrightarrow u'\}.$$

This observation suggests that some arguments in elementary topology may be understood entirely in terms of diagram chasing, see [Gavrilovich, Elementary Topology] for some examples. We hope that this reinterpretation may help clarify the nature of the axioms of a topological space, in particular it offers a constructive approach, may clarify to what extent set-theoretic language is necessary, and perhaps help to suggest an approach to "tame topology" of

Grothendieck, i.e. a foundation of topology “without false problems” and ”wild phenomena” ”at the very beginning”.

Acknowledgements and historical remarks. To be written.

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Much of this paper was done in St.Petersburg; it wouldn’t have been possible without support of family and friends who created an excellent social environment and who occasionally accepted an invitation for a walk or a coffee or extended an invitation; alas, I made such a poor use of it all.

This note is elementary, and it was embarrassing and boring, and embarrassingly boring, to think or talk about matters so trivial, but luckily I had no obligations for a time.

3 Appendix A. Separation axioms as lifting properties (from Wikipedia)

The separation axioms are lifting properties with respect to maps involving up to 4 points and the real line. What follows below is the text of the Wikipedia page on the separation axioms where we added lifting properties formulae expressing what is said there in words.

Let X be a topological space. Then two points x and y in X are *topologically distinguishable* iff the map $\{x \leftrightarrow y\} \longrightarrow X$ is not continuous, i.e. iff at least one of them has an open neighbourhood which is not a neighbourhood of the other.

Two points x and y are *separated* iff neither $\{x\searrow y\} \longrightarrow X$ nor $\{x\searrow y\} \longrightarrow X$ is continuous, i.e each of them has a neighbourhood that is not a neighbourhood of the other; in other words, neither belongs to the other's closure, $x \notin cl\, x$ and $y \notin cl\, x$. More generally, two subsets A and B of X are *separated* iff each is disjoint from the other's closure, i.e. $A \cap clB = B \cap clA = \varnothing$. (The closures themselves do not have to be disjoint.) In other words, the map $i_{AB} : X \longrightarrow \{A \leftrightarrow x \leftrightarrow B\}$ sending the subset A to the point A, the subset B to the point B, and the rest to the point x, factors both as

$$X \longrightarrow \{A \leftrightarrow U_A \searrow x \leftrightarrow B\} \longrightarrow \{A = U_A \leftrightarrow x \leftrightarrow B\}$$

and

$$X \longrightarrow \{A \leftrightarrow x \swarrow U_B \leftrightarrow B\} \longrightarrow \{A \leftrightarrow x \leftrightarrow U_B = B\}$$

here the preimage of x, B, resp. x, A is a closed subset containing B, resp. A, and disjoint from A, resp. B. All of the remaining conditions for separation of sets may also be applied to points (or to a point and a set) by using singleton sets. Points x and y will be considered separated, by neighbourhoods, by closed neighbourhoods, by a continuous function, precisely by a function, iff their singleton sets $\{x\}$ and $\{y\}$ are separated according to the corresponding criterion.

Subsets A and B are *separated by neighbourhoods* iff A and B have disjoint neighbourhoods, i.e. iff $i_{AB} : X \longrightarrow \{A \leftrightarrow x \leftrightarrow B\}$ factors as

$$X \longrightarrow \{A \leftrightarrow U_A \searrow x \swarrow U_B \leftrightarrow B\} \longrightarrow \{A = U_A \leftrightarrow x \leftrightarrow U_B = B\}$$

here the disjoint neighbourhoods of A and B are the preimages of open subsets A, U_A and U_B, B of $\{A \leftrightarrow U_A \searrow x \swarrow U_B \leftrightarrow B\}$, resp. They are *separated by closed neighbourhoods* iff they have disjoint closed neighbourhoods, i.e. i_{AB} factors as

$$X \longrightarrow \{A \leftrightarrow U_A \searrow U'_A \swarrow x \searrow U'_B \swarrow U_B \leftrightarrow B\} \longrightarrow \{A \leftrightarrow U_A = U'_A = x = U'_B = U_B \leftrightarrow B\}.$$

They are *separated by a continuous function* iff there exists a continuous function f from the space X to the real line $\mathbb{R}$ such that $f(A) = 0$ and $f(B) = 1$, i.e. the map i_{AB} factors as

$$X \longrightarrow \{0'\} \cup [0,1] \cup \{1'\} \longrightarrow \{A \leftrightarrow x \leftrightarrow B\}$$

where points $0', 0$ and $1, 1'$ are topologically indistinguishable, and $0'$ maps to A, and $1'$ maps to B, and $[0, 1]$ maps to x. Finally, they are *precisely separated by a continuous function* iff there exists a continuous function f from X to $\mathbb{R}$ such that the preimage $f^{-1}(\{0\}) = A$ and $f^{-1}(\{1\}) = B$. i.e. iff i_{AB} factors as

$$X \longrightarrow [0, 1] \longrightarrow \{A \leftrightarrow x \leftrightarrow B\}$$

where 0 goes to point A and 1 goes to point B.

These conditions are given in order of increasing strength: Any two topologically distinguishable points must be distinct, and any two separated points must be topologically distinguishable. Any two separated sets must be disjoint, any two sets separated by neighbourhoods must be separated, and so on.

The definitions below all use essentially the preliminary definitions above.

In all of the following definitions, X is again a topological space.

X is T0, or Kolmogorov, if any two distinct points in X are topologically distinguishable. (It will be a common theme among the separation axioms to have one version of an axiom that requires T0 and one version that doesn't.) As a formula, this is expressed as

$$\{x \leftrightarrow y\} \longrightarrow \{x = y\} \rightthreetimes X \longrightarrow \{*\}$$

X is R0, or symmetric, if any two topologically distinguishable points in X are separated, i.e.

$$\{x \searrow y\} \longrightarrow \{x \leftrightarrow y\} \rightthreetimes X \longrightarrow \{*\}$$

X is T1, or accessible or Frechet, if any two distinct points in X are separated, i.e.

$$\{x \searrow y\} \longrightarrow \{x = y\} \rightthreetimes X \longrightarrow \{*\}$$

Thus, X is T1 if and only if it is both T0 and R0. (Although you may say such things as "T1 space", "Frechet topology", and "Suppose that the topological space X is Frechet", avoid saying "Frechet space" in this context, since there is another entirely different notion of Frechet space in functional analysis.)

X is R1, or preregular, if any two topologically distinguishable points in X are separated by neighbourhoods. Every R1 space is also R0.

X is weak Hausdorff, if the image of every continuous map from a compact Hausdorff space into X is closed. All weak Hausdorff spaces are T1, and all Hausdorff spaces are weak Hausdorff.

X is Hausdorff, or T2 or separated, if any two distinct points in X are separated by neighbourhoods, i.e.

$$\{x, y\} \hookrightarrow X \rightthreetimes \{x \searrow X \swarrow y\} \longrightarrow \{x = X = y\}$$

Thus, X is Hausdorff if and only if it is both T0 and R1. Every Hausdorff space is also T1.

X is $T2\frac{1}{2}$, or Urysohn, if any two distinct points in X are separated by closed neighbourhoods, i.e.

$$\{x,y\} \hookrightarrow X \rightthreetimes \{x \searrow x' \swarrow X \searrow y' \swarrow y\} \longrightarrow \{x = x' = X = y' = y\}$$

Every T2$\frac{1}{2}$ space is also Hausdorff.

X is completely Hausdorff, or completely T2, if any two distinct points in X are separated by a continuous function, i.e.

$$\{x,y\} \hookrightarrow X \rightthreetimes [0,1] \longrightarrow \{*\}$$

where $\{x,y\} \hookrightarrow X$ runs through all injective maps from the discrete two point space $\{x,y\}$.

Every completely Hausdorff space is also T2$\frac{1}{2}$.

X is regular if, given any point x and closed subset F in X such that x does not belong to F, they are separated by neighbourhoods, i.e.

$$\{x\} \longrightarrow X \rightthreetimes \{x \searrow X \swarrow U \searrow F\} \longrightarrow \{x = X = U \searrow F\}$$

(In fact, in a regular space, any such x and$F$ will also be separated by closed neighbourhoods.) Every regular space is also R1.

X is regular Hausdorff, or T3, if it is both T0 and regular.[1] Every regular Hausdorff space is also $T2\frac{1}{2}$.

X is completely regular if, given any point x and closed set F in X such that x does not belong to F, they are separated by a continuous function, i.e.

$$\{x\} \longrightarrow X \rightthreetimes [0,1] \cup \{F\} \longrightarrow \{x \searrow F\}$$

where points F and 1 are topologically indistinguishable, $[0,1]$ goes to x, and F goes to F.

Every completely regular space is also regular.

X is Tychonoff, or T3$\frac{1}{2}$, completely T3, or completely regular Hausdorff, if it is both T0 and completely regular.[2] Every Tychonoff space is both regular Hausdorff and completely Hausdorff.

X is normal if any two disjoint closed subsets of X are separated by neighbourhoods, i.e.

$$\varnothing \longrightarrow X \rightthreetimes \{x \swarrow x' \searrow X \swarrow y' \searrow y\} \longrightarrow \{x \swarrow x' = X = y' \searrow y\}$$

In fact, by Urysohn lemma a space is normal if and only if any two disjoint closed sets can be separated by a continuous function, i.e.

$$\varnothing \longrightarrow X \rightthreetimes \{0'\} \cup [0,1] \cup \{1'\} \longrightarrow \{0 = 0' \searrow x \swarrow 1 = 1'\}$$

where points $0', 0$ and $1, 1'$ are topologically indistinguishable, $[0,1]$ goes to x, and both $0, 0'$ map to point $0 = 0'$, and both $1, 1'$ map to point $1 = 1'$.

X is normal Hausdorff, or T4, if it is both T1 and normal. Every normal Hausdorff space is both Tychonoff and normal regular.

X is completely normal if any two separated sets A and B are separated by neighbourhoods $U \supset A$ and $V \supset B$ such that U and V do not intersect, i.e.????

$$\varnothing \longrightarrow X \leftthreetimes \{X \swarrow A \leftrightarrow U \searrow U' \swarrow W \searrow V' \swarrow V \leftrightarrow B \searrow X\} \longrightarrow \{U = U', V' = V\}$$

Every completely normal space is also normal.

X is perfectly normal if any two disjoint closed sets are precisely separated by a continuous function, i.e.

$$\varnothing \longrightarrow X \leftthreetimes [0,1] \longrightarrow \{0 \swarrow X \searrow 1\}$$

where $(0,1)$ goes to the open point X, and 0 goes to 0, and 1 goes to 1. Every perfectly normal space is also completely normal.

X is extremally disconnected if the closure of every open subset of X is open, i.e.

$$\varnothing \longrightarrow X \leftthreetimes \{U \searrow Z', Z \swarrow V\} \longrightarrow \{U \searrow Z' = Z \swarrow V\}$$

or equivalently

$$\varnothing \longrightarrow X \leftthreetimes \{U \searrow Z', Z \swarrow V\} \longrightarrow \{Z' = Z\}$$

4 Appendix B. Quotations from sources.

For reader's convenience we quote here from the several sources we use.

[Bourbaki, General Topology, I10.2, Thm.1(d), p.101]:

> THEOREM I. Let $f : X \longrightarrow Y$ be a continuous mapping. Then the following four statements are equivalent:
>
> a) f is proper.
>
> b) f is closed and $f^{-1}(y)$ is quasi-compact for each $y \in Y$.
>
> c) If $\mathfrak{F}$ is a filter on X and if $y \in Y$ is a cluster point of $f(\mathfrak{F})$ then there is a cluster point x of such that $f(x) = y$.
>
> d) If $\mathfrak{U}$ is an ultrafilter on X and if $y \in Y$ is a limit point of the ultrafilter base $f(\mathfrak{U})$, then there is a limit point x of $\mathfrak{U}$ such that $f(x) = y$.

[Engelking, 3.2.1,p.136] ("compact" below stands for "compact Hausdorff"):

> 3.2.1. THEOREM. Let A be a dense subspace of a topological space X and f a continuous mapping of A to a compact space Y. The mapping f has a continuous extension over X if and only if for every pair B_1, B_2 of disjoint closed subsets of Y the inverse images $f^{-1}(B_1)$ and $f^{-1}(B_2)$ have disjoint closures in the space X.

[Hausdorff, Set theory, 40, p.259] ("ε" stands for "ϵ", and "$U_x V_x$" stands for "$U_x \cap V_x$") :

> From the theorems about open sets we derive the following properties of the neighborhoods:
>
> (A) Every point x has at least one neighborhood U_x; and U_x always contains x.
>
> (B) For any two neighborhoods U_x and V_x of the same point, there exists a third, $W_x \le U_x V_x$.
>
> (C) Every point $y \varepsilon U_z$ has a neighborhood $U_y \le U_x$.
>
> It is now again possible to treat neighborhoods as unexplained concepts and to use them as our starting point, postulating Theorems (A), (B), and (C) as neighborhood axioms.[1] Open sets G are then defined as sums of neighborhoods or as sets in which every point $x \varepsilon G$ has a neighborhood $U_x \le G$ (the null set included). Theorems (1), (2), and (3) about open sets are then provable.
>
> ....
>
> [1] Such a program was carried through in the first edition of this book. [Grund- zügeder Mengenlehre. (Leipzig, 1914; repr. New York, 1949),]

5 Appendix C. Examples of lifting properties in other categories.

Here we give a list of elementary notions in algebra defined by lifting properties.

Claim 3. *(i)* $(\varnothing \longrightarrow \{*\})^r$, $(0 \longrightarrow R)^r$, *and* $\{0 \longrightarrow \mathbb{Z}\}^r$ *are the classes of surjections in in tha categories of Sets, R-modules, and Groups, resp., (where* $\{*\}$ *is the one-element set, and in the category of (not necessarily abelian) groups,* 0 *denotes the trivial group)*

(ii) $(\{\star,\bullet\} \longrightarrow \{*\})^l = (\{\star,\bullet\} \longrightarrow \{*\})^r$, $(R \longrightarrow 0)^r$, $\{\mathbb{Z} \longrightarrow 0\}^r$ *are the classes of injections in the categories of Sets, R-modules, and Groups, resp*

(iii) *in the category of R-modules,*
a module P *is projective iff* $0 \longrightarrow P$ *is in* $(0 \longrightarrow R)^{rl}$
a module I *is injective iff* $I \longrightarrow 0$ *is in* $(R \longrightarrow 0)^{rr}$

(iv) *in the category of Groups,*

a finite group H *is nilpotent iff* $H \longrightarrow H \times H$ *is in* $\{0 \longrightarrow G : G \text{ arbitrary}\}^{lr}$

a finite group H *is solvable iff* $0 \longrightarrow H$ *is in* $\{0 \longrightarrow A : A \text{ abelian}\}^{lr} = \{[G,G] \longrightarrow G : G \text{ arbitrary}\}^{lr}$

a finite group H *is of order prime to* p *iff* $H \longrightarrow 0$ *is in* $\{\mathbb{Z}/p\mathbb{Z} \longrightarrow 0\}^r$

a finite group H *is a* p*-group iff* $H \longrightarrow 0$ *is in* $\{\mathbb{Z}/p\mathbb{Z} \longrightarrow 0\}^{rr}$

a group H *is torsion-free iff* $0 \longrightarrow H$ *is in* $\{n\mathbb{Z} \longrightarrow \mathbb{Z} : n > 0\}^r$

a group F *is free iff* $0 \longrightarrow F$ *is in* $\{0 \longrightarrow \mathbb{Z}\}^{rl}$

a homomorphism f *is split iff* $f \in \{0 \longrightarrow G : G \text{ arbitrary}\}^r$

Proof. In (iv), we use that a finite group H is nilpotent iff the diagonal $\{(h,h) : h \in H\}$ is subnormal in $H \times H$. Other examples are straightforward to verify using the definitions. □

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