POINT-SET TOPOLOGY AS DIAGRAM CHASING COMPUTATIONS

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Abstract. We give several examples of properties in point-set topology, such as being Hausdorff or that equalizier $\{x : f(x) = g(x)\}$ is being closed for maps to a Hausdorff space that translate to diagram chasing rules, and where the lifting property is a "category theory negation". This observation can be implemented in a problem solver of a kind suggested by M. Ganesalingam and T. Gowers.

This is a preliminary publication.

Gde-to est' ljudi, dlya kotoryh teorema werna

Somewhere there are people for whom
a theorem is true

1. Introduction

Finite topological spaces and diagram chasing play an important role in this paper, and the first subsection introduces our notation and terminology for them. More background is provided in the following subsection but it is not essential for understanding.

1.1. **Introduction. Notation for finite topological spaces.** A finite topological space defines, and is defined by, a transitive binary relation, or, in another teminology, a partial preorder, the *specialisation preorder*, on its set X of points:

$y \leqslant x$ iff y lies in the least closed set containing the point x, $y \in Clx$.

Downward closed sets are closed, and upward closed sets are open.

1.1.1. *Notation for finite topological spaces and continuous maps.* A list of equalities and inequalities between symbols denotes the partial partial preorder, i.e. a transitive binary relation, obtained by transitive closure of the inequalities and whose elements are equivalence classes generated by the equalities. Thus, $\{x = y\}$ and $\{x \leqslant y = z\}$, $\{x \gtrless y = z < t\}$ denote partial preorders on 1,2 and 3 elements. An arrow $L_1 \longrightarrow L_2$, e.g. $\{a < b, b \gtrless c, c = d, ...\} \longrightarrow \{a < b < b \gtrless c, c = d, d < e, f = g, ...\}$, between such lists L_1 and L_2, denote the order preserving map from the order corresponding to L_1 to the order corresponding to $L_1 \cup L_2$ sending each element to itself, or rather the equivalence class containing the element.

For example, $\{o, x\}$, $\{o \gtrless x\}$ and $\{o > x\}$ respectively denote the 2-point topological space with the discrete topology (all subsets are open), the antidiscrete topology (only the empty set and the whole space are open), and the topology with only one open point o and one closed point x. The spaces $\{x\}$ and $\{x = y\}$ have a single point. Spaces $\{o, x\}$ and $\{o, x = y\}$ are homeomorphic. The arrow $\{y\} \longrightarrow \{o > x = y\}$ is the map sending the point y into the only closed point. The arrow $\{y\} \longrightarrow \{o > x\}$ denotes the same map as the arrow $\{y\} \longrightarrow \{o > x, y\}$, namely the map between a one-point and 3-point spaces.

1.1.2. *Generalities on diagram chasing.* A category $\Re$ consists of a collection of *objects* $\mathcal{O}b\Re$, a collection of morphism, or arrow, sets $\mathcal{M}or(X,Y) = Arr(X,Y)$ for each pair $X,Y \in \mathcal{O}b\Re$ of its objects, a distinguished *identity* element $\mathrm{id}_X \in \mathcal{M}or(X,X)$ for each $X \in \mathcal{O}b\Re$, and a *composition* operation $\circ : \mathcal{M}or(X,Y) \times \mathcal{M}or(Y,Z) \longrightarrow \mathcal{M}or(X,Z)$ for every triple of objects, such that $(f \circ g) \circ h = f \circ (g \circ h)$ and $\mathrm{id}_X \circ f = f = f \circ \mathrm{id}_Y$ for each $f \in Arr(X,Y), g \in Arr(Y,Z), h \in Arr(Z,T)$ which is usually written as $f : X \longrightarrow Y$, $g : Y \longrightarrow Z$, $h : Z \longrightarrow T$ or $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T$. A *diagram* is a directed graph where every vertex is labelled by an object of the category, and each directed edge between vertices labelled X and Y is labelled by an arrow in $\mathcal{M}or(X,Y)$. A diagram *commutes from a vertex x to a vertex y* iff the composition of labels along a path between these vertices does not depend on the path taken from x to y. A *commutative diagram* is a diagram commuting between any two vertices. A helpful topological way to think of categories as of a metric graph (with multiple edges) with surfaces gluing in subgraphs corresponding to commutative diagrams; thus commutative diagrams correspond to contractible subgraphs. *Diagram chasing* refers to a process of category theoretic reasoning and computation involving rules for manipulating commutative diagrams.

1.2. **Background: finite topological spaces as categories, preorders, and homotopy types.** We observe that the following 4 notions are equivalent:

(1) finite topological spaces, and continuous maps
(2) binary transitive reflexive relations on finite sets, and functions respecting the relation
(3) finite partial preorders, and order preserving maps
(4) finite categories such that there is at most one morphisms between any two objects, and (covariant) functors

A finite partial preorder we define as a binary transitive reflexive relation.

1.2.1. *Finite topological spaces as binary transitive relations.* A finite topological space defines, and is defined by, a transitive binary relation, or, in another teminology, a partial preorder, the *specialisation preorder*, on its set X of points:

$$y \leqslant x \text{ iff } y \text{ lies in the least closed set containing the point } x,\ y \in Clx.$$

A subset $Z \subseteq X$ is closed in the topology iff it is downward closed, i.e. $x \in Z$, $y \leqslant x$ implies $y \in Z$. Indeed, a finite union of closed subsets is closed, and this implies a subset Z is closed iff it is the (necessary finite) union of closures of its points. A subset $U \subseteq X$ is open iff it is upward closed, i.e. $u \in U$, $u \leqslant v$ implies $v \in U$. A map $f : X \longrightarrow Y$ is continuous iff the preimage of an open set is open, i.e. iff f preserves the order relation, i.e. $x \leqslant y$ implies $f(x) \leqslant f(y)$. If $f(x) \leqslant g(x)$ for all $x \in X$, then f and g are homotopic.

1.2.2. *Finite topological spaces as categories with unique morphisms.* A finite topological space defines, and is defined by, a category with unique morphisms: its objects are the points of the topological space, and $Mor(x,y) = \{\bullet\}$ iff y lies in the least closed set containing the point x, $y \in Clx$. All diagrams are necessarily commutative. A subset $Z \subseteq X$ is closed iff there are no arrows going outside of the corresponding full subcategory $\mathcal{Z}$, i.e. i.e. $A \in \mathcal{O}b\mathcal{Z}$, $A \longrightarrow B$ implies $B \in \mathcal{O}b\mathcal{Z} = Z$. A subset $U \subseteq X$ is open iff there are no arrows going into the corresponding full subcategory from outside of it, i.e. $A \in \mathcal{O}b\mathcal{U}$, $B \longrightarrow A$ implies $B \in \mathcal{O}b\mathcal{U}$. A map $f : X \longrightarrow Y$ is continuous iff it is a functor, i.e. $A \longrightarrow B$ implies $f(A) \longrightarrow f(B)$, and necessary commutativity conditions hold (trivially, for categories with unique morphisms).

A natural transformation $f \implies g$, i.e. a collection of morphisms $f(A) \xrightarrow{\epsilon_A} g(A)$ for all $A \in X$ with certain commutativity conditions (again trivial for categories with unique morphisms), is a homotopy between maps f and g, but not every homotopy has such a form.

1.2.3. *Finite topological spaces as homotopy types.* Let $X = \bigcup_{0<i<N} D_i$ be a simplicial triangulation of a space X, i.e. $X = \bigcup D_i$, each D_i is a homeomorphic copy of a simplex Δ_n for some $n \in \mathbb{N}$, and for each $0 < i, j < N$ there is $0 < k < N$ such that $D_k = D_i \cap D_j$ unless $D_i \cap D_j = \emptyset$. Let $\mathbb{D}$ be the inclusion partial order on D_i's where $D_i \leqslant D_j$ iff $D_i \subseteq D_j$. Then the map $X \xrightarrow{\tau} \mathbb{D}$ is a weak homotopy equivalence, i.e. map $Z \xrightarrow{f} X$ and $Z \xrightarrow{g} X$ are homotopy equivalent iff $Z \xrightarrow{f} X \xrightarrow{\tau} \mathbb{D}$ and $Z \xrightarrow{g} X \xrightarrow{\tau} \mathbb{D}$ are homotopy equivalent.

The observation above allows one to use simplicial triangulations to find finite analogues of notions in homotopy theory; let us give a couple of examples. Cover a circle by two intervals intersecting at their endpoints. This gives rise to a weak homotopy equivalence $\mathbb{S}^1 \longrightarrow \{x_1 < o_1, x_2 < o_1, x_1 < o_2, x_2 < o_2\}$ to the 4-point space $\{x_1 < o_1, x_2 < o_1, x_1 < o_2, x_2 < o_2\}$ known as *a finite circle*. For a finite topological space X, *non-Hausdorff cone* CX of an order is is the preorder $CX \cup \{\star < x : x \in X\}$ obtained by adding a point $\star$ at the bottom. The non-Hausdorff suspension ΣX is obtained by adding two incomparable points on top, i.e. is the order $\Sigma X \cup \{\star < x, \bullet < x : x \in X\}$. Both maps come with canonical inclusions $X \longrightarrow CX$ and $X \longrightarrow \Sigma X$. An order has a greatest or least element implies it is contractible.

The circle $\mathbb{S}^1$ is the unique 2nd countable space that splits into two connected spaces after removing any two points. An analogous characterisation exists for spheres of any dimension. An interval I with endpoints $a, b \in I$ is the unique 2nd countable space that splits into two connected components after removing any point except a and b, and remains connected after removing either a, b or both.

2. Examples: Lifting properties against counterexamples

In this section we observe that a single categorical construction, the lifting property, defines such properties as *being injective, surjective, connected, separation axioms* T_0*,* T_1*,* T_2*,* T_4*, dense, induced topology; surjective and injective on* π_0. Moreover, as the input data, it uses a minimal or typical counterexample to the property, usually a monotone map between spaces of one and two points or less.

Lifting property also defines *Compactness and preusodocompactness* (for connected spaces).

2.1. **Lifting property: the definition.** We say that *the lifting property holds for morphisms $f : A \longrightarrow B$ and $g : X \longrightarrow Y$, write $f \rightthreetimes g$ or $A \longrightarrow B \rightthreetimes X \longrightarrow Y$*, iff for every morphisms $i : A \longrightarrow X$ and $j : B \longrightarrow Y$, if the square $A \xrightarrow{i} X \xrightarrow{g} Y$, $A \xrightarrow{f} B \xrightarrow{j} Y$ commutes, i.e. $f \circ j = i \circ g$, then there exists a morphism $\tilde{j} : B \longrightarrow X$ making the total diagram $A \xrightarrow{i} X, B \xrightarrow{j} Y$ and $A \xrightarrow{f} B \xrightarrow{\tilde{j}} X \xrightarrow{g} Y$, commute, i.e. $f \circ \tilde{j} = i$ and $\tilde{j} \circ g = j$.

Recall $x \lesseqgtr y$ means that both $x \leqslant y$ and $y \leqslant x$.

2.2. **Lifting properties involving a single point.**

(1) $X \longrightarrow Y$ is surjective iff $\emptyset \longrightarrow \{\bullet\} \rightthreetimes X \longrightarrow Y$
(2) X is non-empty or $X = Y = \emptyset$ iff $X \longrightarrow Y \rightthreetimes \emptyset \longrightarrow \{\bullet\}$

2.3. **Injective, surjective, connected, T_0, T_1, dense, induced topology; surjective and injective on π_0 as lifting properties.** Let us list the lifting properties against all the morphisms involving a non-empty space and a two point space.

(1) $\{x, y\} \longrightarrow \{x = y\}$
  (a) $X \longrightarrow Y$ is injective iff $\{x, y\} \longrightarrow \{x = y\} \rightthreetimes X \hookrightarrow Y$
  (b) X is connected iff $X \longrightarrow \{x = y\} \rightthreetimes \{x, y\} \longrightarrow \{x = y\}$
  (c) $\pi_0(X) \hookrightarrow \pi_0(Y)$ is injective iff $X \longrightarrow Y \rightthreetimes \{x, y\} \longrightarrow \{x = y\}$
(2) $\{x\} \longrightarrow \{x, y\}$
  (a) $X \longrightarrow Y$ is surjective or X (or Y) is empty iff $\{x\} \longrightarrow \{x, y\} \rightthreetimes X \twoheadrightarrow Y$
  (b) $\pi_0(X) \longrightarrow \pi_0(Y)$ is surjective iff $X \longrightarrow Y \rightthreetimes \{x\} \longrightarrow \{x, y\}$
(3) $\{x \lesseqgtr y\} \longrightarrow \{x = y\}$
  (a) $X \longrightarrow Y$ is injective iff $X \longrightarrow Y \rightthreetimes \{x \lesseqgtr y\} \longrightarrow \{x = y\}$
  (b) X is T_0, i.e. at least one of any two distinct points of X has an open neighbourhood which does not contain the other point iff
  $\{x \lesseqgtr y\} \longrightarrow \{x = y\} \rightthreetimes X \longrightarrow \{x = y\}$
  (c) the fibres of $X \longrightarrow Y$ are T_0 iff $\{x \lesseqgtr y\} \longrightarrow \{x = y\} \rightthreetimes X \longrightarrow Y$
(4) $\{x\} \longrightarrow \{x \lesseqgtr y\}$
  (a) $X \longrightarrow Y$ is surjective iff $X \longrightarrow Y \rightthreetimes \{x\} \longrightarrow \{x \lesseqgtr y\}$
  (b) if no open set separates every $y, y' \in Y$, then there exist preimages of y and y' with the same property
  iff $\{x\} \longrightarrow \{x \lesseqgtr y\} \rightthreetimes X \longrightarrow Y$
(5) $\{0 < 1\} \longrightarrow \{0 = 1\}$
  (a) X is T_1, i.e. each one of any two distinct points of X has an open neighbourhood which does not contain the other point iff
  $\{0 < 1\} \longrightarrow \{0 = 1\} \rightthreetimes X \longrightarrow \{0 = 1\}$
  (b) $X \longrightarrow Y$ is a subspace with induced topology
  iff $X \longrightarrow Y \rightthreetimes \{0 < 1\} \longrightarrow \{0 = 1\}$ and $X \longrightarrow Y \rightthreetimes \{0 \lesseqgtr 1\} \longrightarrow \{0 = 1\}$

(c) the topology on X is the coarsest topology such that $X \longrightarrow Y$ is continuous, i.e. topology on X consists of preimages of open subsets of Y iff
$X \longrightarrow Y \rightthreetimes \{0 < 1\} \longrightarrow \{0 = 1\}$

(d) for $X \hookrightarrow Y$, the topology on X is induced from the topology on Y
iff $X \hookrightarrow Y \rightthreetimes \{0 < 1\} \longrightarrow \{0 = 1\}$

(6) $\{0\} \longrightarrow \{0 < 1\}$

(a) the image of X is dense iff $X \longrightarrow Y \rightthreetimes \{0\} \longrightarrow \{0 < 1\}$

(b) for every $x \in X, y \in Y$, $h(x) \in Cl(y)$ implies $x \in Cl(y')$ for some $y', h(y') = y$
iff $\{0\} \longrightarrow \{0 < 1\} \rightthreetimes X \longrightarrow Y$

(7) $\{1\} \longrightarrow \{0 < 1\}$

(a) no proper open subset of Y contains the image of X iff
$X \longrightarrow Y \rightthreetimes \{1\} \longrightarrow \{0 < 1\}$

(b) for every $x \in X, y \in Y$, $y \in Cl_Y(h(x))$ implies $y' \in Cl_X(x)$ for some $y' \in X$, $h(y') = y$
iff $\{1\} \longrightarrow \{0 < 1\} \rightthreetimes X \longrightarrow Y$

Note that roughly a third of these lifting properties correspond to standard notions defined and used in a first course in general topology.

2.4. **More then three points: Hausdorff and T_4.** More points are needed to define Hausdorff and T_4 spaces. Note that $\{a > x < b\}$ and $\{w_1 < u_1 = x = u_2 > w_2\}$ are the smallest non-Hausdorff and non-T_4 but T_3 spaces, resp.

(1) T is T_2, or Hausdorff, iff
$\{a, b\} \hookrightarrow X \rightthreetimes \{a > x < b\} \longrightarrow \{a = x = b\}$

(2) X is normal, or T_4, i.e. for every two closed sets $W_1, W_2 \subseteq X$ there exists non-intersecting open neighbourhoods $U_1 \supseteq W_1$ and $U_2 \supseteq W_2$, such that $U_1 \cap U_2 = \emptyset$, iff $\emptyset \longrightarrow X \rightthreetimes \{w_1 > u_1 < x > u_2 < w_2\} \longrightarrow \{w_1 < u_1 = x = u_2 > w_2\}$

We also remark that X is T_4 iff for every closed embedding $A \hookrightarrow X$, it holds $A \hookrightarrow X \rightthreetimes \mathbb{R} \longrightarrow \{\bullet\}$.[1]

2.5. **Compactness and pseudocompactness as a lifting property.** We observe that both being pseudocompact and being compact (not necessarily Hausdorff) are lifting properties with respect to simple counterexamples.

(1) Every continuous function $X \longrightarrow \mathbb{R}$ is bounded iff
$\emptyset \longrightarrow X \rightthreetimes \coprod\{[-n, n] : n \in \mathbb{Z}\} \longrightarrow \mathbb{R}$

(2) X is compact (not necessarily Hausdorff) iff for every ordinal α,
$\emptyset \longrightarrow X \rightthreetimes \coprod\{\beta : \beta \in \alpha\} \longrightarrow \alpha$

2.6. **Closed subsets of metric spaces as a lifting property.** Let $\mathbb{N}$ denote the discrete space with countably many points, and let $\mathbb{N} \cup \{\infty\}$ denote its one-point compactification, i.e. a subset of $\mathbb{N} \cup \{\infty\}$ is closed iff it is either finite or contains ∞.

[1] We thank http://math.stackexchange.com/questions/434312/are-there-useful-categorical-characterisations-of-the-topological-separation-axi for this remark

(1) In a metric space X, a subset $A \subseteq X$ is closed iff $\mathbb{N} \longrightarrow \mathbb{N} \cup \{\infty\} \rightthreetimes A \hookrightarrow X$

3. Extracting diagrammatic "meaning"

We observe that several basic definitions in a first course of topology correspond to diagram chasing (computational) rules; here we use two examples to explain our naive and straightforward approach how to extract a commutative diagram out of a definition in point-set topology.

3.1. **Diagram chasing reformulation of Hausdorff separation property.** Recall the definition of a Hausdorff space: a topological space T is Hausdorff, or T_2, iff for every two different points $x, y \in T$ there exist open neighbourhoods $U_x \ni x$ and $U_y \ni y$ which do not intersect, $U_x \cap U_y = \emptyset$. Now we need to figure out a way to draw arrows corresponding to the x, y, U_x, U_y and their relationships.

A point $x \in T$ gives us an arrow, meaning a (necessarily continuous) map(morphism), from a single point space $\{x\} \longrightarrow T$. A pair of points $x \neq y \in T$ gives us an injective arrow $\{x, y\} \hookrightarrow T$ in the category of topological spaces.

Non-intersecting subsets $U_x, U_y \subseteq T$ give us a set-theoretic single arrow $T \longrightarrow \{x, y, \star\}$ defined by $U_x = f^{-1}(x), U_y = f^{-1}(y)$. A map is continuous iff the preimage of every open set is open, and we want to say that U_x and U_y are open: thus we define the topology on a space of three points $x, y, \star$ where the open subsets are singleton sets $\{x\}$ and $\{y\}$, as well as the empty set $\emptyset$ and the whole space $\{x, y, \star\}$. Thus, in notation explained above, we get an arrow $T \longrightarrow \{x > \star < y\}$.

Finally, we need to say that $x \in U_x$ and $y \in U_y$: this means that the two arrows $\{x, y\} \longrightarrow T$ and $T \longrightarrow \{x > \star < y\}$ above and the arrow $\{x, y\} \longrightarrow \{x > \star < y\}$ form a commutative triangle.

Now we are able to rewrite the definition: T is Hausdorff iff

$T_2(i)$ for every injective arrow $\{x, y\} \longrightarrow T$ there is an arrow $T \longrightarrow \{x > \star < y\}$ making a commutative triangle $\{x, y\} \longrightarrow T$ and $T \longrightarrow \{x > \star < y\}$, $\{x, y\} \longrightarrow \{x > \star < y\}$ commute.

We can reformulate this:

$T_2(ii)$ the arrow $\{x, y\} \longrightarrow \{x > \star < y\}$ factors through every injective arrow $\{x, y\} \longrightarrow T$

$T_2(iii)+$ given arrows $\{x\} \longrightarrow T$, $\{y\} \longrightarrow T$, $\{x\} \longrightarrow \{x > \star < y\}$, $\{y\} \longrightarrow \{x > \star < y\}$ add either of

$T_2(iii')-$ $\{x\} \longrightarrow \{y\}$

$T_2(iii'')-$ $\{x\} \longrightarrow \{x > \star < y\}$, $\{y\} \longrightarrow \{x > \star < y\}$, and $T \longrightarrow \{x > \star < y\}$

and mark the diagram of all these arrows as commutative

In fact, one may reformulate this as a lifting property; the proof requires a combinatorial check.

$T_2(iv)$ $\{x, y\} \hookrightarrow T \rightthreetimes \{x > \star < y\} \longrightarrow \{x = y = \star\}$.

3.2. **Limits are unique in Hausdorff spaces.** Now let us explain how to read diagrams and diagram chasing rules from the theorem that in a Hausdorff space any sequence has at most one limit, to reformulate it in our diagram chasing way.

3.2.1. *Limits are unique in Hausdorff spaces: rewriting the statement.* Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of points of a topological space T. A point $x \in X$ is a limit of the sequence if for any neighborhood $U \ni x$ of x there exists a number N such that $a_n \in U$ for every $n > N$.

Recall we treat a point of a space as an arrow from a singleton. Here we see arrows $\{a_n\}_{n\in\mathbb{N}} \longrightarrow T$ and arrow $\{x\} \longrightarrow T$, or rather more correctly $\mathbb{N} \xrightarrow{\{a_n\}_{n\in\mathbb{N}}} T$ and $\{\bullet\} \xrightarrow{x} T$. However, we choose to denote there arrows as $\{a_n\text{''}\}_{n\in\mathbb{N}} \xrightarrow{\{a_n\}_{n\in\mathbb{N}}} T$ and $\{\text{``}x\text{''}\} \xrightarrow{x} T$, or even $\{\text{``}a_n\text{''}\}_{n\in\mathbb{N}} \to T$ and $\{\text{``}x\text{''}\} \to T$: quotation marks '“”' ensure that $\{a_n\text{''}\}_{n\in\mathbb{N}}$ and $\mathbb{N}$ are isomorphic. This notation enables one to guess what the arrow is only from knowing its ends, and makes the diagram we build closer to the text we analyse.

Now we start to read a definition of topology from the definition of limit above: a subset $U \ni \text{``}x\text{''}$ of $\{\text{``}a_n\text{''}\}_{n\in\mathbb{N}} \cup \{\text{``}x\text{''}\}$ is open iff "x" $\in U$ and there exist a number N such that "a_n" $\in U$ for every $n > N$. The definition places no restriction on preimages of open subsets of T not containing x (except that they do not contain the preimage of x), and we read that off as every subset $U \not\ni \text{``}x\text{''}$ not containing "x" is open.

With help of this topology, the definition is reformulated as follows: x is the limit of sequence $\{a_n\}_{n\in\mathbb{N}} \subseteq T$ iff the ("obvious" from notation) map $\{\text{``}a_n\text{''}\}_{n\in\mathbb{N}} \cup \{\text{``}x\text{''}\} \xrightarrow{\{a_n\}_{n\in\mathbb{N}}\cup\{x\}} T$ is continuous, in this topology. There is also an "obvious" arrow $\{\text{``}a_n\text{''}\}_{n\in\mathbb{N}} \longrightarrow \{\text{``}a_n\text{''}\}_{n\in\mathbb{N}} \cup \{\text{``}x\text{''}\}$ where every point goes into itself.

Now, the uniqueness of limits theorem says that if $x \in T$ and $y \in T$ both are limits of the sequence $\{a_n\}_{n\in\mathbb{N}}$, then $x = y$. We read off that we have arrows $\{\text{``}a_n\text{''}\}_{n\in\mathbb{N}} \cup \{\text{``}x\text{''}\} \xrightarrow{\{a_n\}_{n\in\mathbb{N}}\cup\{x\}} T$ and $\{\text{``}a_n\text{''}\}_{n\in\mathbb{N}} \cup \{\text{``}y\text{''}\} \xrightarrow{\{a_n\}_{n\in\mathbb{N}}\cup\{y\}} T$. To formulate that a_n's in both arrows are the same, we require that the composition arrows $\{\text{``}a_n\text{''}\}_{n\in\mathbb{N}} \longrightarrow \{\text{``}a_n\text{''}\}_{n\in\mathbb{N}} \cup \{\text{``}x\text{''}\} \xrightarrow{\{a_n\}_{n\in\mathbb{N}}\cup\{y\}} T$ and $\{\text{``}a_n\text{''}\}_{n\in\mathbb{N}} \longrightarrow \{\text{``}a_n\text{''}\}_{n\in\mathbb{N}} \cup \{\text{``}y\text{''}\} \xrightarrow{\{a_n\}_{n\in\mathbb{N}}\cup\{x\}} T$ coincide. We also have "obvious" arrows $\{\text{``}x\text{''}\} \to \{\text{``}a_n\text{''}\}_{n\in\mathbb{N}} \cup \{\text{``}x\text{''}\}$ and $\{\text{``}y\text{''}\} \to \{\text{``}a_n\text{''}\}_{n\in\mathbb{N}} \cup \{\text{``}y\text{''}\}$ and composition arrows $\{\text{``}x\text{''}\} \xrightarrow{x} T$ and $\{\text{``}y\text{''}\} \xrightarrow{y} T$. By definition of "obvious"ness, all the "obvious" arrows defined above form a commutative diagram: namely the arrows

(L_{hyp}) the following arrows form a commutative diagram:

$$\{\text{``}x\text{''}\} \xrightarrow{x} T,\ \{\text{``}y\text{''}\} \xrightarrow{y} T, \{\text{``}a_n\text{''}\}_{n\in\mathbb{N}} \xrightarrow{\{a_n\}_{n\in\mathbb{N}}} T$$

$$\{\text{``}x\text{''}\}_{n\in\mathbb{N}} \longrightarrow \{\text{``}a_n\text{''}\}_{n\in\mathbb{N}} \cup \{\text{``}x\text{''}\} \xrightarrow{\{a_n\}_{n\in\mathbb{N}}\cup\{x\}} T$$

$$\{\text{``}y\text{''}\}_{n\in\mathbb{N}} \longrightarrow \{\text{``}a_n\text{''}\}_{n\in\mathbb{N}} \cup \{\text{``}y\text{''}\} \xrightarrow{\{a_n\}_{n\in\mathbb{N}}\cup\{y\}} T$$

$$\{\text{``}a_n\text{''}\}_{n\in\mathbb{N}} \longrightarrow \{\text{``}a_n\text{''}\}_{n\in\mathbb{N}} \cup \{\text{``}x\text{''}\} \xrightarrow{\{a_n\}_{n\in\mathbb{N}}\cup\{x\}} T$$

$$\{\text{``}a_n\text{''}\}_{n\in\mathbb{N}} \longrightarrow \{\text{``}a_n\text{''}\}_{n\in\mathbb{N}} \cup \{\text{``}y\text{''}\} \xrightarrow{\{a_n\}_{n\in\mathbb{N}}\cup\{y\}} T$$

That is the hypothesis of our theorem.

Now the conclusion is that $x = y$. A point being a morphism, this means the arrows $\{“x”\} \xrightarrow{x} T$ and $\{“y”\} \xrightarrow{b} T$ coincide. In the words, the arrows $\{“x”\} \xrightarrow{x} T$, $\{“y”\} \xrightarrow{y} T$ and $\{“x”\} \longrightarrow \{“y”\}$ form a commutative diagram. In fact, that means that all the arrows mentioned above form a commutative diagram.

Finally, we see that our theorem says that adding the (isomorphism) arrow $\{“x”\} \longrightarrow \{“y”\}$ preserves the commutativity of the diagram (L_{hyp}). That is, it became the following diagram chasing rule:

(L_{rule}) given commutative diagram (L_{hyp}), add the (isomorphism) arrow $\{“x”\} \longrightarrow \{“y”\}$ and mark the total diagram commutative

We remark that this reformulation is not purely diagrammatic: it is still extra knowledge what the topology on the space $\{“a_n”\}_{n\in\mathbb{N}} \cup \{“x”\}$ is.

3.2.2. *Limits are unique in a Hausdorff space: writing the proof.* By assumption we have a commutative diagram (L_{hyp}). Apply the Hausdorff property rule $T_2(iii)$: in (L_{hyp}) we have arrows $\{x\} \longrightarrow T$, and $\{y\} \longrightarrow T$, so either $T_2(iii')$ we may add $\{x\} \longrightarrow \{y\}$ or $T_2(iii'')$ we may add $\{x\} \longrightarrow \{x > \star < y\}$, $\{y\} \longrightarrow \{x > \star < y\}$, and $T \longrightarrow \{x > \star < y\}$. In case $T_2(iii')$, the theorem is proved. In $T_2(iii'')$, as a subdiagram, we obtain a commutative diagram (L_{hyp}) with T replaced by $\{x > \star < y\}$, while $\{x\} \longrightarrow T$ and $\{y\} \longrightarrow T$ become $\{x\} \longrightarrow \{x > \star < y\}$, $\{y\} \longrightarrow \{x > \star < y\}$. Thus we only need to check that (L_{hyp}) does not hold for that particular choice of $T = \{x > \star < y\}$ and maps $\{x\} \longrightarrow T$, and $\{y\} \longrightarrow T$. To do so we need to know the topology on the infinite space $\{“a_n”\}_{n\in\mathbb{N}} \cup \{“x”\}$, and thus we use the usual argument and not diagram chasing.

Let us remark that the arrow $\{“a_n”\}_{n\in\mathbb{N}} \longrightarrow \{“a_n”\}_{n\in\mathbb{N}} \cup \{“x”\}$ is one-point compactification of the discrete topological space $\{“a_n”\}_{n\in\mathbb{N}}$, and this construction is functorial. In particular, an automorphism of $\{“a_n”\}_{n\in\mathbb{N}}$ extends uniquely to an automorphism of $\{“a_n”\}_{n\in\mathbb{N}} \cup \{“x”\}$.

3.3. **A closed subset of a complete metric space is complete.** Consider the problem that a closed subset of a complete metric space is complete. [GG,§2.2] considers it as example of “a routine problem and examine[s] how a human mathematician would typically solve it”. Let us now translate it to diagram chasing; the reader may find it helpful to have a look at the diagrams on page 22 first.

More formally the problem is stated as follows. Let X be a complete metric space and let A be a closed subset of X. Prove that A is complete. A metric space is complete iff every Cauchy sequence $\{a_n\} \subseteq X$ converges to a point $b \in X$. A subset $A \subseteq X$ is closed iff for each sequence $\{a_n\} \subseteq A$ converges to $b \in X$, then $b \in A$. Let us introduce a label $(Cauchy)$ to talk about Cauchy sequences, and let us introduce (lim) label for the arrows $\{“a_n”\}_{n\in\mathbb{N}} \xrightarrow{(lim)} \{“a_n”\}_{n\in\mathbb{N}} \cup \{“b”\}$ for the limit arrows described in the previous subsection. In that notation, the definition of completeness translates to each arrow X is complete iff each $\{“a_n”\}_{n\in\mathbb{N}} \xrightarrow{(Cauchy)} X$ factors via $\{“a_n”\}_{n\in\mathbb{N}} \xrightarrow{(lim)} \{“a_n”\}_{n\in\mathbb{N}} \cup \{“b”\}$, i.e.

$$\{``a_n"\}_{n\in\mathbb{N}} \xrightarrow{(Cauchy)} X \text{ always factors as } \{``a_n"\}_{n\in\mathbb{N}} \xrightarrow{(lim)} \{``a_n"\}_{n\in\mathbb{N}} \cup \{``b"\} \longrightarrow X.$$

A is complete iff each $\{``a_n"\}_{n\in\mathbb{N}} \xrightarrow{(Cauchy)} A$ factors via $\{``a_n"\}_{n\in\mathbb{N}} \xrightarrow{(lim)} \{``a_n"\}_{n\in\mathbb{N}} \cup \{``b"\}$, i.e.

$$\{``a_n"\}_{n\in\mathbb{N}} \xrightarrow{(Cauchy)} A \text{ always factors as } \{``a_n"\}_{n\in\mathbb{N}} \xrightarrow{(lim)} \{``a_n"\}_{n\in\mathbb{N}} \cup \{``b"\} \longrightarrow A.$$

$A \subseteq X$ is closed iff $\{``a_n"\}_{n\in\mathbb{N}} \xrightarrow{(lim)} \{``a_n"\}_{n\in\mathbb{N}} \cup \{``b"\} \rightthreetimes A \hookrightarrow X$, or, in different notation, $\mathbb{N} \xrightarrow{(lim)} \mathbb{N} \cup \{\infty\} \rightthreetimes A \hookrightarrow X$. Finally, we need to say that A is a subset of X. Above we identified a subset A of X with its characteristic function $X \xrightarrow{A} \{0 \leqslant 1\}$. Here a more complicated approach is necessary: for a subset $A \subseteq X$, the arrow $A \longrightarrow X$ is the pullback of its characteristic function $X \xrightarrow{A} \{0 \leqslant 1\} \longleftarrow \{0\}$. That is, to give an arrow $Z \longrightarrow A$ into A is equivalent to giving an arrow $Z \longrightarrow X$ making the diagram into the diagram $Z \longrightarrow X \xrightarrow{A} \{0 \leqslant 1\} \longleftarrow \{0\} \longleftarrow Z$ commute.

Now the proof reduces to diagram chasing.

3.4. $f(x) = g(x)$ **defines a closed subset for** $f, g : X \longrightarrow Y$ **and** Y **Hausdorff.** Now let us prove that the coincidence set $Eq_{f=g} = \{x \in X | f(x) = g(x)\}$ of two continuous maps from an arbitrary space to a Hausdorff space is closed.

The property translates to the following diagram chasing rule:

(0+) add a rule $T_2(Y)$ saying that Y is Hausdorff, and add arrows $f, g : X \longrightarrow Y$

(0−) construct an arrow $X \xrightarrow{``f(x)=g(x)"} \{o > x\}$, and derive, for that arrow, the rule that $(0a)$ is equivalent to $(0b)$ where:

- $(0a)$ $\{x\} \xrightarrow{x} X \xrightarrow{f} Y$ and $\{x\} \xrightarrow{x} X \xrightarrow{g} Y$ commute at $\{x\}$
- $(0b)$ $\{x\} \xrightarrow{x} X \xrightarrow{``f(x)=g(x)"} \{o > x\}$ and $\{x\} \longrightarrow \{o < x\}$ form a commutative diagram

We consider the subset $Eq_{f=g} = \{x \in X | f(x) = g(x)\}$. That is, by a form of extensionality we add the arrow $X \xrightarrow{``f(x)=g(x)"} \{o \gtrless x\}$ and the rule:

(1+) add arrow $X \xrightarrow{``f(x)=g(x)"} \{o \gtrless x\}$, and the rule that either of $(1a)$ or $(1b')$ implies the other:

- $(1a)$ $\{x\} \xrightarrow{x} X \xrightarrow{f} Y$ and $\{x\} \xrightarrow{x} X \xrightarrow{g} Y$ commute at $\{x\}$
- $(1b')$ $\{x\} \xrightarrow{x} X \xrightarrow{``f(x)=g(x)"} \{o \gtrless x\}$ and $\{x\} \longrightarrow \{o \gtrless x\}$ form a commutative diagram

We need to prove that $Eq_{f=g}$ is closed. That is, we need to factor the arrow $\{x\} \xrightarrow{x} X \xrightarrow{``f(x)=g(x)"} \{o \gtrless x\}$ via the arrow $\{o > x\} \longrightarrow \{o \gtrless x\}$. In the diagram chasing way, we

(2+) add the arrow $\{o > x\} \longrightarrow \{o \gtrless x\}$ to $X \xrightarrow{``f(x)=g(x)"} \{o \gtrless x\}$ we have already

(2−) construct the arrow $X \xrightarrow{``f(x)=g(x)"} \{o > x\}$ such that the these arrows form a commutative triangle.

To prove the set is closed, we prove its complement $X \setminus Eq_{f=g}$ is open. To prove that, it is enough to pick an arbitrary point $o \in X \setminus Eq_{f=g}$ such that $f(o) \neq g(o)$, and find an open neighbourhood $o \subseteq U_o \subseteq X \setminus Eq_{f=g}$. That is,

(3+) add new arrows $\{o\} \longrightarrow X$ and $\{o\} \longrightarrow \{o \gtreqless x\}$ to the arrows $\{o > x\} \longrightarrow \{o \gtreqless x\}$ and $X \xrightarrow{\text{“}f(x)=g(x)\text{”}} \{o \gtreqless x\}$ marking the triangle commutative

(3−) construct an arrow $X \longrightarrow \{o > x\}$ such that
  (3.1−) the diagram commutes at $\{o\}$
  (3.2−) add a new arrow $\{o\}' \longrightarrow X$ and $\{o\}' \longrightarrow \{o > x\}$, mark the diagram commutative from $\{o\}'$ to $\{o > x\}$, and derive that the diagram is commutative from $\{o\}'$ to $\{o \gtreqless x\}$

(3.3+) fulfill goal (2−) above by adding the? arrow $X \xrightarrow{\text{“}f(x)=g(x)\text{”}} \{o > x\}$ and marking the required diagram commutative

We need to use the assumption that Y is Hausdorff. Note $f(o) \neq g(o)$ and thus there exist open neighbourhoods $f(o) \in U_f \subseteq Y$ and $g(o) \in U_g \subseteq Y$ such that $U_f \cap U_g = \emptyset$. That is,

(4+) add arrows $X \xrightarrow{f} Y$ and $X \xrightarrow{g} Y$

In the diagram so far, there are only two arrows from a point to Y:

(5+) $\{o\} \longrightarrow X \xrightarrow{f} Y$ and $\{o\} \longrightarrow X \xrightarrow{g} Y$.

Apply Y is Hausdorff: we have either $T_2(iii')$ or $T_2(iii'')$. However, $T_2(iii')$ contradicts commutativity of the diagram; thus $T_2(iii'')$. That is,

(6+) add the following arrows marking the appropriate triangles commute

$$\{o\} \xrightarrow{o \mapsto o_f} \{o_f > \star < o_g\},\ \{o\} \xrightarrow{o \mapsto o_g} \{o_f > \star < o_g\},\ Y \longrightarrow \{o_f > \star < o_g\}$$

Finally, note that for every point in $f^{-1}(U_f) \cap g^{-1}(U_g)$, $f(z) \neq g(z)$. That is, do:

(7+) add $X \longrightarrow \{o_f > \star < o_g\} \times \{o_f > \star < o_g\} \xrightarrow{(o_f,o_g) \mapsto o} \{o > x\}$

where the first arrow is the product $X \longrightarrow \{o_f > \star < o_g\} \times \{o_f > \star < o_g\}$ of the arrows $X \xrightarrow{f} Y \longrightarrow \{o_f > \star < o_g\}$ and $X \xrightarrow{g} Y \longrightarrow \{o_f > \star < o_g\}$ to finite spaces.

(8+) fulfill the goal (3.2−) and (3.1−) by a diagram chasing argument

To show (8+) is admissible, apply the rule added in (1+): check that $(1a)$ fails by considering cases where $\{o\}$ might map in the finite topological spaces $\{o_f > \star < o_g\}$, $\{o_f > \star < o_g\}$ and $\{o > x\}$ using that we know explicitly the maps between them. Then, by $(1b')$ in the diagram $\{o\}$ may not go into $x \in \{o > x\}$, and thus the diagram commutes at $\{o\}$. The theorem is proven.

3.5. **A diagram chasing definition of compactness.** Recall a space X is compact iff for an arbitrary directed covering S of X by open subsets, X is an element of S.

A teacher sometimes chooses to informally phrase compactness as a rule: given a compact space X, to prove that (*top*+) X has a certain property, first prove that (*dir*−) open subsets of X with this property form a directed open covering of X, i.e. for every U,V open in X

with the property, there is $U,V \subseteq W$ open in X satisfying the property, and that $(cov-)$ for every point x of X, there is an open neighbourhood $x \in U_x \subseteq X$ satisfying the property.

Let us translate that. We know an open subset of X is an arrow $X \longrightarrow \{o > x\}$. How do we talk about subsets having a certain property? We allow ourselves to write labels on arrows, and allow diagram chasing rules to manipulating these labels.

With that in mind, the following is a literal rendering of the teacher's explanation above. The primes in $\{o > x\}'$ and $\{o > x\}''$ indicate that these are different vertices in the diagram we construct, although the spaces are canonically isomorphic.

Thus, X is compact iff the following rule is admissible.

(K) given a label (s)

$(K+)$ add a rule that

rule $(cov-)$ and rule $(dir-)$ imply rule $(top+)$

where

$(cov-)$ given arrows $\{o > x\} \longleftarrow \{o\} \longrightarrow X$, construct an arrow $X \xrightarrow{(s)} \{o > x\}$ making the triangle commutative

$(dir-)$ given arrows $X \xrightarrow{(s)} \{o > x\}'$ and $X \xrightarrow{(s)} \{o > x\}''$, add an arrow $X \xrightarrow{(s)} \{o > x\}$ and rules

$(dir)'$ $\{o\} \longrightarrow X \xrightarrow{(s)} \{o > x\}'$ and $\{o\} \longrightarrow \{o > x\}'$ commute at $\{o\}$ to $\{o > x\}'$ implies the same and $\{o\} \longrightarrow X \xrightarrow{(s)} \{o > x\} \longleftarrow \{o > x\}'$ commute at $\{o\}$ to $\{o > x\}$

$(dir)''$ $\{o\} \longrightarrow X \xrightarrow{(s)} \{o > x\}''$ and $\{o\} \longrightarrow \{o > x\}''$ commute at $\{o\}$ to $\{o > x\}''$ implies the same and $\{o\} \longrightarrow X \xrightarrow{(s)} \{o > x\} \longleftarrow \{o > x\}''$ commute at $\{o\}$ to $\{o > x\}$

$(top+)$ add a commutative diagram $X \xrightarrow{(s)} \{o > x\} \longleftarrow \{o\}$ and $X \longrightarrow \{o\}$

We may omit "to $\{o > x\}$", as it is the only vertex where paths from $\{o\}$ may converge.

Note that $(top+)$ immediately implies $(cov-)$ and $(dir-)$ by a diagram chasing argument whether $\{o\}$ might go to o or x in $\{o > x\}$ and using the commutativity of the triangle in $(top+)$.

To use that X is compact, we consider the compound rule (K) above admissible, i.e. we allow ourselves to use it in derivations. To prove that X is compact, we pick a new label (s) which does not occur elsewhere in the proof, and add rules $(cov-)$ and $(dir-)$ to the list of admissible rules, and try to derive $(top+)$. We then consider the compound rule (K) admissible if we are able to derive $(top+)$.

4. Algorithmic aspects

We make a couple of remarks and speculations about our proofs.

4.1. **Structure of a proof: list of goals and derivation rules.** These proofs go as follows: add vertices(objects), arrows between objects, add a vertex(object), an arrow between objects, mark a pair of vertices and a subcollection of arrows as commutative from one vertex to another, add new derivation rules to the list of *admissible* derivation rules. Add and remove *goals* from a list of goals we maintain; sometimes we have to consider options and create branches. To apply an admissible derivation rule, one checks its hypothesis using a special kind of pattern matching.

A goal may be either to add an arrow of a particular kind, to mark a certain collection of arrows commutative from one vertex to another, or any other rule.

An application of a derivation rule consists of adding an arrow, marking a diagram commutative from a vertex to a vertex, or adding a new derivation rule to the list of admissible derivation rules, adding goals to or removing goals from the list of goals.

A theorem becomes proved when the list of goals becomes empty.

4.2. **Goals and subgoals.** Often, a rule introduces more assumptions, e.g. by adding arrows and objects, or relaxing commutativity conditions, e.g. to only require commutativity of arrows to or from finite spaces. A goal is replaced by subgoals easily implied by it. Importantly, the arrows to construct may remain the same, and the new and the old diagrams are reducts of each other in some sense.

4.3. **Reinterpreting arrows. Arrows between finite spaces.** Let us see how the property that a subset is open iff each point has an open neighbourhood, provides an example of both phenomena. A subset V is open iff every point $u \in V$ has an open neighbourhood $u \in U \subseteq V$ where U is open.

(V) given (i) $X \xrightarrow{V} \{o \gtreqless x\} \longleftarrow \{o > x\}$, add ($ii$) $X \xrightarrow{V} \{o > x\}$ such that the triangle commutes

(U) given (i) and $X \longleftarrow \{o\} \longrightarrow \{o > x\}$, add ($ii$) such that the diagram commutes from $\{o\}$ to $\{o > x\}$

Note that the arrow $X \longrightarrow \{o > x\}$ is interpreted differently in $(V)(ii)$ and $(U)(ii)$, namely $(V)(ii)$ means "V is open" while $(U)(ii)$ means "there is an open neighbourhood in V of an arbitrary point $u \in V$".

Note also that the commutativity conditions now concern only arrows between finite spaces.

In the proof in §3.4, the arrow $X \longrightarrow \{o > x\}$ gets reinterpreted 3 times: rule (0−) as $X \xrightarrow{\text{“}f(x)=g(x)\text{”}} \{o > x\}$ as the subset $\{x : f(x) = g(x)\}$, rule (2−) as the open neighbourhood U_o inside of $\{x : f(x) = g(x)\}$, and rule (7−) as $f^{-1}(U) \cap g^{-1}(V)$.

Similar reinterpretations occur in the the definition of compactness of §3.5, and the commutativity conditions in the introduced subgoals ($cov-$) and ($dir-$) concern only arrows between finite spaces $\{o\}$ and $\{o > x\}$.

4.4. **No loops or automorphisms: a decidable fragment.** There are no loops or automorphisms in the examples we saw; this leads to the following observation.

Call a collection of diagrams, or, more generally, a collection of rules, *stratifiable* iff the following relation on the (names of) variables occurring in the collection defines an anti-reflexive partial order $X \lneq Y$

$X \lneq Y$ iff there is an occurrence of an arrow from X to Y

This means that the formal transitive closure under composition of all the arrows is necessarily finite; in particular, there is no loop $f : X \longrightarrow X$ giving rise to the infinitely many automorphisms $f, f \circ f, f \circ f \circ f, ...$

Moreover, according to an observation of N.Durov, in a similar manner one may formally generate from a stratified collection of arrows and a collection of commutativity conditions on these arrows, a category with finitely many objects and arrows that satisfies those commutativity conditions. This should allow to define a decidable fragment of category theory, to be published in a forthcoming paper.

4.5. **Reading a natural-language proof to assist proof searching.** Our diagrammatic rendering follows a (simple) proof or definition as it is written line by line. Moreover, our diagrams seems to be well-suited for brute force proof search. This raises the question whether the (natural language) text of a proof can guide the proof search, by limiting brute force searching only to those branches that share some combinatorial patterns with the natural language text. For example, by looking at what variables occur close to each other, and choosing rules that involve those variables. Arguably, this is how a human skims proofs of simple statements - just by looking for keywords to guide their own attempt at a proof.

In fact, one may hope that our diagram chasing is just complicated enough that one may formulate a well-defined refutable conjecture relating to these diagram chasing techniques and the human mind. As a first attempt, consider the following experiment. Find a pair of complicated enough proofs in point-set topology such that their diagram chasing translation share a non-obvious trick but little else. Then explain one of the arguments in words, as it usually would be explained in a point-set textbook, to a 1st year mathematics student, and then check whether the other proof has become easier for the student. If the proof does become easier, interpret this as evidence that understanding the argument involves translating the argument into category theory. The check should probably be done in a week's time reflecting the usual regularity of lectures and seminar meetings: internalizing a proof takes time.

We also observe that sometimes drawings used to demonstrate a proof correspond quite closely to a diagram chasing argument. For example, the definition of Hausdorff one draws an oval representing the topological space X, then draws two points in it, and then draws small ovals around them. In diagram chasing terms, drawing the two points correspond to drawing an injective arrow $\{\bullet, \bullet\} \longrightarrow X$, and drawing the two small ovals correspond to drawing all the open sets in a 3-point space where each oval represent an open point, and the third closed point is drawn as a complement of the two ovals in the big one.

4.6. **Use of trivial examples in proof searching.** Another feature of our diagrammatic derivation system is a good notion of inheritance: after applying a rule, there is a good notion of what arrows remained the same, which new vertices correspond to the old ones. This might enable use of trivial examples in the following way. Pick an arrow and "contract" it, i.e. label it an isomorphism, or mark a subdiagram commutative, or perhaps add more assumptions until the number of unknown arrows becomes manageable. Use brute force to find a proof, and trim it only to rule applications actually used. Perhaps do this several times using different assumptions, and find rule applications which occur many times. Then try to apply the same rules in the original diagram under no assumptions.

4.7. **Lifting property as a negation in a category: examples.** We notice a pattern in the examples of §2: these properties are defined as a lifting property with respect to the simplest or an otherwise archetypal counterexample. In particular, being *injective, surjective, connected,* T_0*,* T_1*, dense, induced topology; surjective and injective on* π_0 are all defined by a lifting property with respect to a map between 1-point and 2-point spaces not having the property.

A demonstrative example is provided by the morphism $\{x, y\} \longrightarrow \{\bullet\}$: it is both the simplest (counter)example of a unconnected space, and of a non-injective map. Accordingly, the left lifting property $X \longrightarrow \{\bullet\} \rightthreetimes \{x, y\} \longrightarrow \{\bullet\}$ defines X is connected, and $\{x, y\} \longrightarrow \{\bullet\} \rightthreetimes X \longrightarrow Y$ defines $X \longrightarrow Y$ is injective.

Compactness provides a less trivial example: a standard example of a non-compact space, say in a course of analysis, is the space $\mathbb{R}$ and its decomposition $\mathbb{R} = \cup_{n\in N}(-n, n)$ as a union of intervals. Accordingly, the often used characterisation of compactness (for metrizable spaces), known as pseudocompactness, that every continuous function to $\mathbb{R}$ is bounded, is, for X connected, equivalent to the lifting property $\emptyset \longrightarrow X \rightthreetimes \sqcup_{n\in N}(-n, n) \longrightarrow \mathbb{R}$. A similar but less ¿obvious? characterisation $\emptyset \longrightarrow X \rightthreetimes \sqcup_{\beta\in\alpha}\beta \longrightarrow \alpha$ for all ordinals α, works for arbitrary (not necessarily Hausdorff) connected spaces.

4.8. **Lifting property as a negation in a category: a diagram chasing explanation.** A diagram chasing explanation, cf. [GH-I§3.4,Remark 27], for the examples above is that for the lifting property with respect to a morphism h is a category-theoretically usable definition of a class of morphisms not containing h: $h \notin \{h\}^{\rightthreetimes} := \{g : g \rightthreetimes h\}$ unless h is an isomorphism. More generally, given a diagram chasing property K, lifting property allows to give a diagram chasing definition of a class $K^{\rightthreetimes}$ of morphisms not having the property: namely, $h \in K \cap K^{\rightthreetimes}$ implies h is an isomorphism where $K^{\rightthreetimes} := \{g : g \rightthreetimes h \text{ for all } h \in K\}$. The dual class ${}^{\rightthreetimes}K := \{g : h \rightthreetimes g \text{ for all } h \in K\}$ has the same properties.

Thus we see that we have a left negation and a right negation which allow to define morphisms without a property in a form suitable for diagram chasing.

4.9. **Lifting property as a rule to linearise the order.** We find this very trivial and short remark noteworthy. After applying the lifting property $A \longrightarrow B \rightthreetimes X \longrightarrow Y$ to a square $A \longrightarrow B \longrightarrow Y, A \longrightarrow X \longrightarrow Y$ the order on the four vertices becomes linear: $A \longrightarrow B \longrightarrow X \longrightarrow Y$. This reduces the number of arrows needed to represent the order, for example in the chain-merge data structure.

4.9.1. *Models of Meaning.* We propose that these diagram chasing rules may serve as a useful model of meaning of sentences in a point-set textbook: the reason to utter a sentence in a proof is to transfer the (mathematical structure of) the corresponding diagram to the listener. The features of this model is that it ignores the grammar (and allows for meaning of ungrammatical, incomplete sentences) and it is aware of the shared context, namely that (we assume that) both parties share knowledge of point-set topology.

4.9.2. *A fully automatic problem solver with human-style output of Ganesalingam-Gowers.* We wonder if our approach can be used to write an automatic theorem prover (problem solver) similar in purpose to that of Ganesalingam-Gowers, albeit in a different and more restricted mathematical domain. In their own words, they “describe a program that solves elementary mathematical problems, mostly but not exclusively in metric space theory, and presents the solutions in a form that is hard to distinguish from solutions that human mathematicians might write. [...] It would not be able to produce human-style output if it did not mirror very closely the way that human mathematicians think.” Our program shall aim to solve elementary problems in point-set topology and use diagram chasing as its internal data format. One hopes that for elementary problems, diagram chasing proofs remain close enough to human-style proofs that an automatic translation is possible, essentially reversing the process done in §3. We also note that the solution of the second problem presented on the first page of [GG] is in point-set topology and, from our point of view, employs an argument using the lifting property $\{\bullet,\bullet\} \longrightarrow \{\bullet\} \rightthreetimes X \xrightarrow{f} Y$ of an injection. In §3.3 we discuss a problem analysed by [GG,§2.2].

Should such a program indeed be written, it would then be very interesting to try to extend it into a a program able to verify boring technical and computational diagram chasing parts of a human written mathematical proof. There is no attempt to verify a whole proof, but rather only a small computation inside of it, at most to list explicitly all the assumptions used in the proof of a small computational lemma, with a view that a human reader can then check these assumptions if necessary.

5. A computer syntax

In this section we make a couple of suggestions of how one may want to define a computer-like syntax for writing diagram chasing rules. These suggestions are incomplete and somewhat inconsistent, and perhaps best ignored.

We aim not to define a complete formal syntax and its semantics; but rather a sketch sufficient to write down formal proofs informally.

We have an infinite supply of variables of types `Point`, `Object` and `Label`; usually words of Latin letters. A term is either (*i*) a variable, or (*ii*) a constant term of type `Object` that is a list of equalities and inequalities between points, e.g. `{a<b, c<>d, e=f,...}`, or (*iii*) the same followed by several “'” (where `c<>d` means that both `c` and `d` are greater or equal to each other).

There is special notation for arrows. Expression `X-->Y` denotes a variable ranging over arrows from X to Y. Expression `X--(s,..,q)-->Y` denotes a variable ranging over arrows from X to Y carrying each of the labels `s,..,q` listed.

Note this gives no convenient way to talk about loops. This is a feature as loops give rise to non-decidability. To talk about loops and multiple arrows, we have to introduce new objects `X`, `X'`, `X''` and arrows `X-->A`, `X'-->A`, `X''-->A` and new admissible rules about `(id)`-labels `X-(id)->X'-(id)->X''`.

A diagram is a collection of arrows. We have a notion of a current context containing variables and rules.

Expression `@X<Y` is a predicate stating that the diagram commutes from `X` to `Y` where the diagram consists of the arrows, i.e. variables of type `Arrow`, in the current context. More generally, you specify explicitly a list of arrows on the left, and an arbitrary partial order on the right.

We also have operators `:match` and `:create`, also denoted as `*` and `_`. Operator `:match Exp` matches `Exp` and variables therein against current context, and operator `:create Exp` add to the current context the expression `Exp` with substitutions made.

There should be an explicit *dependency order* on variables: we often need an introduce a variable, say x, required to be new, say with with respect to variables a, b, c. This is equivalent to having a function $x = x(a, b, c)$ but it can also be stated as $x >_{dep} a, x >_{dep} b$ and $x >_{dep} c$.

There should be syntax sugar added.

Let us give the definition of compactness in §3.5 as an example.

Thus, X is compact iff the following rule is admissible.

```
*s:Label;
 :create{
  :match{ *{o>x}<--{o}-->X;  _X--(s)-->{o>x}; _@{o}<X } ;
  :match{ *X--(s)-->{o>x}'; *X--(s)-->{o>x}''; _X--(s)-->{o>x};
         _{  *{o}-->X; *[[ {o}-->X--(s)-->{o>x}'; {o}-->{o>x}'; ]]@{o}<{o>x}';
               _[[ {o}-->X--(s)-->{o>x}'; {o}-->{o>x}'; {o}-->X--(s)-->{o>x}-->{o>x}' ]]
          };

         _{  *{o}-->X; *[[ {o}-->X--(s)-->{o>x}''; {o}-->{o>x}''; ]]@{o}<{o>x}'';
               _[[ {o}-->X--(s)-->{o>x}''; {o}-->{o>x}''; {o}-->X--(s)-->{o>x}-->{o>x}''
          };
         }
  :create{ [[ X--(s)-->{o>x}<--{o}; X-->{o} ]]@X<{o<x}
         }
         }
```

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No attempt have been made to provide a complete bibliography. The author shall be happy to receive, and then add, suggestions of missing references and references to relevant work.

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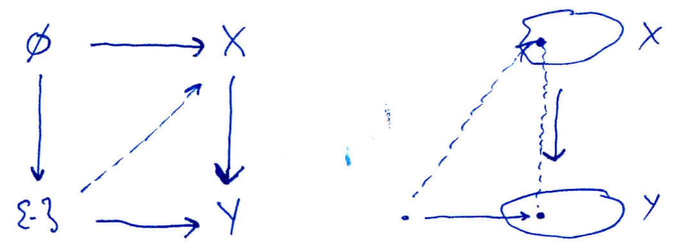
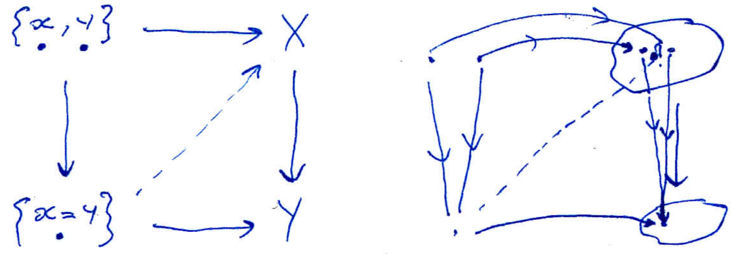
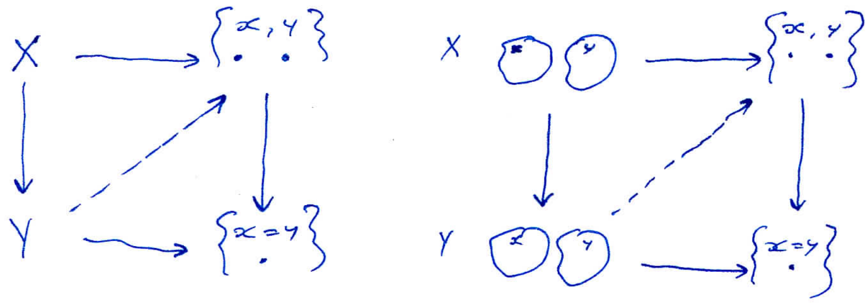


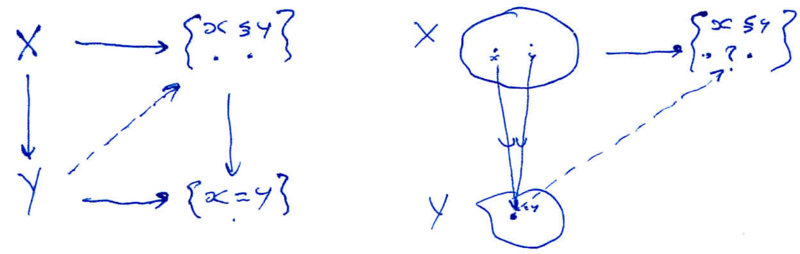
$X \twoheadrightarrow Y$



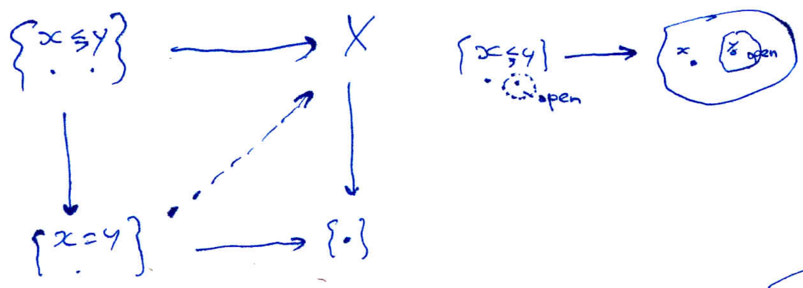
$X \hookrightarrow Y$



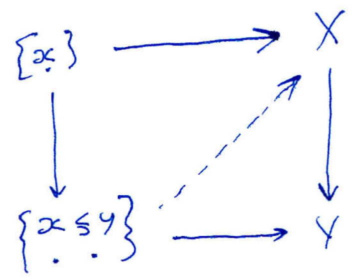
$\pi_0(X) \hookrightarrow \pi_0(Y)$

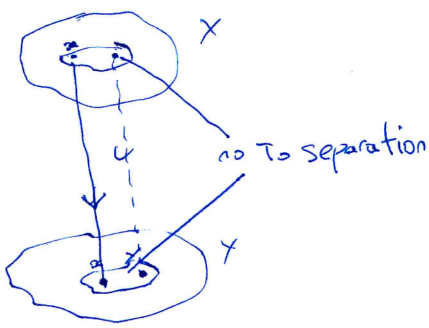


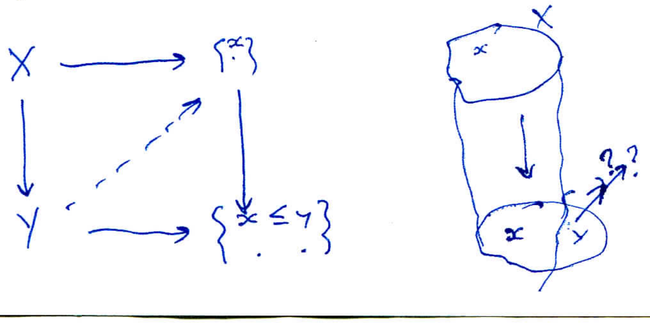
$X \hookrightarrow Y$



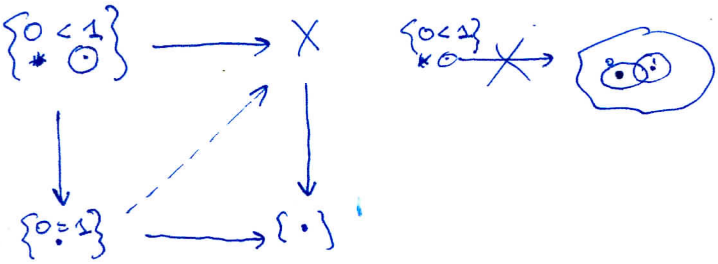
X is T_0



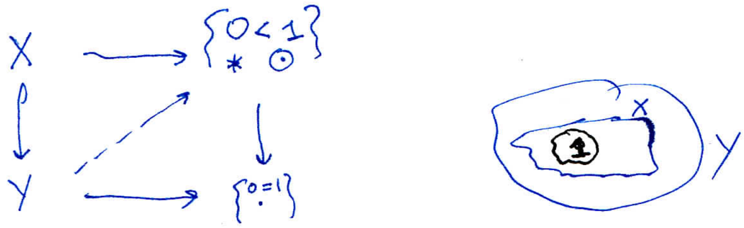




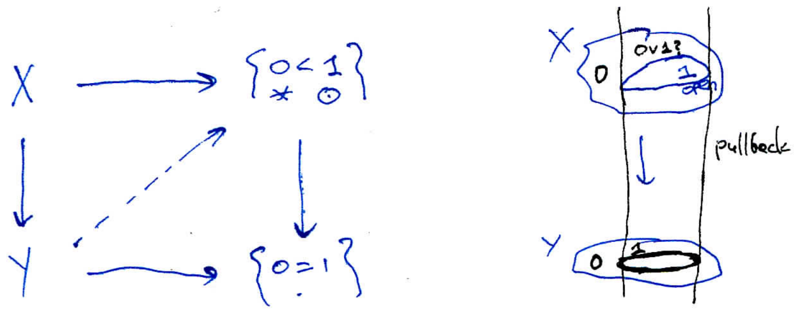
$X \twoheadrightarrow Y$



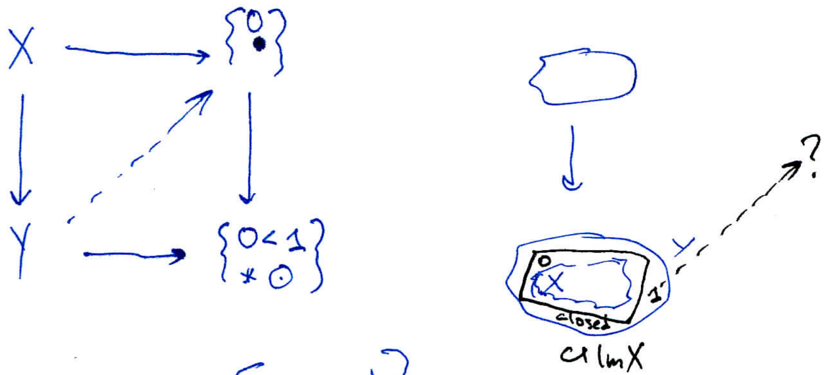
X is T_1



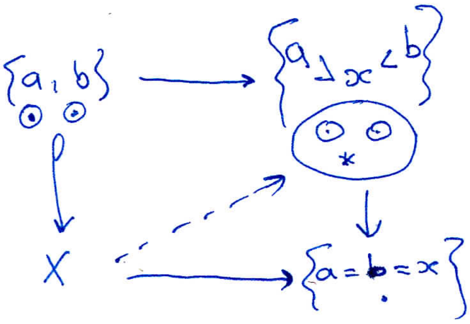
induced topology



the coarsest topology

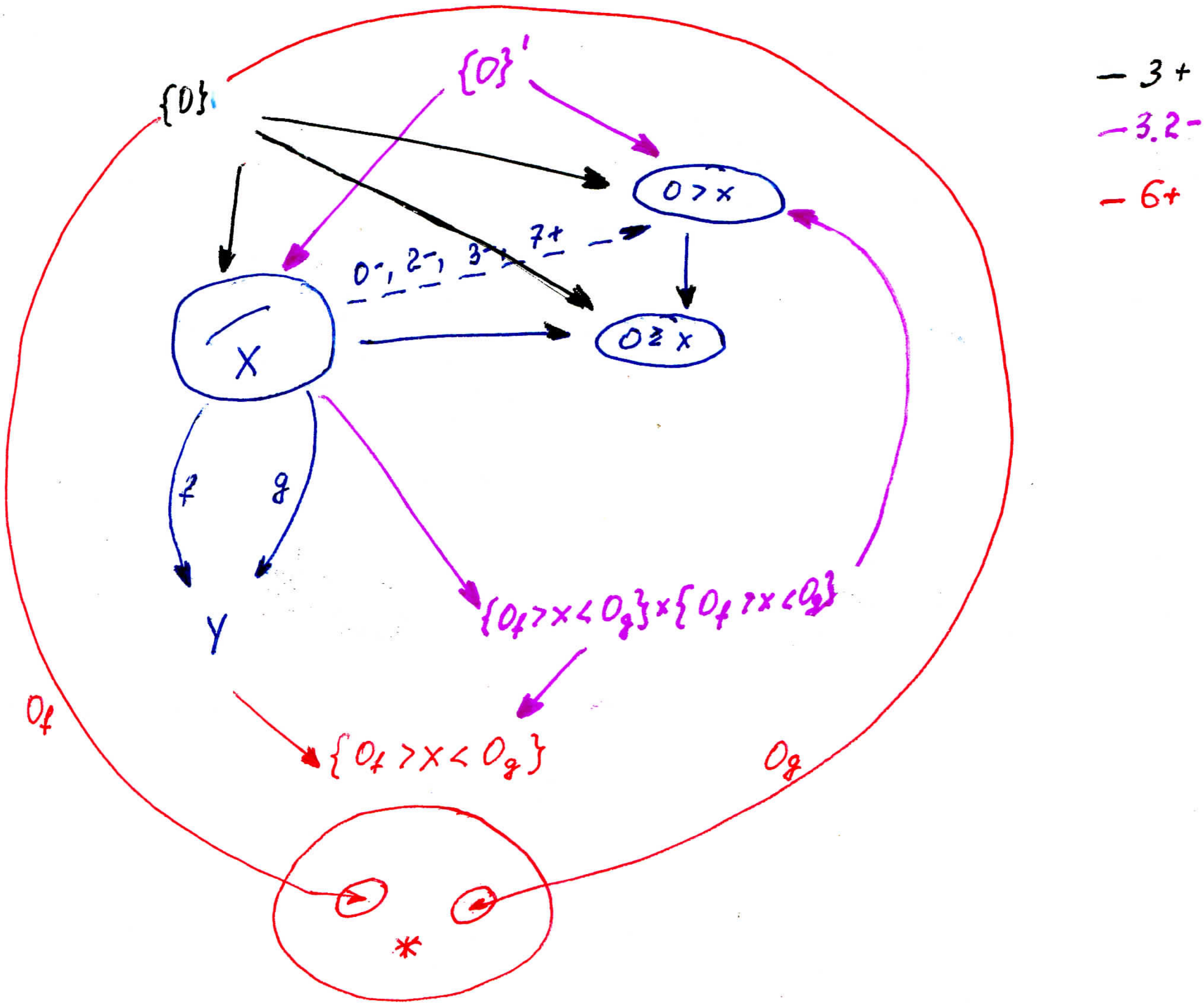


Im X is dense

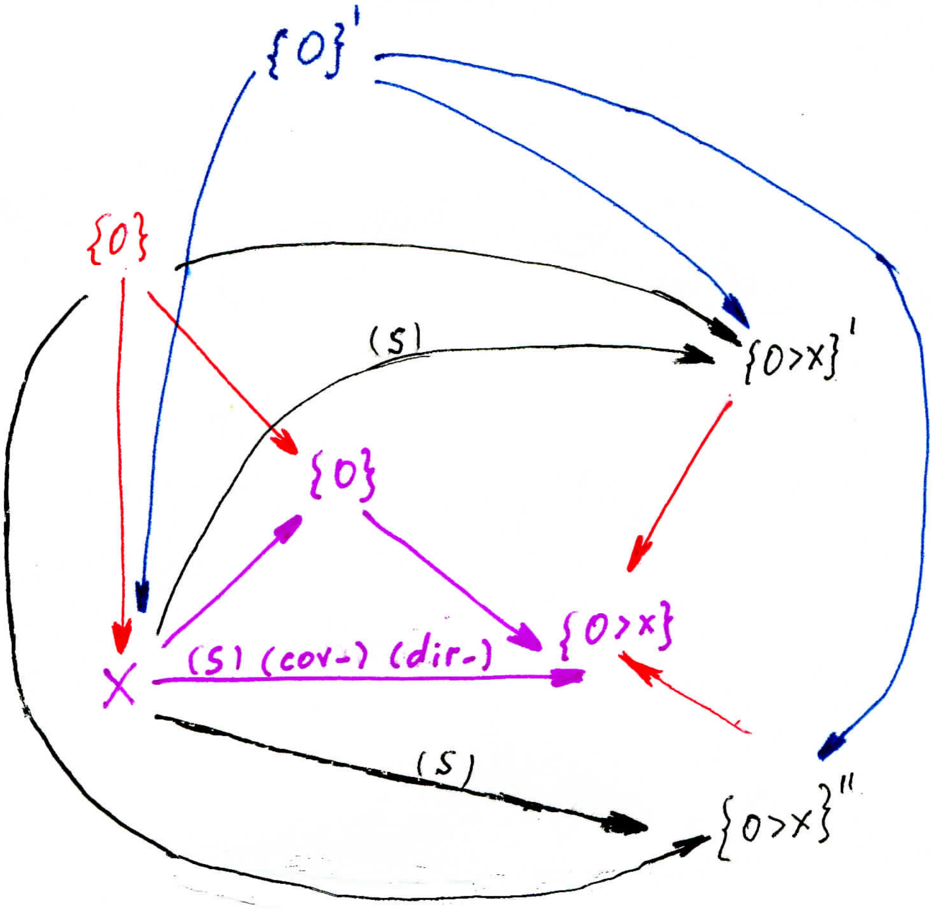


X is T_2 (Hausdorff)

3.3.



3.5.

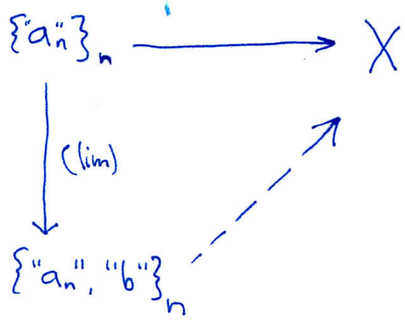


— (cov-) @ {0} → {0>x}'

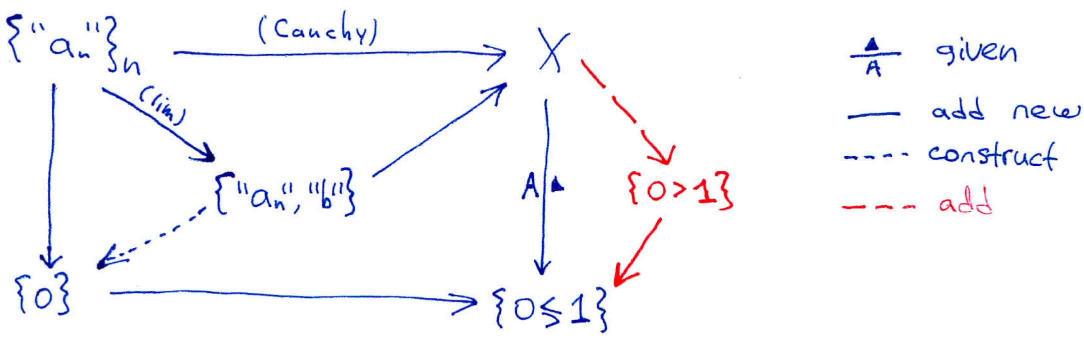
— (dir-) {0} new

@ {0} → {0>x}

X is complete



$A \subseteq X$ is closed



A is complete