A DECIDABLE EQUATIONAL FRAGMENT OF CATEGORY THEORY WITHOUT AUTOMORPHISMS

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Abstract. We record an observation of Nikolai Durov that there is a decidable equational fragment of category theory without automorphisms.

PRELIMINARY DRAFT

1. Introduction

The purpose of this note is to state an observation that there is a decidable equational fragment of category theory without automorphisms. The condition 'without automorphisms' is to ensure that the fragment does not interpret (via the equational theory of a [finitely presented] group of automorphisms of an object of the category) the word problem for finitely presented groups which is well-known to be algorithmically undecidable.

Let us first informally sketch our definitions and results.

Fix a language L_A extending a language appropriate to describe a category such as the language L_A defined in the next section. Call a formula φ *stratified* iff there is an ordering $X_1, \ldots, X_m$ of (both free and bound) variables in φ such that it has the following property:

(1) if φ mentions a morphism from an object X_i to an object X_j, then $i < j$.

Call a formula *implication-like* iff it is an implication between conjunctions of identities between compositions of morphisms and limits and colimits of commutative diagrams.

In this note we observe that it is decidable whether an implication-like stratified formula in the language of categories is valid in any category. We do so by showing that there is a finite free category satisfying a finite number of commutativity conditions and identities between objects, morphisms and limits of diagrams occurring in a stratified formula.

This simple observation raises several obvious questions concerning decidability of fragments consisting of stratified formulae. We take the opportunity to list a few relevant to the work of the current author.

Are positive stratified $\forall\exists$-formulae in the language L_A of categories decidable?

Axioms M1–M6 of model categories (see [Qui]) can be expressed as positive stratifiable $\forall\exists$-formulae in the language of labelled categories, namely the language extending L_A by three unary predicates. There is also an unstratifiable axioms M0 requiring existence of all finite limits and colimits. However, there is an obvious stratified part of the axiom requiring existence only of those finite limits and colimits which agree with a given ordering of variables.

> Does this lead to a decidable stratified fragment of the theory of model categories? How expressive is this fragment?

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There is a simpler version of this question. [GH-I] defines a model category whose underlying category is a partial order. This naturally leads to a stratifiable set of axioms. (fixme: does it)

Is there a decidable stratified fragment of the theory of posetal model categories?

A draft [G] observes that some definitions and theorems in general topology can be expressed as diagram chasing computations in labelled categories and finite preorders. It appears that sometimes these computations correspond to stratified formulae. It is tempting to ask whether this leads to a decidable stratified fragment.

2. A stratified fragment of category theory in a first order language.

We consider a category as a (single-sorted) structure in the language with the following predicates:

(1) 3-ary predicate $\mathrm{id}(X, Y, f)$
(2) 5-ary predicate $\mathrm{Comp}(X, Y, Z, f, g, h)$
(3) a $2v + e + 1$-ary predicate $\mathrm{Limit}_D(X_0, X_1, \ldots, X_v, \iota_1, \ldots, \iota_v, f_1, \ldots, f_e)$ for each oriented graph (possibly with multiple edges) having v vertices numbered $1, \ldots, v$ and e edges labelled $1, \ldots, e$.
(4) a $2v+e+1$-ary predicate $\mathrm{Colimit}_D(X_0, X_1, \ldots, X_v, \iota_1, \ldots, \iota_v, f_1, \ldots, f_e)$ for each oriented graph (possibly with multiple edges) having v vertices numbered $1, \ldots, v$ and e edges labelled $1, \ldots, e$.

Given a category $\mathcal{C}$, let $\mathcal{M}_{\mathcal{C}}$ be the following model. Its universe is $\mathcal{O}b\mathcal{C} \cup \bigcup_{X,Y \in \mathcal{O}b\mathcal{C}} \mathcal{M}or(X, Y)$.

(i) $\mathrm{id}(X, Y, f)$ interpreted as "X,Y are objects, and f is the identity morphism $id : X \longrightarrow Y$"
(ii) $\mathrm{Comp}(X, Y, Z, f, g, h)$ interpreted as "X, Y, Z are objects, and $f : X \longrightarrow Y$, $g : Y \longrightarrow Z$ and $h : X \longrightarrow Z$ are morphisms as indicated, and $h = fg$"
(iii) $\mathrm{Limit}_D(X_0, X_1, \ldots, X_v, \iota_1, \ldots, \iota_v, f_1, \ldots, f_e)$ interpreted as "$(X_0, \iota_1, \ldots, \iota_v)$ is a limit of (necessarily commutative) diagram corresponding to the graph D with vertices marked by $X_1, \ldots, X_v$ and edges labelled by morphisms $f_1, \ldots, f_e$." In more detail, it says
   (a) $X_0, X_1, \ldots, X_v$ are objects, and $X_0 \xrightarrow{\iota_1} X_1, \ldots, X_0 \xrightarrow{\iota_v} X_v$ are morphisms as indicated
   (b) $f_i : X_{b_i} \longrightarrow X_{e_i}$ for $1 \leqslant i \leqslant e$ where i is the number of the edge of D from b_i to e_i.
   (c) $X_0 \xrightarrow{\iota_1} X_1, \ldots, X_0 \xrightarrow{\iota_v} X_v, X_{b_1} \xrightarrow{f_1} X_{e_1}, \ldots, X_{b_e} \xrightarrow{f_e} X_{e_e}$ form a commutative diagram
   (d) for arbitrary object X_0' and arrows $X_0' \xrightarrow{\iota_1'} X_1, \ldots, X_0' \xrightarrow{\iota_v'} X_v$, if

$$X_0' \xrightarrow{\iota_1'} X_1, \ldots, X_0' \xrightarrow{\iota_v'} X_v, X_{b_1} \xrightarrow{f_1} X_{e_1}, \ldots, X_{b_e} \xrightarrow{f_e} X_{e_e}$$

   form a commutative diagram, then there is a unique morphism $X_0 \xrightarrow{\iota} X_0'$ such that the arrows

$$X_0 \xrightarrow{\iota} X_0', X_0 \xrightarrow{\iota_1} X_1, \ldots, X_0 \xrightarrow{\iota_v} X_v, X_0 \xrightarrow{\iota_1'} X_1, \ldots, X_0 \xrightarrow{\iota_v'} X_v, X_{b_1} \xrightarrow{f_1} X_{e_1}, \ldots, X_{b_e} \xrightarrow{f_e} X_{e_e}$$

   form a commutative diagram
(iv) an analogous condition for colimit predicate $\mathrm{Colimit}_D(-, -, \ldots, -)$ with direction of all arrows reversed

Let $X_1, \ldots, X_n$ be an *ordered* list of variables. We say that a formula φ is *stratified* iff for every occurrence of

(1) $\mathrm{id}(X_i, Y_j, f)$, it holds $i < j$;
(2) $\mathrm{Comp}(X_i, X_j, X_k, f, g, h)$, it holds $i < j < k$;

(3) $\mathrm{Limit}_D(X_{i_0}, X_{i_1}, \ldots, X_{i_v}, \iota_1, \ldots, \iota_v, f_1, \ldots, f_e)$, it holds $i_0 < i_1, \ldots, i_0 < i_v$ and $b_1 < e_1, \ldots, b_e < e_e$ in notation above

(4) $\mathrm{Colimit}_D(X_{i_0}, X_{i_1}, \ldots, X_{i_v}, \iota_1, \ldots, \iota_v, f_1, \ldots, f_e)$, it holds $i_0 > i_1, \ldots, i_0 > i_v$ and $b_1 < e_1, \ldots, b_e < e_e$ in notation above

Notice that there are stratified formulae defining equalizers and coequalizers.

3. A category as a data structure. Axioms as rules for manipulating the data structure.? >Axioms as allowed operations.?

As a data structure, a category is a directed graph with multiple edges with a certain? class of distinguished subgraphs satisfying certain axioms: its vertices are objects of the category, its edges are morphisms (and usually are called arrows), and the distinguished subgraphs correspond to commutative diagrams. Further, note that the axioms may be viewed algorithmically as rules for adding new edges and adding new distinguished subgraphs.

The axiom of existence of an identity morphism is a rule to

(id_1) add a loop around a vertex and mark that loop as a distinguished subgraph, provided no such loop already exists

(id_2) Take two distinguished subgraphs one of which is a loop. Mark their union as a distinguished subgraph.

The axiom of existence and associativity of composition is the following rule:

($\to\to$) Take two edges $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$ such that the second one leaves the end of the first one. Then add an edge $X \xrightarrow{h} Z$ from the beginning of the first edge to the end of the second edge, and mark the resulting triangle as a distinguished subgraph, provided there is no distinguished triangle containing edges $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$

($\to\to\to$) given edges $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T$ and $X \xrightarrow{fg} Z$, $Y \xrightarrow{gh} T$, $X \xrightarrow{fgh} T$ such that triangles $X \xrightarrow{f} Y \xrightarrow{g} Z$, $X \xrightarrow{fg} Z$, and triangle $Y \xrightarrow{g} Z \xrightarrow{h} T$, $Y \xrightarrow{gh} T$, and triangle $X \xrightarrow{f} Y \xrightarrow{gh} T$, $X \xrightarrow{fgh} T$ are distinguished, mark the triangle $X \xrightarrow{f} Y \xrightarrow{gh} T$, $X \xrightarrow{fgh} T$ as distinguished.

Given a commutative diagram D in a category, existence of a limit of a diagram can also be thought of as a rule:

(lim) given a distinguished subgraph D whose vertices are $X_1, \ldots, X_n$ and whose edges are $f_1, \ldots, f_m$,
   (a) add a *limit* vertex L_D and edges $L_D \xrightarrow{\iota_1} X_1, \ldots, L_D \xrightarrow{\iota_n} X_n$, and mark commutative the subgraph $D \cup \{L_D\} \cup \{\iota_1, \ldots, \iota_n\}$
   (b) add the rule that
      (i) given a vertex L and edges $L \xrightarrow{i_1} X_1, \ldots, L \xrightarrow{i_n} X_n$, and a commutative subgraph $D \cup \{L\} \cup \{i_1, \ldots, i_n\}$, add an arrow $L_D \xrightarrow{\iota} L$ and mark the total subgraph

$$D \cup \{L_D\} \cup \{\iota_1, \ldots, \iota_n\} \cup \{L\} \cup \{i_1, \ldots, i_n\} \cup \{L_D \xrightarrow{\iota} L\}$$

      commutative.
      (ii) given a vertex L and edges $L \xrightarrow{i_1} X_1, \ldots, L \xrightarrow{i_n} X_n$, and arrows $L_D \xrightarrow{\iota} L$ $L_D \xrightarrow{\iota'} L$, mark commutative the subgraph

$$D \cup \{L_D\} \cup \{\iota_1, \ldots, \iota_n\} \cup \{L\} \cup \{i_1, \ldots, i_n\} \cup \{L_D \xrightarrow{\iota} L, L_D \xrightarrow{\iota'} L\}$$

provided both subgraphs $D \cup \{L_D\} \cup \{\iota_1, \dots, \iota_n\} \cup \{L\} \cup \{i_1, \dots, i_n\} \cup \{L_D \xrightarrow{\iota} L\}$ and $D \cup \{L_D\} \cup \{\iota_1, \dots, \iota_n\} \cup \{L\} \cup \{i_1, \dots, i_n\} \cup \{L_D \xrightarrow{\iota'} L\}$ are marked commutative

The rule for colimits is dual and we do not state it.

4. Theorem and proof

Call a formula *implication-like* iff it is an implication between conjunctions of positive atomic formulae.

Suppose $\mathcal{C}$ is a category, $D_1, \dots, D_n$ are diagrams in $\mathcal{C}$ and $L_1, \dots, L_n$ are objects of $\mathcal{C}$ such that $\mathrm{Hom}_{\mathcal{C}}(L_i, X) = \{*\}$ for any $X \in D_i$.

We will call a pair $(\{L_i\}, \{D_i\})$ a *stratified formula with limits* (in $\mathcal{C}$) if $\mathrm{Hom}_{\mathcal{C}}(L_i, X) = \varnothing$ for all X from the definition of D_j whenever there exists a sequence $j = a_1, \dots, a_k = i$ of indexes such that $\mathrm{Hom}_{\mathcal{C}}(L_{a_k}, Y) \neq \varnothing$ for all Y from the definition of $D_{a_{k+1}}$ for all i and j. This condition means that there is a preorder on diagrams ($D_i \leq D_j$ iff there exists such a sequence of indexes from j to i).

We will call the system of diagrams $\{D_i\}$ *independent* if for every i and j there exists $X \in D_j$ such that $\mathrm{Hom}_{\mathcal{C}}(Y, X) = \varnothing$ for every Y with $\mathrm{Hom}_{\mathcal{C}}(Y, X') \neq \varnothing$ for any $X' \in D_i$. This condition can be thought as there are "almost no maps" from D_i to D_j for any i and j. Note that the independence of diagrams implies that for any set of L_i such that $\mathrm{Hom}\,(L_i, X) \neq \varnothing$ for any $X \in D_i$ a pair $(\{L_i\}, \{D_i\})$ is a stratified formula with limits.

Construction of the localisation. Let $\mathcal{C}$ be a category, S be a set of morphisms in $\mathcal{C}$, $G \in \mathrm{Func}(\mathcal{C}, Sets)$. We denote by $F[S^{-1}]$ the initial object in the category of all natural transformations from F to functors that invert S and call it a *localisation of* F. Note that it doesn't follow from the definition that such a functor exists. By abuse of notation the codomain of this universal morphism will also be denoted by $F[S^{-1}]$.

The localisation of the category $\mathcal{C}$ by the set of morphisms S is canonically isomorphic to the full subcategory of $\mathrm{Func}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sets})$ whose objects are functors $\mathrm{Hom}_{\mathcal{C}}(-, X)[S^{-1}]$ for every X. Indeed, let us show that $\mathrm{Hom}_{\mathcal{C}[S^{-1}]}(-, X) \cong \mathrm{Hom}_{\mathcal{C}}(-, X)[S^{-1}]$ as functors from $\mathcal{C}$ to $Sets$. We need to check that the natural transformation $\mathrm{Hom}_{\mathcal{C}}(-, X) \to \mathrm{Hom}_{C\mathcal{C}[S^{-1}]}(-, X)$ is universal among all natural transformations to functors that invert S. Then the result will follow from the Yoneda embedding.

Suppose a functor $F \in \mathrm{Func}(\mathcal{C}^{\mathrm{op}}, Sets)$ sends objects in S to isomorphisms. Denote by $\widetilde{F}$ the corresponding functor on $\mathcal{C}[S^{-1}]^{\mathrm{op}}$. We must show that the map

$$\mathrm{Nat}_{\mathcal{C}}(\mathrm{Hom}_{\mathcal{C}[S^{-1}]}(-, X), F) \to \mathrm{Nat}_{\mathcal{C}}(\mathrm{Hom}_{\mathcal{C}}(-, X), F)$$

is a bijection. Note that the domain of this map is isomorphic to $\mathrm{Nat}_{\mathcal{C}}[S^{-1}](\mathrm{Hom}_{\mathcal{C}[S^{-1}]}(-, X), \widetilde{F}) \cong F(X)$. The codomain is also $F(X)$ and the map coincides with $\mathrm{id}_{F(X)}$ under these identifications. So the universal property is satisfied.

Lemma 1. *Let $\mathcal{C}$ be a category, X, Y be objects of $\mathcal{C}$, $\varphi_i \in \mathrm{Hom}_{\mathcal{C}}(X_i, Y_i)$ be a set of morphisms in $\mathcal{C}$. Let B be an object of $\mathcal{C}$ such that $\mathrm{Hom}_{\mathcal{C}}(X_i, B) = \varnothing$ for every i. Then $\mathrm{Hom}_{\mathcal{C}}(-, B) \cong \mathrm{Hom}_{\mathcal{C}[\varphi^{-1}]}(-, B)$ as functors from $\mathcal{C}$ to* Sets.

Proof. By the construction of the localisation of a category via localisation of its representable functors (see above) $\mathrm{Hom}_{\mathcal{C}[S^{-1}]}(-, B) \cong \mathrm{Hom}_{\mathcal{C}}(-, B)[S^{-1}]$. Note that $\mathrm{Hom}_{\mathcal{C}}(X_i, B) = \varnothing$ implies

$\mathrm{Hom}_{\mathcal{C}}(Y_i, B) = \varnothing$. So $\mathrm{Hom}_{\mathcal{C}}(-, B)$ converts φ into $id_\varnothing$, hence inverts φ. This implies that $\mathrm{Hom}_{\mathcal{C}}(-, B) \cong \mathrm{Hom}_{\mathcal{C}}(-, B)[S^{-1}]$. □

Lemma 2. *Let L be a limit of a diagram D in the category $\mathcal{C}$. Suppose S is a set of morphisms such that $\mathrm{Hom}_{\mathcal{C}}(\mathrm{dom}(f), L) = \varnothing$ for any $f \in S$. Suppose also that there exists an object X of D such that $\mathrm{Hom}_{\mathcal{C}}(Y, X) = \varnothing$ for any Y such that $\mathrm{Hom}_{\mathcal{C}}(Y, \mathrm{codom}(f)) \neq \varnothing$ for some $f \in S$. Then the localisation functor $\mathcal{C} \to \mathcal{C}[S^{-1}]$ preserves the limit of D.*

Proof. By Lemma 1 we have $\mathrm{Hom}_{\mathcal{C}[S^{-1}]}(-, L) \cong \mathrm{Hom}_{\mathcal{C}}(-, L)$. Let A be an object of $\mathcal{C}$. Then one of the following holds: either $\mathrm{Hom}_{\mathcal{C}}(A, \mathrm{dom}(f)) \neq \varnothing$ for some $f \in S$, or $\mathrm{Hom}_{\mathcal{C}}(A, \mathrm{dom}(f)) = \varnothing$ for any $f \in S$. If the second condition is satisfied then by the dual version of Lemma 1 we have

$$\varprojlim \mathrm{Hom}_{\mathcal{C}[S^{-1}]}(A, D) \cong \varprojlim \mathrm{Hom}_{\mathcal{C}}(A, D) \cong \mathrm{Hom}_{\mathcal{C}}(A, L) \cong \mathrm{Hom}_{\mathcal{C}[S^{-1}]}(A, L).$$

Otherwise there exists an object X in D such that $\mathrm{Hom}_{\mathcal{C}}(A, X) = \varnothing$ and $\mathrm{Hom}_{\mathcal{C}}(dom(f), X) = \varnothing$ for any $f \in S$. In this case $\mathrm{Hom}_{\mathcal{C}[S^{-1}]}(A, X) = \mathrm{Hom}_{\mathcal{C}[S^{-1}]}(A, X) = \varnothing$. Hence

$$\varnothing = \varprojlim \mathrm{Hom}_{\mathcal{C}[S^{-1}]}(A, D) = \varprojlim \mathrm{Hom}_{\mathcal{C}}(A, X) = \mathrm{Hom}_{\mathcal{C}}(A, L) = \mathrm{Hom}_{\mathcal{C}[S^{-1}]}(A, L).$$

So we obtain the result. □

Lemma 3. *Let $\mathcal{C}$ be a finite category, and suppose that $D_1, \ldots, D_n$ are finite diagrams in $\mathcal{C}$.*

1. *The completion $\widehat{\mathcal{C}}$ of the category $\mathcal{C}$ with respect to the diagrams $D_1, \ldots, D_n$ is finite. The completion functor preserves limits and colimits.*
2. *Let $\tilde{\mathcal{C}}$ be a finite category with $\mathrm{Obj}\, \widetilde{\mathcal{C}} = \mathrm{Obj}\, \mathcal{C} \cup \{L_1, \ldots, L_n\}$. Suppose that the morphisms in $\mathcal{C}$ are also morphisms of $\widetilde{\mathcal{C}}$, and that $\mathrm{Hom}\,(L_i, X) = \{*\}$ for X from definition of D_i. Suppose also that the pair $(\{D_i\}, \{L_i\})$ is a stratified formula with limits. We also will think that the order of indexes agrees with the order defined by the condition of being stratified. Then all the operations of completion at D_i and localisation of the completed category by the canonical morphism from L_i to the limit of D_i give us the finite category.*
3. *In context of the previous statement suppose also that the system $\{D_i\}$ is independent. Then there exists an initial object in the category of functors F to the categories in which $\varprojlim F(D_i)$ exists such that F preserves functors and the canonical morphism $F(B_i) \xrightarrow{\varphi_i} \varprojlim F(D_i)$ is an isomorphism for all i. Moreover the codomain of this functor is a finite category.*

Proof. 1. Firstly, note that for any functors $F, G : \mathcal{C}^{\mathrm{op}} \to \mathrm{FinSets}$ the set $\mathrm{Nat}(F, G)$ is finite since $\mathrm{Nat}(F, G)$ is a subset of the finite set $\prod_{X \in \mathrm{Obj}\, \mathcal{C}} \mathrm{Hom}\,(F(X), G(X))$. Construct $\widehat{\mathcal{C}}$ as the subcategory of $\mathrm{Func}(\mathcal{C}^{\mathrm{op}}, \mathrm{FinSets})$ consisting of representable functors and (pointwise) limits of diagrams of representable functors corresponding to D_i. It is finite since it contains a finite number of objects and all the hom-sets are finite. Note also that the canonical functor $\mathcal{C} \to \widehat{\mathcal{C}}$ preserves limits and colimits.

2. Let $\mathcal{E}$ be the completion of the category $\widetilde{\mathcal{C}}$ with respect to the diagrams $D_1, \ldots, D_n$. This category is finite by the first part of the lemma. Denote by S the set of morphisms $L_i \xrightarrow{\varphi_i} \varprojlim D_i$. We need to prove that $\mathcal{E}[S^{-1}]$ is finite. Firstly, we obtain that $\mathcal{E}[\varphi_i^{-1}]$ is finite. Every morphism f in $\mathcal{E}[\varphi_i^{-1}]$ can be presented as a certain zig-zag of the minimal length i.e. a morphism of the form $f_1 \circ \varphi_i^{-1} \circ \ldots f_m \circ \varphi_i^{-1} \circ g$ of minimal length. The only morphism from L_i to $\varprojlim D_i$ is φ_i, so m should be less or equal to 1 by the minimality of the presentation. So the set of morphisms between objects X

and Y is a quotient-set of a finite set $(\mathrm{Hom}_{\mathcal{E}}(X, \varprojlim D_i) \times \mathrm{Hom}_{\mathcal{E}}(L_i, Y)) \amalg \mathrm{Hom}_{\mathcal{E}}(X, Y)$ and hence, is a finite set itself. To do the inductive step we check that $\mathrm{Hom}_{\mathcal{E}[\{\varphi_1,\ldots,\varphi_k\}^{-1}]}(B_{k+1}, \varprojlim D_{k+1}) = \{*\}$ and use the argument given above. Suppose $f = f_1 \circ \varphi_{a_1}^{-1} \circ \ldots f_m \circ \varphi_{a_m}^{-1} \circ g$ is an element of $\mathrm{Hom}_{\mathcal{E}[\{\varphi_1,\ldots,\varphi_k\}^{-1}]}(B_{k+1}, X)$ for some object X from the diagram D_{k+1}. By the definition of the order on diagrams the codomain of f_j can't be X so $m = 0$ and hence $f \in \mathrm{Hom}_{\mathcal{E}}(B_{k+1}, X) = \{*\}$.

Now it suffices to prove that the given localisation functor preserves the limits of the diagrams D_i. Let $\mathcal{E}$ be a category and let $(\{L'_i\}, \{D'_i\})$ be a system of diagrams and objects (see the notation above) such that the system of diagrams $\{D'_i\}$ is independent. And suppose also that limits of D'_i exist in $\mathcal{E}$. Let ψ_i denote the canonical morphism from L'_i to $\varprojlim D'_i$. We are going to show that the localisation functor $\mathcal{E} \to \mathcal{E}[{\psi_1}^{-1}]$ preserves limits of D'_i and the system $\{D'_i\}$ is independent in $\mathcal{E}[\{\psi_1, \ldots, \psi_k\}^{-1}]$ for any possible k. The first statement follows from Lemma 2 and the second statement can be checked directly: suppose $A \in \mathrm{Obj}\,\mathcal{E}$, $f \in \mathrm{Hom}_{\mathcal{E}[\{\psi_1,\ldots,\psi_k\}^{-1}]}(A, X)$ for some $X \in D'_i$. It can be presented as a certain zig-zag $f_1 \circ \psi_{a_1} \circ \ldots \circ \psi_{a_h} \circ g$ of composable morphisms. The codomain of g is in D'_{a_h}, so there are no morphisms in $\mathcal{E}$ from A to some object $Y \in D_j$ as well as there are no morphisms in $\mathcal{E}$ from L'_r to Y for any r. Summarising, there are no zig-zags i.e. morphisms in $\mathcal{E}[\{\psi_1, \ldots, \psi_k\}^{-1}]$ from A to Y. So the image of a system of diagrams $\{D_i\}$ in the localized category is independent.

The above observation implies by induction argument that the localisation functor from $\mathcal{C}$ to $\mathcal{C}[\{\varphi_1, \ldots, \varphi_m\}^{-1}]$ preserves limits of D_i.

□

Theorem 4. *It is decidable whether a quantifier-free implication-like stratified formula in the language $\mathcal{L}$ above holds in an arbitrary category.*

Proof. The formula is implication-like positive, so the hypothesis of the implication corresponds to a directed graph whose vertices and edges are labelled by the variables, certain triangles are marked commutative, and certain vertices are marked as limits and colimits of certain distinguished subgraphs. This structure is finite; take the closure of the structure under the rules above. By construction? the closure is a category satisfying the assumption of the implication. Moreover, there is a natural functor from this category into any category satisfying the assumption; we may say this is a free category satisfying the assumption of the implication.

However, as the formula is stratified, the closure is still finite. Check whether the conclusion holds in this category. It is easy to check it does so iff it holds in any category satisfying the assumption of the implication.

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