Tame topology: a naive elementary approach via finite topological spaces an unproofread draft

To Evgenii Shurygin In memoriam

Misha Gavrilovich mishap@sdf.org *

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1 Idea

We observe that several elementary properties can be defined starting from a single morphism by iterating the Quillen lifting property and restricting to size at most 4. This includes the properties of a topological space being compact or being discrete, the separation axioms, a finite group being solvable, a p-group, a group of order prime to p.

We also observe that several elementary notions in topology can be expressed in a very concise and uniform manner using the Quillen lifting property and finite topological spaces, and, further, that parts of several standard elementary arguments in topology correspond to manipulations (calculations) with arrows and labels, of category-theory style.

We suggest it is worthwhile to try to develop a set of rules for manipulating arrows and labels which allow to represent standard elementary arguments in topology as calculations, both human readable and computer verifiable.

Should this be possible, we suggest it is then worthwhile to think whether these rules lead to an approach to the tame topology of Grothendieck, i.e. a foundation of topology "without false problems" and "wild phenomena" "at the very beginning".

It appears that our observations suggest it is worthwhile to try to develop the abstract theory the Quillen lifting property and examples in specific categories including the categories of finite groups, topological spaces, and metric spaces.

An earlier draft [Lifting Properties] tries to show how to "read off" our observations from the text of [Bourbaki, General Topology].

Structure of the paper. §2 and §3 introduce notaton and some generalities; §4 lists examples of lifting properties in various categories; §5 discusses several definitions of compactness; §6 observes that a uniform structure gives rise to a simplicial object; §7 is devoted to open questions and speculations.

§5.1 is the heart of the paper: it presents the ultrafilter definition of compactness as a lifting property.

2 Lifting property.

Let $f \rightthreetimes g$ denote, for $f : A \longrightarrow B$, $g : X \longrightarrow Y$, that for each $i : A \longrightarrow X$, $j : B \longrightarrow Y$ such that $ig = fj$ there is $j' : B \longrightarrow X$ such that $fj' = i$ and $j'g = j$.

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*http://mishap.sdf.org/mints-lifting-property-as-negation
Higher School of Economics, Soyza Pechatnikov str., 16, St.Petersburg, Russia.
St. Petersburg Institute for Economics and Mathematics of the Russian Academy of Sciences.

For a class (property) C of arrows in a category, let

$$C^l := \{f^l : f^l \rightthreetimes g \text{ for each } g \in C\}$$

$$C^r := \{g^r : f \rightthreetimes g^r \text{ for each } f \in C\}$$

Note $C^l \cap C \subset Isomorphisms$, $C^r \cap C \subset Isomorphisms$, and $C \subset C^{rl}$, $C \subset C^{lr}$, $C^{rlr} = C^r$ and $C^{lrl} = C^l$, and that C^l, C^{rl} are subcategories closed under limits existing in the ambient category, and similarly C^r, C^{lr} is subcategories closed under colimits existing in the ambient category.

We call C^l and C^r *left- and right-negation* of property C, or simply $\rightthreetimes$*-negation* ; sometimes we call double negations C^{lr} and C^{rl} $\rightthreetimes$*-generalisations* of property C, or say that property C^{lr} and C^{rl} is *exemplified by property C.*

Perhaps the easiest category theoretic way to define a class of morphism without a given property is by taking left or right negation (i.e. lifting property) against all morphisms with the property, hence the terminology.

3 Diagram chasing as a computation

We like to think of $f \rightthreetimes g$ as a diagram chasing rule which turns a commutative diagram "with a hole" into a linearly ordered commutative diagram

$$A \xrightarrow{f} B \xrightarrow{j'} X \xrightarrow{g} Y$$

hence simplifying the diagram chasing computation.

We like to think of properties of morphisms as *labels on arrows.*

We like to think of a property (C) of a morphism as *a label (c) on an arrow* denoted

$$X \xrightarrow{(c)} Y.$$

We like to think of *diagram chasing* as a computation adding *arrows* and *labels* to commutative diagrams, and adding rules for doing so.

For example, *taking right negation* of a label (C) consists of introducing the following rules:

(i) given a commutative square labelled as shown, add diagonal arrow $B \longrightarrow X$

$$\begin{tikzcd} A \arrow[r] \arrow[d, "(C)"'] & X \arrow[d, "(C^r)"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

(ii) given an arbitrary arrow $X \longrightarrow Y$, you may add new objects A,B and new arrows $A \longrightarrow X$, $A \longrightarrow B \longrightarrow Y$ forming a commutative square as shown, and add the rule:

(ii)' if there is an arrow $B \longrightarrow X$ making the diagram commute, put label (C^r) on $X \longrightarrow Y$

Note self-reference: by (ii) label (C^r) says you can *prove* there is an arrow $B \longrightarrow X$ satisfying certain conditions.

4 Examples of lifting properties

See [Lifting property] for a discussion how these lifting properties can be "read off" from the text of standard definitions.

In the category of Sets,

$\{\} \longrightarrow \{o\} \rightthreetimes g$ iff g is surjective
$\{o, o\} \longrightarrow \{o\} \rightthreetimes g$ iff g is injective
$f \rightthreetimes \{o, o\} \longrightarrow \{o\}$ iff f is injective

In Rings,

$0 \longrightarrow R \rightthreetimes g$ iff g is surjective $R \longrightarrow 0 \rightthreetimes g$ iff g is injective

In R-mod, for a commutative ring R,

$0 \longrightarrow P$ in $(0 \longrightarrow R)^{rl}$ iff P is projective
$I \longrightarrow 0$ in $(R \longrightarrow 0)^{rr}$ iff I is injective

In the category of metric spaces with either Lipsitz or uniformly continuous maps,

$\{1/n\}_{n\in\mathbb{N}} \longrightarrow \{0\} \cup \{1/n\}_{n\in\mathbb{N}} \rightthreetimes X \longrightarrow \{o\}$ iff X is complete
$\{1/n\}_{n\in\mathbb{N}} \longrightarrow \{0\} \cup \{1/n\}_{n\in\mathbb{N}} \rightthreetimes A \longrightarrow X$ iff A is closed in X, for A a subset of X
here the metric on $\{1/n\}_{n\in\mathbb{N}} \longrightarrow \{0\} \cup \{1/n\}_{n\in\mathbb{N}}$ is induced from the real line

In the category of finite groups,

the order of G is prime to p iff $0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \rightthreetimes G \longrightarrow 1$

G is a p-group iff $G \longrightarrow 1$ in $(0 \longrightarrow \mathbb{Z}/p\mathbb{Z})^{rr}$

H is soluble

iff $1 \longrightarrow H$ is in $(\{1 \longrightarrow A : A \text{ abelian }\})^{lr}$
iff $1 \longrightarrow H$ is in $(\{[G,G] \longrightarrow G \ : \ G \text{ arbitrary }\})^{lr}$

H is nilpotent iff $H \longrightarrow H \times H$ in $(1 \longrightarrow *)^{lr}$

here $(1 \longrightarrow *)$ denotes the class of maps $\{1 \longrightarrow G \ : \ G \text{ arbitrary }\}$ and $H \longrightarrow H \times H$, $h \mapsto (h,h)$ is the diagonal map

B is the normal closure of the image of A in B iff $A \longrightarrow B \rightthreetimes 1 \longrightarrow *$ for any group $*$

$Im\, A$ is subnormal in B iff $A \longrightarrow B$ in $(1 \longrightarrow *)^{lr}$

G is perfect iff $1 \longrightarrow G$ in $\{1 \longrightarrow A \ : \ A \text{ is abelian }\}^{l}$

G is soluble iff $1 \longrightarrow G$ in $\{1 \longrightarrow G \ : \ G \text{ is perfect }\}^{r}$

In the category of groups,

$\mathbb{Z} * \mathbb{Z} \longrightarrow \mathbb{Z} \times \mathbb{Z} \rightthreetimes A \longrightarrow 1$ iff A is abelian
here $\mathbb{Z} * \mathbb{Z}$ is the free non-abelian group on two generators, and $\mathbb{Z} \times \mathbb{Z}$ is the free abelian group on two generators

In the category of topological spaces.

First we need to introduce notation for finite topological spaces and their maps. We view a finite topological space as a finite category whose objects are points of X and there is a morphism $x \longrightarrow y$, necessarily unique, iff $y \in cl(x)$; a continuous map is a functor. A category is denoted by a list of its morphisms; '=' denotes an identity morphism. Unless otherwise indicated, a functor sends each object into itself.

For example, $\{x \leftrightarrow y\}$ denotes the antidiscrete space on two points, $\{x \rightarrow y\}$ denotes the Sierpinski space with point x open and point y open, and $\{x = y\}$ denotes the single point. There are maps

$$\{x, y\} \longrightarrow \{x \rightarrow y\} \longrightarrow \{x \leftrightarrow y\} \longrightarrow \{x = y\}$$

the last map denotes the map gluing the two points x and y.

Separation Axioms

T0 (Kolmogorov) $\quad \{x \leftrightarrow y\} \longrightarrow \{x = y\} \rightthreetimes X \longrightarrow \{o\}$

R0 (symmetric) $\{x \to y\} \longrightarrow \{x \leftrightarrow y\} \,\rightthreetimes\, X \longrightarrow \{o\}$

T1 (Frechet) $\{x \to y\} \longrightarrow \{x = y\} \,\rightthreetimes\, X \longrightarrow \{o\}$

T2 (Hausdorff) $\{x, y\} \hookrightarrow X \,\rightthreetimes\, \{x \to o \leftarrow y\} \longrightarrow \{x = o = y\}$

where $\{x, y\} \hookrightarrow X$ runs over all injective maps

T2.5 (Uryhson) $\{x, y\} \hookrightarrow X \,\rightthreetimes\, \{x \to x' \leftarrow o \to y' \leftarrow y\} \longrightarrow \{x = x' = o = y' = y\}$

T3 (regular) $\{x\} \longrightarrow X \,\rightthreetimes\, \{x \leftrightarrow u \to o \leftarrow v \to b\} \longrightarrow \{x = u = o = v \to b\}$

(completely regular) $\{x\} \longrightarrow X \,\rightthreetimes\, [0, 1] \cup \{1'\} \longrightarrow \{0 = x = 1 \to 1'\}$

here point 1' is closed, $cl(1) = \{1, 1'\}$ the map sends $[0, 1]$ to x, and $1'$ maps to point $1'$.

T4 (normal)

(i) $\{\} \longrightarrow X \,\rightthreetimes\, \{a \leftarrow u \to o \leftarrow v \to b\} \longrightarrow \{a \leftarrow u = o = v \to b\}$

(ii) $\{\} \longrightarrow X \,\rightthreetimes\, \{0'\} \cup [0, 1] \cup \{1'\} \longrightarrow \{0' \leftarrow 0 = x = 1 \to 1'\}$

here points 0' and 1' are closed, $cl(0) = \{0, 0'\}$, $cl(1) = \{1, 1'\}$ the map sends $[0, 1]$ to x, and $0'$ maps to point $0'$, and $1'$ maps to point $1'$. Note (ii) implies (i) as the map $\{0'\} \cup [0, 1] \cup \{1'\} \longrightarrow \{0' \leftarrow 0 = x = 1 \to 1'\}$ factors via $\{a \leftarrow u \to o \leftarrow v \to b\} \longrightarrow \{a \leftarrow u = o = v \to b\}$; and the classical proof of Uryhson Lemma that (i) implies (ii) is an infinite iteration of (i) and passing to the limit

(perfectly normal) $\{\} \longrightarrow X \,\rightthreetimes\, [0, 1] \longrightarrow \{0 \leftarrow x \to 1\}$

here points 0 and 1 are closed, the map sends $(0, 1)$ to x, and 0 maps to point 0, and 1 maps to point 1.

(extremally disconnected) $\{\} \longrightarrow X \,\rightthreetimes\, \{x \to x' \leftarrow o \to y' \leftarrow y\} \longrightarrow \{x = x' \to o \leftarrow y = y'\}$

Various properties

X is connected iff $\{\} \longrightarrow X \,\rightthreetimes\, \{x, y\} \longrightarrow \{x = y\}$

X is discrete iff

$\{\} \longrightarrow X$ in $(\{\} \longrightarrow \{o\})^{rl}$

$\{\} \longrightarrow X \,\rightthreetimes\, \{x \to y\} \longrightarrow \{x \leftrightarrow y\}$

$\{\} \longrightarrow X \,\rightthreetimes\, \{x, y\} \longrightarrow \{x \leftrightarrow y\}$

A Hausdorff space K is compact iff $\{\} \longrightarrow K$ in

$$(\{z \leftarrow u \leftrightarrow v \to t\} \longrightarrow \{x \to z = u = v = t\})^{lr} =$$

$\{\, \{y\} \longrightarrow \{o \to y\}; \{x \leftrightarrow y\} \longrightarrow \{x = y\}; \{o \to y\} \longrightarrow \{o = y\}; \{z \leftarrow o \to t\} \longrightarrow \{z = o = t\}\}^{lr}$

(see below for explanation)

For a connected space K, each real-valued function on K bounded iff

$$\{\} \longrightarrow K \,\rightthreetimes\, \cup_{n \in \mathbb{N}}(-n, n) \longrightarrow \mathbb{R}$$

here $\cup_{n \in \mathbb{N}}(-n, n) \longrightarrow \mathbb{R}$ denotes the map to the real line from the disjoint union of intervals (-n,n) which cover it.

$A \longrightarrow B$ is injective

iff $A \longrightarrow B \,\rightthreetimes\, \{x \leftrightarrow y\} \longrightarrow \{x = y\}$

iff $\{x, y\} \longrightarrow \{x = y\} \,\rightthreetimes\, A \longrightarrow B$

topology on A is induced from B iff $A \longrightarrow B \,\rightthreetimes\, \{x \to y\} \longrightarrow \{x = y\}$

$Im\, A$ is dense in B iff $A \longrightarrow B \,\rightthreetimes\, \{y\} \longrightarrow \{x \to y\}$

X is a closed subset of Y iff $X \longrightarrow Y$ in $(\{y\} \longrightarrow \{x \to y\})^{rl}$

$(\{y\} \longrightarrow \{x \to y\})^{rl}$ is the class of inclusions $X \longrightarrow Y$ where X is a closed subset of Y

$(\{y\} \longrightarrow \{x \leftrightarrow y\})^{rl}$ is the class of inclusions $X \longrightarrow Y$ where X is a subset of Y

$Im\, A$ is open in B iff $A \longrightarrow B \rightthreetimes \{o \to x \leftrightarrow y\} \longrightarrow \{o = x \leftrightarrow y\}$

$A \longrightarrow B \rightthreetimes \{x \leftarrow o \to y\} \longrightarrow \{x = o = y\}$ iff for each disjoint closed subsets Z',Z'' of A, the closures of their images in B are also disjoint.

$A \longrightarrow B \rightthreetimes \{z \leftarrow x \leftarrow o \to y \to z\} \longrightarrow \{o \to x = y \to z\}$ for each closed subsets Z',Z'' of A, $cl_B(ImZ') \cap cl_B(ImZ'') = cl_B(ImZ' \cap ImZ'')$

5 Compactness.

Here we discuss 3 or 4 definitions of compactness: via ultrafilters, finite subcovers, and universally closed. We also discuss compactness as being uniform, i.e. it is allowed to change of order of quantifiers $\forall\exists \longrightarrow \exists\forall$.

The only (almost) satisfactory reformulation is that of the definition via ultrafilters. Our discussion of the other definitions is preliminary.

5.1 Compactness via ultrafilters.

Mathematically, this reformulation is the definition of compactness via ultrafilters [Bourbaki, General Topology, I§10.2, Thm. 1(d), p.101] and a Taimanov's 1952 theorem [Taimanov] on extending functions to compacta from dense subspaces.

Let $\{0\} \longrightarrow \{0 \to 1\}$ denote the map sending a point into the open point of the Sierpinski two-point space.

$$(c_n) := \{g : X \longrightarrow Y : \{0\} \longrightarrow \{0 \to 1\} \rightthreetimes g, |X|, |Y| < n+1\}$$

Theorem 1. *The class $(c_n)^{lr}$, $n > 0$, is contained in the class of proper maps. A Hausdorff space K is compact iff the map $K \longrightarrow \{o\}$ sending K to a single point is in $(c_3)^{lr}$. Moreover, a map $f : X \longrightarrow Y$ of normal (T4) spaces X,Y is proper iff f is in $(c_4)^{lr}$.*

Proof. Note that for a map $g : X \longrightarrow Y$ of finite spaces X,Y, $\{0\} \longrightarrow \{0 \to 1\} \rightthreetimes g$ means g is closed and hence proper.

Let F be an ultrafilter on the set of points of a discrete topological space A. Define topology on $A \cup_F \{\infty\}$ as follows: a subset is closed iff it either contains ∞ or is an F-small subset of A.

Let $A \longrightarrow A \cup_F \{\infty\}$ be the obvious embedding.

Lemma 1. *A map $g : X \longrightarrow Y$ is proper iff $A \longrightarrow A \cup_F \{\infty\} \rightthreetimes g$ for each map associated with an ultrafilter.*

Hence, maps $A \longrightarrow A \cup_F \{\infty\}$ are $(c_3)^l$, and thus $(c_3)^{lr}$-maps are necessarily proper.

Proof. This is a characterisation of (not necessarily Hausdorff) compacta and proper maps via ultrafilters (Bourbaki, General Topology, I§10.2,Th.1(d)). For example, for Y a point, it says $X \longrightarrow \{o\}$ is proper, i.e. X is compact, iff any ultrafilter F on X converges, and that can be expressed as $A \longrightarrow A \cup_F \{\infty\} \rightthreetimes g$ where A is the set of points of X, $A \longrightarrow X$ to be the obvious map. □(Lemma)

Lemma 2. *$A \longrightarrow B$ is $(c_4)^l$ implies A is an open dense subset of B, and*

(t) for each disjoint closed subsets Z',Z'' of A, their closures in B are also disjoint.

(t') for each Z',Z'' closed subsets of A, $cl_B(Z') \cap cl_B(Z'') = cl_B(Z \cap Z'')$

Proof. See the section on examples. □(Lemma)

Now let $i : A \longrightarrow B$, $g : X \longrightarrow Y$, $f : A \longrightarrow X$, $j : B \longrightarrow Y$. Necessarily for the lifting function $f' : B \longrightarrow X$ it holds

$$f'(b) \in \cap\{cl_X(f(Z)) \ : \ Z \subset A \text{ closed}, b \in cl_B(i(Z))\}$$

Let us first prove that this intersection is non-empty, and, assuming X and Y normal, is exactly one point and f so defined is continuous.

As $g : X \longrightarrow Y$ is closed,

$$j(b) \in j(cl_B(i(Z))) \subset cl_Y(j(i(Z)) = cl_Y(g(f(Z)) = g(cl_X(f(Z)),$$

hence

$$\cap\{cl_X(f(Z)) \ : \ Z \subset A \text{ closed, } b \in cl_B(i(Z))\} \cap g^-1(b)$$

is non-empty as a directed family of closed subsets of a compact space $g^{-1}(b)$.

To prove the intersection consists of a single point and the function f' is continuous, notice that each neighbourhood U of a point x in X contains a closed subset $V \subset U$ such that Inn V is an open neighbourhood of x. This set V is part of the intersection above.

$\square$(Theorem).

Question 1. It is really necessary to assume T4 in the theorem, e.g. does $(c_5)^{lr}$ coincide with proper maps for non-Hausdorff etc spaces?

5.2 "An open covering has a finite subcovering"; "an open neighbourhood U(x) of a point x as a function of the point"

Mathematically, this reformulation is based on the following observation:

> a space K is compact iff for each open covering U of K, K is closed in $K \cup \{\infty\}$ in the topology generated elements of U as *closed* subsets.

This lets us express being *finite* with the help of the notion of the topology generated by a family of sets.

[Hausdoff, Set theory] denotes by U(x) a neighbourhood of a point x, which suggests viewing $U(x)$ as a (possibly multivalued) function of a point x ; in our arrow notation this would correspond to

$$\{x\} \longrightarrow K \xrightarrow{(U(x))} \{x \to y\} \qquad (*)$$

here it is implicit that x maps to x in the composition; (x) in $U(x)$ signifies that $U(x)$ depends on x.

Changing a single symbol "$\to$" into "$\leftarrow$" leads us to consider elements of U as closed subsets of K:

$$\{x\} \longrightarrow K \xrightarrow{(U(x))} \{x \leftarrow y\} \quad (**)$$

How would appear *the topology generated by*? Manipulating a definable family of arrows from an X, by its nature, is similar to working with the topology on X satisfying properties which reflect what we use about the family.

Of course, this change of a symbol would only make sense within a context of a formal calculus which we do not have yet. In our calculus, the arrows $(**)$ should inherit some properties from $(*)$, e.g. a family of arrows $(**)$ commutes iff the corresponding family of arrows $(*)$ commutes. ..A collection of arrows of form $(**)$ (or $(*)$) should define a topology generated by sets in U.... ....

5.3 "the image of a closed set is closed"

We shall see that this can be understood as an instance of being uniform, i.e. a change of order of quantifiers $\forall\exists \longrightarrow \exists\forall$:

K is compact iff the following implication holds for each set X and each subset $Z \subset X \times K$:

$$\frac{\forall y \in K \exists U \exists V (U \subset X \text{ open and } V \subset K \text{ open and } a \in U \text{ and } y \in V \text{ and } U \times V \subset Z)}{\exists U \exists V \forall y \in K (U \subset X \text{ open and } V \subset K \text{ open and } a \in U \text{ and } y \in V \text{ and } U \times V \subset Z)}$$

The hypothesis says Z contains a rectangular open neighbourhood of each point of the line $\{a\} \times K$; the conclusion says that Z contains a rectangular open neighbourhood of the whole line $\{a\} \times K$.

5.4 Being uniform — Changing the order of quantifiers $\forall\exists \longrightarrow \exists\forall$

We give three more examples where a use of compactness is changing the order of quantifiers. Later we see that paracompactness can also be expressed this way.

Question 2. Describe a logic and a *syntactic* class of formulae where such exchange $\forall\exists \longrightarrow \exists\forall$ of order quantifiers is permissible.

Is there a treatment of compactness in terms of changing the order of quantifiers ?

Find more interesting examples where a use of compactness is expressed as a change of order of quantifiers; Martin Bays noted that in a sense, most of our examples are not very interesting as there is a directed system associated with the quanfifier free formula. Find a general theorem covering these examples.

5.4.1 Each real-valued function on a compact set is bounded

$$\frac{\forall x \in K \exists M (f(x) \leq M)}{\exists M \forall x \in K (f(x) \leq M)}$$

Note that the hypothesis holds trivially: take $M := f(x)$.

Note this is a lifting property, for K connected:

$$\{\} \longrightarrow K \,\rightthreetimes\, \cup_{n\in\mathbb{N}}(-n,n) \longrightarrow \mathbb{R}$$

here $\cup_n(-n,n) \longrightarrow \mathbb{R}$ denotes the map to the real line from the disjoint union of intervals $(-n,n)$ which cover it. Note this is a standard example of an open covering of $\mathbb{R}$ which shows it is not compact.

5.4.2 A Hausdorff compact is necessarily normal.

The application of compactness in the usual proof of this amounts to the following change of order of quantifiers:

For each pair of closed disjoint compact subsets A and B of a Hausdorff space K, it holds:

$$\frac{\forall a \in A \forall b \in B \exists U \exists V (a \in U \text{ and } b \in V \text{ and } U \cap V = \{\} \text{ and } U \subset K \text{ open and } V \subset K \text{ open})}{\exists U \exists V \forall a \in A \forall b \in B (a \in U \text{ and } b \in V \text{ and } U \cap V = \{\} \text{ and } U \subset K \text{ open and } V \subset K \text{ open})}$$

5.4.3 Lebesgue's number Lemma

Let S be a family of (arbitrary!) subsets of a metric space X.

$$\frac{\forall x \in X \exists \delta > 0 \exists U \in S \forall y \in X (dist(x,y) < \delta \implies y \in U)}{\exists \delta > 0 \forall x \in X \exists U \in S \forall y \in X (dist(x,y) < \delta \implies y \in U)}$$

The hypothesis says that $\{Inn\,U \ : \ U \in S\}$ is an open cover of X; the conclusion is as usually stated, that each set of diameter $< \delta$ is covered by a single member of the cover.

Note that this lemma may be expressed in terms of uniform structures; see our remarks on uniform structures as simplicial objects

6 A uniform structure as a simplicial object

Let X be a set.

Recall a simplicial object in a category C is a contravariant functor

$$F : \{o \to o \to .. \to o\}^{op} \longrightarrow C$$

where $\{o \to o \to ... \to o\}$ denotes the full subcategory of the category of categories generated by the finite linear orders $o \to o \to ... \to o$ viewed as categories.

With a set X, one can associate the trivial simplicial object $(X)_\times$ in Sets

$$F(o \to o \to ... \to o) = X \times X \times ... \times X$$

whose maps correspond to removing and repeating coordinates, i.e. in other words, face and degeneration maps correspond to coordinate projections and diagonal embeddings.

Theorem 2. *A uniform structure $\mathcal{U}$ on a set X correspond to a simplicial object $(U)_o$ in the category of topological spaces such that*

(i) the underlying sets form the trivial simplicial object $(X)_\times$ in Sets associated to X

(ii) topology on X_0 is antidiscrete

(iii) the topology on X^{n+1} is the pullback of the topologies on X^n and X^{n-1} with any choice of face and degeneration maps satisfying the obvious restrictions

(iv) the involution $X \times X \longrightarrow X \times X$ is a homeomorphism

Proof. Equip $X \times X$ with the topology generated by the subsets ($\mathcal{U}$-entourages) of $X \times X$ in $\mathcal{U}$. As each entourage $V \subset X \times X$ contains the diagonal, the map $X_0 \longrightarrow X \times X$ is continuous. Equip X^n with the topology according to (iii). This is well-defined because for each W there is V such that $V \circ V \subset W$. (iv) is satisfied because entourages are assumed symmetric. □(Theorem)

Question 3 (Alexandroff)**.** writes "as it seems to me, one of the deepest and most interesting properties of paracompacts" is the following theorem of A.Stone: that

> A T_1-space is *paracompact* iff for each open covering α of X there is an open covering β such that for each x in X there is U in A such that $\cup\{V \in B : x \in V\} \subset U$

As quantifier exchange, this is:

$$\frac{\text{for each open covering } \alpha \text{ exists open covering } \beta.\ \forall x \in X \forall V \in \beta \exists U \in \alpha (x \in V \implies V \subset U)}{\text{for each open covering } \alpha \text{ exists open covering } \beta.\ \forall x \in X \exists U \in \alpha \forall V \in \beta (x \in V \implies V \subset U)}$$

The hypothesis holds trivially: take $\beta = \alpha, V = U$. Reformulate this property in simplicial terms. More generally, develop the theory of uniform structure in simplicial terms.

7 Open questions and speculations.

We sketch several rather speculative research directions and a few more concrete questions and conjectures.

Problem 1. Develop a calculus based on the lifting properties, arrows, labels and finite topological spaces.

1. Standard arguments and definitions in elementary topology should be represented by short formal calculations which are both human readable and computer verifiable.
2. In particular, the calculus should express concisely all the three definitions of compactness, and prove their equivalence by short formal calculations.
3. It should include a formulation of the Arzela-Ascoli theorem.

Problem 2. Rewrite the theory of uniform structures in terms of the corresponding simplicial objects. In particular, reformulate in simplicial terms the Lebesgue's number lemma, partition of unity, and the characterisation of paracompactness by A.Stone mentioned by [Alexandroff].

Problem 3. Write a first year course introducing elementary topology and category theory ideas at the same time, based on the observations above and the calculus to be developed. Compactness would be explained with help of all the definitions above; Tychonoff theorem is immediate via the lifting property definition $(c_3)^{rl}$ of compactness; $\forall\exists \longrightarrow \exists\forall$ definitions would give students some intuition.

As a first step, write an exposition aimed at students of the separation axioms and Uryhson Lemma in terms of the lifting properties.

Problem 4. Develop the theory of real numbers using these observations. Is it possible? Kline characterisation of a sphere may be of use.

Question 4. Calculate left and right $\rightthreetimes$-negations and generalisations, e.g. $(C)^r$, $(C)^l$, $(C)^{rl}$, $(C)^{ll}$, $(C)^{rr}$, $(C)^{llr}$, ... for various simple classes of morphisms in various categories, e.g. morphisms of finite topological spaces or finite groups.

Develop abstract theory of the lifting property.

For example, is the class of finite CA-groups or CN-groups defined by a natural lifting property? Recall that a group is a CA-group, resp. CN-group, iff the centraliser of a non-identity element is necessarily abelian, resp. nilpotent.

Does triviality of $C^r \cap C^l$ imply that either C^r or C^l is trivial?

Question 5. We give several examples where uses of compactness are expressed as change of order of quantifiers $\forall\exists \longrightarrow \exists\forall$. Find more examples. Is there a theorem generalising these examples?

7.1 Speculations

Question 6. Does topological intuition (as developed by a first year student) relate to the formal calculus we'd like to develop? Note that this might be testable by an experiment, namely it might be possible to test whether mistakes of intuition correspond to mistakes of calculation. This might even be used to develop the calculus.

Question 7. Write a *very short* program which would "invent" (generate) the (very) basic theory of topology, possibly using unstructured input such as the text of (Bourbaki, General Topology). Our examples suggest that iterating right and left $\rightthreetimes$-negation up to 3 times (i.e. the Quillen lifting property) and restricting size to 3 or 4 is enough to generate, but not single out, the notions of compactness, connectedness, a subset, a closed subset, separation axioms, and some implications between them.

What is the length of a shortest such program? To what extent have the axioms of topology to be hardcoded rather than generated?

7.2 Concrete questions

Conjecture 1. *In the category of topological spaces, $(c_5)^{rl}$ is the class of proper maps.*

Question 8. Prove that each Hausdorff compact is normal using the definitions in terms of $\rightthreetimes$ properties. Is there a proof using only diagram chasing with finite preorders, or are additional axioms necessary? The easiest way to do so is probably to write an automatic prover (diagram chaser) or a computer algebra system.

Question 9. Calculate $(C)^r$, $(C)^l$, $(C)^r \cap (C)^l$, $(C)^{rl}$, $(C)^{ll}$, $(C)^{rr}$, $(C)^{llr}$, ... for the classes of morphisms used in the examples in this short note. Do you get interesting classes this way ?

Question 10. Can either the class of finite CA-groups or CN-groups be defined as a class of finite groups satisfying certain lifting properties in the category of all groups?

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This note is elementary, and it was embarrassing and boring, and embarrassingly boring, to think or talk about matters so trivial, but luckily I had no obligations for a time.

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