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POINT-SET TOPOLOGY AS DIAGRAM CHASING COMPUTATIONS

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To Grigori Mints Z"L in memoriam

> 2. Hawk/Goose effect. A baby chick does not have any built-in image of "deadly hawk" in its head but distinguishes frequent, hence, harmless shapes, sliding overhead from potentially dangerous ones that appear rarely. Similarly to "first", "frequent" and "rare" are universal concepts that were not specifically designed by evolution for distinguishing hawks from geese. This kind of universality is what, we believe, turns the hidden wheels of the human thinking machinery.
>
> Misha Gromov, *Math Currents in the Brain.*

Abstract

We observe that some natural mathematical definitions are lifting properties relative to simplest counterexamples, namely the definitions of surjectivity and injectivity of maps, as well as of being connected, separation axioms T_0 and T_1 in topology, having dense image, induced (pullback) topology, and every real-valued function being bounded (on a connected domain).

We also offer a couple of brief speculations on cognitive and AI aspects of this observation, particularly that in point-set topology some arguments read as diagram chasing computations with finite preorders.

1. *Introduction. Structure of the Paper*

This note is written for *The De Morgan Gazette* to demonstrate that some natural definitions are lifting properties relative to the simplest counterexample, and to suggest a way to "extract" these lifting properties from the text of the usual definitions and proofs. The exposition is in the form of a story and aims to be self-contained and accessible to a first year student who has taken some first lectures in naive set theory, topology, and who has heard a definition of a category. A more sophisticated reader may find it more illuminating to recover our formulations herself from reading either the abstract, or the abstract and the opening sentence of the next section. The displayed formulae and

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Figure 1(a) defining the lifting property provide complete formulations of our theorems to such a reader.

Our approach naturally leads to a more general observation that in basic point-set topology, a number of arguments are computations based on symbolic diagram chasing with finite preorders; because of lack of space, we discuss it in a separate note [G0].

2. *Surjection and injection*

We try to find some "algebraic" notation to (re)write the *text* of the definitions of surjectivity and injectivity of a function, as found in any standard textbook. We want something very straightforward and syntactic—notation for what we (actually) say, for the text we write, and not for its meaning, for who knows what meaning is anyway?

(*)$_{\mathbf{words}}$ "A function f from X to Y is *surjective* iff for every element y of Y there is an element x of X such that $f(x) = y$."

A function from X to Y is an arrow $X \longrightarrow Y$. Grothendieck taught us that a point, say "x of X", is (better viewed as) as $\{\bullet\}$-valued point, that is an arrow

$$\{\bullet\} \longrightarrow X$$

from a (the?) set with a unique element; similarly "y of Y" we denote by an arrow

$$\{\bullet\} \longrightarrow Y.$$

Finally, make dashed the arrows required to "exist". We get the diagram Fig. 1(b) without the upper left corner; there "$\{\}$" denotes the empty set with no elements listed inside of the brackets.

(**)$_{\mathbf{words}}$ "A function f from X to Y is *injective* iff no pair of different points is sent to the same point of Y."

"A function f from X to Y" is an arrow $X \longrightarrow Y$. "A pair of points" is a $\{\bullet,\bullet\}$-valued point, that is an arrow

$$\{\bullet,\bullet\} \longrightarrow X$$

from a two element set; we ignore "different" for now. "the same point" is an arrow $\{\bullet\} \longrightarrow Y$. Represent "sent to" by an arrow

$$\{\bullet,\bullet\} \longrightarrow \{\bullet\}.$$

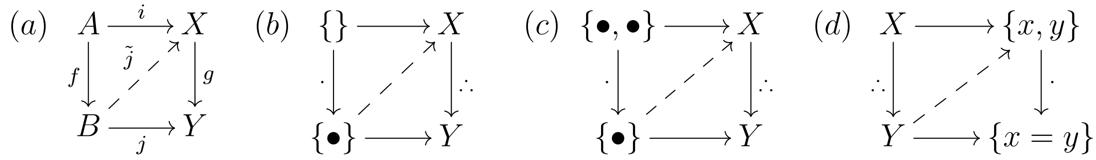

FIGURE 1: *Lifting properties. Dots ∴ indicate free variables, i.e. a property of what is being defined. (a) The definition of a lifting property* $f \rightthreetimes g$*: for each* $i : A \longrightarrow X$ *and* $j : B \longrightarrow Y$ *making the square commutative, i.e.* $f \circ j = i \circ g$*, there is a diagonal arrow* $\tilde{j} : B \longrightarrow X$ *making the total diagram* $A \xrightarrow{f} B \xrightarrow{\tilde{j}} X \xrightarrow{g} Y, A \xrightarrow{i} X, B \xrightarrow{j} Y$ *commutative, i.e.* $f \circ \tilde{j} = i$ *and* $\tilde{j} \circ g = j$*. (b)* $X \longrightarrow Y$ *is surjective (c)* $X \longrightarrow Y$ *is injective;* $X \longrightarrow Y$ *is an epicmorphism if we forget that* $\{\bullet\}$ *denotes a singleton (rather than an arbitrary object and thus* $\{\bullet, \bullet\} \longrightarrow \{\bullet\}$ *denotes an arbitrary morphism* $Z \sqcup Z \xrightarrow{(id,id)} Z$*) (d)* $X \longrightarrow Y$ *is injective, in the category of Sets;* $\pi_0(X) \longrightarrow \pi_0(Y)$ *is injective, in the category of topological spaces.*

What about "different"? If the points are not "different", then they are "the same" point, that is an arrow

$$\{\bullet\} \longrightarrow X.$$

Now all these arrows combine nicely into diagram Figure 1(c). How do we read it? We want this diagram to have the meaning of the sentence $(**)_{\text{words}}$ above, so we interpret such diagrams as follows:

($\rightthreetimes$) "for every commutative square (of solid arrows) as shown there is a diagonal (dashed) arrow making the total diagram commutative" (see Fig. 1(a)).

(recall that "commutative" in category theory means that the composition of the arrows along a directed path depends only on the endpoints of the path)

Property ($\rightthreetimes$) has a name and is in fact quite well-known [Qui]. It is called *the lifting property*, or sometimes *orthogonality of morphisms*, and is viewed as the property of the two downward arrows; we denote it by $\rightthreetimes$.

Now we rewrite $(*)_{\text{words}}$ and $(**)_{\text{words}}$ as:

$$(*)_{\rightthreetimes} \qquad \{\} \longrightarrow \{\bullet\} \rightthreetimes X \longrightarrow Y$$

$$(**)_{\rightthreetimes} \qquad \{\bullet, \bullet\} \longrightarrow \{\bullet\} \rightthreetimes X \longrightarrow Y$$

So we rewrote these definitions without any words at all. Our benefits? The usual little miracles happen:

Notation makes apparent a similarity of $(*)_{\text{words}}$ and $(**)_{\text{words}}$: they are obtained, in the same purely formal way, from the two simplest arrows (maps, morphisms) in the category of Sets. More is true: it is also apparent that these arrows are the simplest *counterexamples* to the properties, and this suggests that we think of the lifting property as a category-theoretic (substitute for) negation. Note also that a non-trivial (=non-isomorphism) morphism never has the lifting property relative to itself, which fits with this interpretation.

Now that we have a formal notation and the little observation above, we start to play around looking at simple arrows in various categories, and also at not-so-simple arrows representing standard counterexamples.

You notice a few words from your first course on topology: (*i*) *connected,* (*ii*) *the separation axioms* T_0 *and* T_1*,* (*iii*) *dense,* (*iv*) *induced (pullback) topology*, and (*v*) *Hausdorff* are, respectively,

(i):

$$X \longrightarrow \{\bullet\} \rightthreetimes \{\bullet,\bullet\} \longrightarrow \{\bullet\}$$

(ii):

$$\{\bullet \gtrless \star\} \longrightarrow \{\bullet\} \rightthreetimes X \longrightarrow \{\bullet\}$$

and

$$\{\bullet < \star\} \longrightarrow \{\bullet\} \rightthreetimes X \longrightarrow \{\bullet\}$$

(iii):

$$X \longrightarrow Y \rightthreetimes \{\bullet\} \longrightarrow \{\bullet \to \star\}$$

(iv):

$$X \longrightarrow Y \rightthreetimes \{\bullet < \star\} \longrightarrow \{\bullet\}$$

(v):

$$\{\bullet,\bullet'\} \longrightarrow X \rightthreetimes \{\bullet > \star < \bullet'\} \longrightarrow \{\bullet\}$$

See the last two pages for illustrations how to read and draw on the blackboard these lifting properties in topology; here

$$\{\bullet < \star\},\ \{\bullet \gtrless \star\},\ \ldots$$

denote finite preorders, or, equivalently, finite categories with at most one arrow between any two objects, or finite topological spaces on their elements or objects, where a subset is closed iff it is downward closed (that is, together with each element, it contains all the smaller elements). Thus

$$\{\bullet < \star\},\ \{\bullet \gtrless \star\}\ \text{ and }\ \{\bullet > \star < \bullet'\} \longrightarrow \{\bullet\}$$

denote the connected spaces with only one open point $\bullet$, with no open points, and with two open points $\bullet, \bullet'$ and a closed point $\star$. Line (v) is to be interpreted somewhat differently: we consider *all* the arrows of form

$$\{\bullet, \bullet'\} \longrightarrow X.$$

We mentioned that the lifting property can be seen as a kind of negation. Confusingly, there are *two* negations, depending on whether the morphism appears on the left or right side of the square, that are quite different: for example, both the pullback topology and the separation axiom T_1 are negations of the same morphism, and the same goes for injectivity and injectivity on π_0 (see Figure 1(c,d)).

Now consider the standard example of something non-compact: the open covering

$$\mathbb{R} = \bigcup_{n\in\mathbb{N}} \{\, x \,:\, -n < x < n \,\}$$

of the real line by infinitely many increasing intervals. A related arrow in the category of topological spaces is

$$\bigsqcup_{n\in\mathbb{N}} \{\, x \,:\, -n < x < n\} \longrightarrow \mathbb{R}.$$

Does the lifting property relative to that arrow define compactness? Not quite but almost:

$$\{\} \longrightarrow X \rightthreetimes \bigsqcup_{n\in\mathbb{N}} \{x \,:\, -n < x < n\,\} \longrightarrow \mathbb{R}$$

reads, for X connected, as "every continuous real-valued function on X is bounded", which is an early characterisation of compactness taught in a first course on analysis.

A category theorist would rewrite $(**)_{\rightthreetimes}$ as

$$\textbf{(**)}_{\textbf{mono}} \qquad \bullet \vee \bullet \longrightarrow \bullet \rightthreetimes X \longrightarrow Y$$

denoting by $\vee$ and $\bullet \vee \bullet \longrightarrow \bullet$ the coproduct and the codiagonal morphism, respectively, and then read it as follows: given two morphisms

$$\bullet \xrightarrow{\text{left}} X \quad \text{and} \quad \bullet \xrightarrow{\text{right}} X,$$

if the compositions

$$\bullet \xrightarrow{\text{left}} X \longrightarrow Y \;=\; \bullet \xrightarrow{\text{right}} X \longrightarrow Y$$

are equal (both to $\bullet \xrightarrow{\text{down}} Y$), then

$$\bullet \xrightarrow{\text{left}} X \;=\; \bullet \xrightarrow{\text{right}} X$$

are equal (both to $\bullet \xrightarrow{\text{down}} X$). Naturally her first assumption would be that $\bullet$ denotes an *arbitrary* object, as that spares the extra effort needed to invent the axioms particular to the category of sets (or topological spaces) that capture that $\bullet$ denotes a single element, i.e. allow to treat $\bullet$ as a single element. (A logician understands "arbitrary" as "we do not know", "make no assumptions", and that is how formal derivation systems treat "arbitrary" objects.) Thus she would read $(**)_{\rightthreetimes}$ as the usual category theoretic definition of a monomorphism. Note this reading doesn't need that the underlying category has coproducts: a category theorist would think of working inside a larger category with formally added coproducts $\bullet \vee \bullet$, and a logician would think of working inside a formal derivation system where "$\bullet$" is either a built-in or "a new variable" symbol, and "$\bullet \vee \bullet \longrightarrow \bullet$" (or "$\{\bullet, \bullet\} \longrightarrow \{\bullet\}$") is (part of) a well-formed term or formula.

And of course, nothing prevents a category theorist to make a dual diagram

$$(**)_{\mathbf{epi}} \qquad X \longrightarrow Y \rightthreetimes \bullet \longrightarrow \bullet \times \bullet, \qquad \bullet \text{ runs through all the objects}$$

and read it as:

$$X \longrightarrow Y \xrightarrow{\text{left}} \bullet \;=\; X \longrightarrow Y \xrightarrow{\text{right}} \bullet \text{ implies } Y \xrightarrow{\text{left}} \bullet \;=\; Y \xrightarrow{\text{right}} \bullet$$

which is the definition of an epimorphism.

3. *Speculations.*

Does your brain (or your kitten's) have the lifting property built-in? Note [G0] suggests a broader and more flexible context making contemplating an experiment possible. Namely, some standard arguments in point-set topology are computations with category-theoretic (not always) commutative diagrams of preorders, in the same way that lifting properties define injection and surjection. In that approach, the lifting property is viewed as a rule to add a new arrow, a computational recipe to modify diagrams.

Can one find an experiment to check whether humans *subconsciously* use diagram chasing to reason about topology?

Does it appear implicitly in old original papers and books on point-set topology?

Would teaching diagram chasing hinder or aid development of topological intuition in a first course of topology? Say if one defines connected, dense, Hausdorff et al via the lifting properties (i–v)?

Is diagram chasing with preorders too complex to have evolved? Perhaps; but note the self-similarity: preorders are categories as well, with the property that there is at most one arrow between any two objects; in fact sometimes these categories are thought of as 0-categories. So essentially your computations are in the category of (finite 0-) categories.

Is it universal enough? Diagram chasing and point-set topology, arguably a formalisation of "nearness", is used as a matter of course in many arguments in mathematics.

Finally, isn't it all a bit too obvious? Curiously, in my experience it's a party topic people often get stuck on. If asked, few if any can define a surjective or an injective map without words, by a diagram, or as a lifting property, even if given the opening sentence of the previous section as a hint. No textbooks seem to bother to mention these reformulations (why?). An early version of [GH-I] states $(*)_{\rightthreetimes}$ and $(**)_{\rightthreetimes}$ as the simplest examples of lifting properties we were able to think up; these examples were removed while preparing for publication.

No effort has been made to provide a complete bibliography; the author shall happily include any references suggested by readers in the next version [G].

Acknowledgments and historical remarks

It seems embarrassing to thank anyone for ideas so trivial, and we do that in the form of historical remarks. Ideas here have greatly influenced by extensive discussions with Grigori Mints, Martin Bays, and, later, with Alexander Luzgarev and Vladimir Sosnilo. At an early stage Ksenia Kuznetsova helped to realise an earlier reformulation of compactness was inadequate and that labels on arrows are necessary to formalise topological arguments. "A category theorist [that rewrote] (**) as" the usual category theoretic definition of a monomorphism, is Vladimir Sosnilo. Exposition has been polished in the numerous conversations with students at St. Peterburg and Yaroslavl'2014 summer school.

Reformulations $(*)_{\rightthreetimes}$ and $(**)_{\rightthreetimes}$ of surjectivity and injectivity, as well as connectedness and (not quite) compactness, appeared in early drafts of a paper [GH-I] with Assaf Hasson as trivial and somewhat curious examples of a lifting property but were removed during preparation

for publication. After $(**)_{\rightthreetimes}$ came up in a conversation with Misha Gromov the author decided to try to think seriously about such lifting properties, and in fact gave talks at logic seminars in 2012 at Lviv and in 2013 at Munster and Freiburg, and 2014 at St. Petersburg. At a certain point the author realised that possibly a number of simple arguments in point-set topology may become diagram chasing computations with finite topological spaces, and Grigori Mints insisted these observations be written. Ideas of [ErgB] influenced this paper (and [GH-I] as well), and particularly our computational approach to category theory. Alexandre Borovik suggested to write a note for *The De Morgan Gazette* explaining the observation that 'some of human's "natural proofs" are expressions of lifting properties as applied to "simplest counterexample"'.

I thank Yuri Manin for several discussions motivated by [GH-I].

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I wish to express my deep thanks to Grigori Mints, to whose memory this paper is dedicated …

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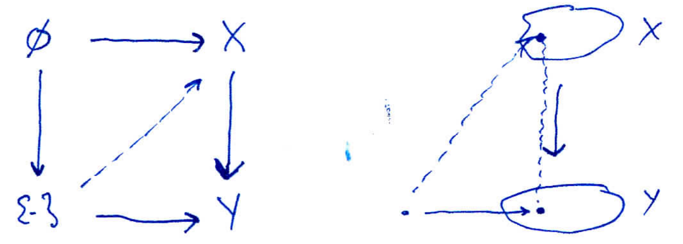
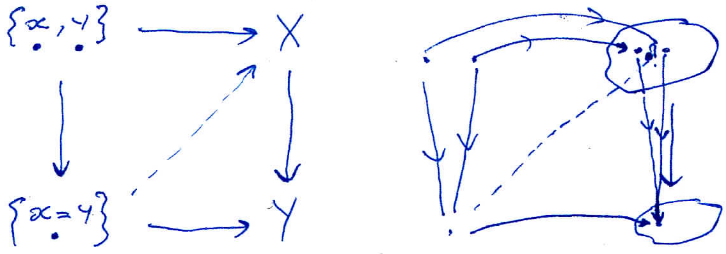
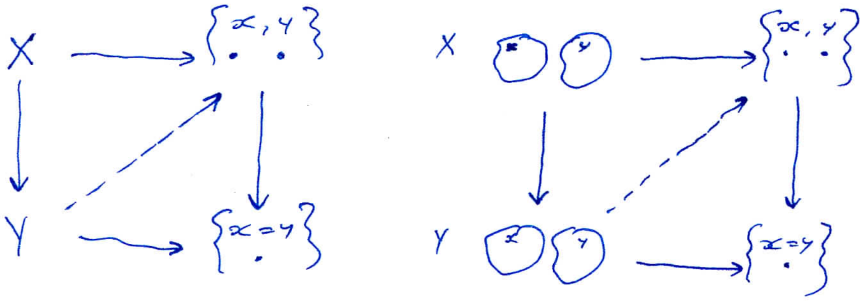


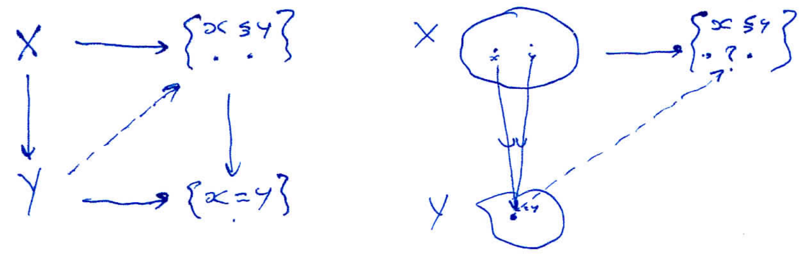
$X \twoheadrightarrow Y$



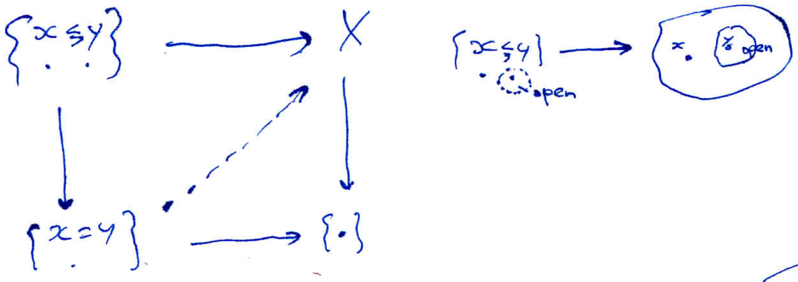
$X \hookrightarrow Y$



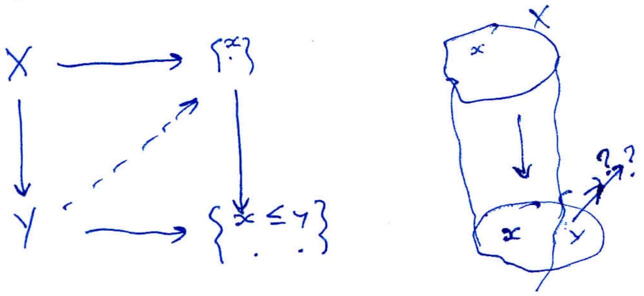
$\pi_0(X) \hookrightarrow \pi_0(Y)$



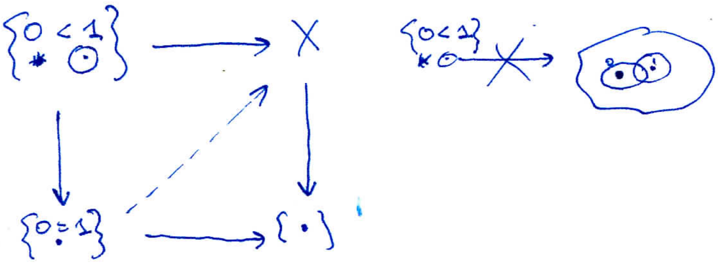
$X \hookrightarrow Y$



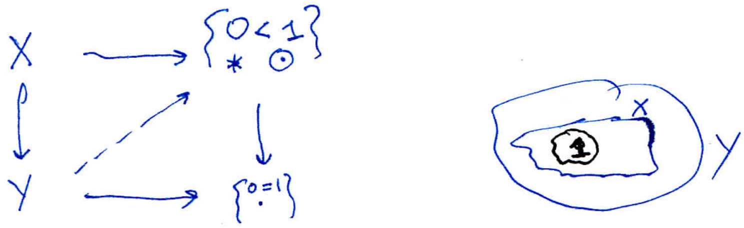
X is T_0



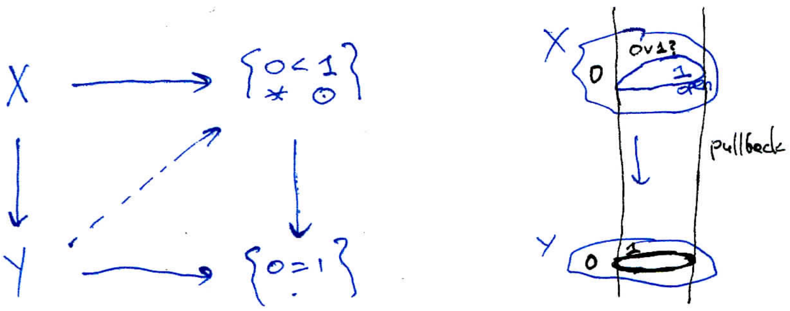
$X \twoheadrightarrow Y$



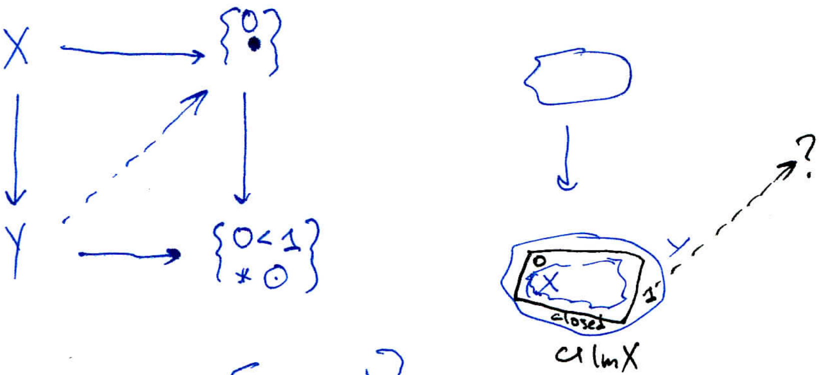
X is T_1



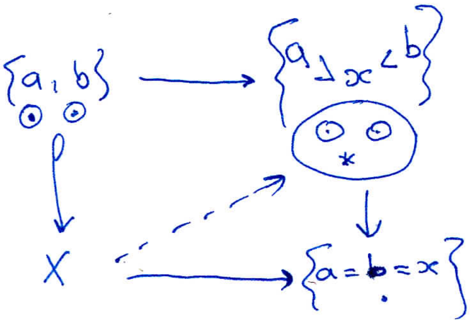
induced topology



the coarsest topology



Im X is dense



X is T_2 (Hausdorff)