A SUGGESTION TOWARDS A FINITIST'S REALISATION OF TOPOLOGY

to Vita Kreps Z"L in memoriam

This kind of universality is what, we believe, turns the hidden wheels of the human thinking machinery.

Abstract. — We observe that the notion of a trivial Serre fibration, a Serre fibration, and being contractible, for finite CW complexes, can be defined in terms of the Quillen lifting property with respect to a single map $M \to \Lambda$ of finite topological spaces (preorders) of size 5 and 3, one of the simplest examples of a map contracting something (namely, the V in M), and of a trivial Serre fibration. In particular, we observe that the double Quillen orthogonal $\{M \to \Lambda\}^{lr}$ is precisely the class of trivial Serre fibrations if calculated in a certain category of nice topological spaces. This suggests a question whether there is a finitistic/combinatorial definition of a model structure on the category of topological spaces entirely in terms of the single morphism $M \to \Lambda$, apparently related to the Michael continuous selection theory.

1. Introduction

Being contractible, compact (for nice spaces), trivial Serre fibration (for nice spaces, with caveats), connected, dense, extremally disconnected, zero-dimensional, and separation axioms T_0, T_1, T_4, T_5, can each be defined in terms of the Quillen lifting property [**1**] and a single map of topological spaces (preorders), usually with less than 7 points [**2, 3**], and related either to the definition or a simple example of the property. This suggests a combinatorial, computational notation for these topological properties, which could perhaps be of use in computer algebra and proof verification. This notation shows there is *finite combinatorics implicit in the basic definitions of topology*—what does it tell us ?

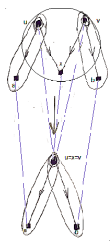

Fig.1.Arrow $u \to a$ means $a \in cl\, u$.

$$\left\{ {}_a \swarrow {}^u \searrow {}_x \swarrow {}^v \searrow {}_b \right\}$$
$$\Downarrow$$
$$\left\{ {}_a \swarrow {}^{u=x=v} \searrow {}_b \right\}$$

In this note we show the finite combinatorics implicit in the basic definitions of *contractible*, *trivial fibrations*, and *fibrations*. We observe that for a certain $M \to \Lambda$ of finite topological spaces (see Fig. 1), one of the simplest examples of a map contracting something (namely, the V in M), and of a trivial Serre fibration, its double Quillen orthogonal (negation) $\{M \to \Lambda\}^{lr}$, defined below, is exactly the class of trivial Serre fibrations if calculated in a certain category of nice spaces. If we calculate it in the category of (all) topological spaces, we only prove that $\{M \to \Lambda\}^{lr}$ is a class of trivial Serre fibrations, and

$$\text{a finite CW complex } X \text{ is contractible iff } X \to \{o\} \ \in \ \{M \to \Lambda\}^{lr}$$

Michael selection theorem ([**6**, Thm.1.2], see §3.1) implies that, if calculated in the category of paracompact spaces of finite Lebesgue dimension, $\{M \to \Lambda\}^{lr}$(1) is a class of trivial Serre fibrations containing many natural examples, for example all maps of completely metrisable spaces with weakly connected locally uniformly weakly connected fibres, e.g. locally trivial maps with weakly connected locally weakly connected fibres. This category is large enough not to affect the calculations of $\left(\{\{o\} \longrightarrow \{o{\searrow}c\}\}^{r}_{<5}\right)^{lr}$ (compactness), $\{\varnothing \to \{o\}\}^{rll}$ (connected), and $\{\varnothing \to \{o\}\}^{lrrrl}$ (quotient).

These observations allow us to state in §2.4 a conjectural definition of a model structure in what looks almost like a computer syntax and does not mention the real numbers. We follow the notation for finite spaces of [**2**], cf. Fig. 1.

Conjecture **($M \to \Lambda$).** — *A closed model structure on the category of topological spaces is defined as follows:*

(c) $\left\{\left\{{}_{a}\swarrow^{u}\searrow{}_{x}\swarrow^{v}\searrow{}_{b}\right\} \longrightarrow \left\{{}_{a}\swarrow^{u=x=v}\searrow{}_{b}\right\}\right\}^{l}$ *is the class of cofibrations.*

(wf) $\left\{\left\{{}_{a}\swarrow^{u}\searrow{}_{x}\swarrow^{v}\searrow{}_{b}\right\} \longrightarrow \left\{{}_{a}\swarrow^{u=x=v}\searrow{}_{b}\right\}\right\}^{lr}$ *is the class of trivial fibrations.*

(f) *a map $p : Y \to B$ is a fibration iff for any commutative diagram of solid arrows labelled as shown there exist an arrow $X \to Y$ making the total diagram commutative:*

$$\begin{tikzcd} A & A \arrow[l, "\mathrm{id}"'] \arrow[r] \arrow[d, "(c)"] & Y \arrow[d, "p"] \\ & X \arrow[ul, "(wf)"] \arrow[r] \arrow[ur, dotted] & B \end{tikzcd}$$

(wc) *a map is a weak cofibration iff it is has the right lifting property with respect to all fibrations: (wc):=(f)^l*

(w) *a weak equivalence is the composition of a trivial cofibration with a trivial fibration: (w):=(wc)(wf)*

In §2.5 we show that the expression below describes(2)(3) the lifting property defining microfibrations, thereby suggesting to replace (f) by (f)$_{\text{micro}}$

(f)$_{\text{micro}}$ a map is a fibration iff it admits a decompostion

$$\cdot \xrightarrow{\{\{o\to_{a} \leftrightarrow b\}\longrightarrow\{o=a\leftrightarrow b\}\}^{l}} \cdot \xrightarrow{\{\{o\to_{a} \leftrightarrow b\}\longrightarrow\{o=a\leftrightarrow b\}\}^{lr},\{\{_{a}\leftarrow u\to_{x}\leftarrow v\to_{b}\}\longrightarrow\{_{a}\leftarrow u=x=v\to_{b}\}\}^{lr}} \cdot$$

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(1)Recall that a morphism i in a category has the *left lifting property* with respect to a morphism p, and p also has the *right lifting property* with respect to i, denoted $i \rightthreetimes p$, iff for each $f : A \to X$ and $g : B \to Y$ such that $p \circ f = g \circ i$ there exists $h : B \to X$ such that $h \circ i = f$ and $p \circ h = g$.
For a class P of morphisms in a category, its *left orthogonal* $P^{\rightthreetimes l}$ with respect to the lifting property, respectively its *right orthogonal* $P^{\rightthreetimes r}$, is the class of all morphisms which have the left, respectively right, lifting property with respect to each morphism in the class P. In notation,

$$P^{\rightthreetimes l} := \{i \ : \ \forall p \in P \ i \rightthreetimes p\}, P^{\rightthreetimes r} := \{p \ : \ \forall i \in P \ i \rightthreetimes p\}, P^{lr} := (P^{l})^{r}, ..$$

Taking the orthogonal of a class P is a simple way to define a class of morphisms excluding non-isomorphisms from P, in a way which is useful in a diagram chasing computation, and is often used to define properties of morphisms starting from an explicitly given class of (counter)examples. For this reason, it is convenient and intuitive to refer to P^{l} and P^{r} as *left, resp. right, Quillen negation* of property P. See [**1**] for a quick explanation and some examples.

(2)We write $p : Y \xrightarrow{(P)} B$ or simply $p : Y \xrightarrow{P} B$ to mean that the morphism $X \to Y$ has property P. Thus (f)$_{\text{micro}}$ means to say that a mophism p is a fibration iff $p = p_{l} \circ p_{lr}$ for some $p_{l} \in \{\{o\to_{a} \leftrightarrow b\} \longrightarrow \{o = a \leftrightarrow b\}\}^{l}$ and $p_{lr} \in \{\{o\to_{a} \leftrightarrow b\} \longrightarrow \{o = a \leftrightarrow b\}\}^{lr} \cap \{M \to \Lambda\}^{lr}$

(3)This expression almost says the the non-Hausdorff mapping cone of the map is in $\{M \to \Lambda\}^{lr}$. The left orthogonal $\{...\}^{l}$ is the class of open inclusions.

A Serre microfibration with weakly contractible fibres is necessarily a Serre fibration [**18**], and this suggests that (f) and $(\mathrm{f})_{\mathrm{micro}}$ might be consistent.

Obviously, we may also consider an analogue of this conjecture in another category such as the of posets, simplicial sets, or the category of simplicial objects in the category of filters [**21**] extending the category of topological spaces, of simplicial sets, and of uniform spaces.

Formal topological spaces: a model category of diagram chasing constructions. — The language of this conjecture is purely combinatorial. Can we define a model category of "formal" topological spaces ("formal" as in formal power series), i.e. a model category whose objects and arrows belong to a calculus of diagram chasing computations, so to say ? A naïve hope is that the size of spaces appearing in the Quillen orthogonals (negations) representing basic notions of topology [**2, 3**] is small enough (< 7) to make feasible the exponential growth in the computer processing of such a calculus.

The following conjecture would represent a rule in such a diagram chasing calculus of formal topological spaces,[4] and perhaps might imply the previous conjecture.

***Conjecture* (M2).** — *For each finite set P of maps of finite spaces, and each string consisting of letters l and r, each map in the category of topological spaces*[5] *decomposes as a map in $(P)^{sl}$ followed by a map in $(P)^{slr}$, and as a map in $(P)^{srl}$ followed by a map in $(P)^{sr}$:*

$$\begin{tikzcd} & \cdot \arrow[dr, dotted, "(P)^{slr}"] & \\ \cdot \arrow[ur, dotted, "(P)^{sl}"] \arrow[rr, "\forall"'] & & \cdot \end{tikzcd} \qquad \begin{tikzcd} & \cdot \arrow[dr, dotted, "(P)^{sr}"] & \\ \cdot \arrow[ur, dotted, "(P)^{srl}"] \arrow[rr, "\forall"'] & & \cdot \end{tikzcd}$$

The reformulations in [**2, 3**] show that this conjecture captures a few standard constructions in topology, e.g. it would imply that each topological space can be "approximated" by a compact space (Stone-Cech compactification, up to separation axioms, see Lemma 3.2); by a space satisfying separation axiom T_0,T_1, T_4, T_5, totally disconnected, extremally disconnected, Lebesgue zero-dimensional, ultranormal, discrete; that it is well-defined to take a connected component; add an open/closed subset to a topology, take a non-Hausdorff mapping cone/cylinder, etc.

Logical ideas.— Of course, the real temptation is to develop a computer algebra system doing topology using a syntax extending the concise syntax for topology we discuss, and to use it in teaching a first year course combining topology and category theory. To find a diagram chasing calculation deriving the axioms of a model category from Conjecture M2 and other diagram chasing rules of topology such as [**19**, 2.3],

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(4)See [**19**, §2.3] for a discussion how to view the axioms of topology as rules of diagram chasing with preorders.

(5)Here it may be important (and, inasmuch, in Conjecture 2.4.1) to consider the category of *all* topological spaces rather than a convenient category of topological spaces, for the orthogonal (Quillen negation) for compactness is a natural example which, in a sense, captures the definition of compactness via ultrafilters, and, accordingly, its calculation requires considering topological spaces associated with ultrafilters. However, note that these spaces are (non-Hausdorff) paracomact of finite Lebesgue dimension, simply because they have very few open covers.

arguably, might clarify the topological notions/intution and be an indication that our observations lead to a tame topology of [**4**].

Structure of the paper. — As a warm-up the reader may want to skip, §2.1-2.2 define *connected*, *quotient*, and *compact* in terms of maps of spaces with at most 2 points, as

$$\{\varnothing \to \{o\}\}^{rll},\ \{\varnothing \to \{o\}\}^{lrrrl},\ \text{and}\ \left(\{\{o\} \longrightarrow \{o{\to}c\}\}^{r}_{<5}\right)^{lr}$$

In §2.1 we also define a few other notions starting with the simplest possible map, the inclusion of the empty space into a singleton, and in Appendix §3.3 we list a few more. In §2.2 we define the class of proper maps of nice spaces.

§2.3 and §2.4. is the main body of the paper. In §2.3 we discuss the definition of trivial fibrations, and in §2.4 state the conjecture proposing a "finitistic"/computational model structure. The reader may want to skip or skim though the vague discussion in §2.5 of the definitions of fibrations, trivial fibrations, and Michael selection theory, attempting to provide some intuitions and context. This includes a diagram chasing expression for the non-Hausdorff mapping cone and an explanation of $(\mathrm{f})_{\mathrm{micro}}$ as defining property of a microfibration.

In Appendix §3.1 we state [**6**, Thm.1.2] of continuous selection theory we use, and Appendix §3.2 we state the theorems of [**14**] we use for compactness.

A vague discussion of intuitions behind the conjecture. — We now make a number of rather vague remarks in an attempt to explain and motivate the proposed definition of the model structure.

In fact, the precise choice of the map $M \to \Lambda$ in the double Quillen orthogonal (negation) $\{M \to \Lambda\}^{lr}$ is a way to add precise "niceness" assumptions to the "naïve" lifting property defining fibrations:

(wf) a map $p : Y \to B$ is a trivial fibration iff the lifting property $A \to X \rightthreetimes Y \xrightarrow{p} B$ holds whenever $A \subset X$ is a "nice" closed subset of a "nice" space X.

(f) a map $p : Y \to B$ is a fibration iff, whenever $A \subset X$ closed and "nice", for any lifting problem $A \to X \rightthreetimes Y \xrightarrow{p} B$, there exists a diagonal lifting defined on some open neighbourhood of A.(6)

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(6) Formally in notation, for any commutative square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & Y \arrow[d, "p"] \\ X \arrow[r, "\phi"] & B \end{tikzcd}$$

there is an open $A \subset U \subset X$ and a map $\tilde{f}_U : U \to Y$ such that the diagram

$$\begin{tikzcd} & A \arrow[r, "f"] \arrow[d] \arrow[dl, dotted] & Y \arrow[d, "p"] \\ U \arrow[r, dotted] \arrow[urr, dotted, "\tilde{f}_U"] & X \arrow[r, "\phi_{|U}"] & B \end{tikzcd}$$

commutes. This is similar to a lifting property $A \to X \rightthreetimes Y_{p\searrow B} \to Y$ where $Y_{p\searrow B}$ is the non-Hausdorff mapping cone of $Y \xrightarrow{p} B$, see footnote(16) for the definition and a diagram-chasing characterisation.

We use the word "nice", in this paper, to mean various precise assumptions of the kind made to avoid spurious difficulties related to wild phenomena such as curves cheerfully filling cubes, which are irrelevant from the point of view of the topological intuition of shapes, cf. [**4**, §5, pp.28/29].

The definition of trivial Serre fibration in (wf) and of a Serre microfibration in (f) chooses the nicest possible $A \subset X$ – the inclusions of a sphere as the boundary of a ball. Michael continuous selection theory [**6**, Thm.1.2] chooses least(?) nice ones: an arbitrary closed subset of a Hausdorff paracompact space of finite Lebesgue dimension, and implies that, when calculated in the full subcategory of paracompact spaces of finite Lebesque dimension, $\{M \to \Lambda\}^{lr}$ is a class of trivial Serre fibrations containing all locally trivial maps (fibre bundles) of completely metrisable spaces with contractible and locally contractible base and fibres (see §3.1, esp. Thm.3.1.2, for a summary of [**6**, Thm.1.2] of Michael continuous selection theory; also see Lemma 2.1.1(4), §2.5(ii), and Conjecture 2.4.1).

The map $M \to \Lambda$ does capture the implicit combinatorics of the definition of a trivial fibration in presence of the right "niceness" assumptions, and by saying this we mean that the double orthogonal (negation) of $M \to \Lambda$ leads to the right notion if calculated in a certain subcategory of nice spaces, but it is not clear to us whether this implicit combinatorics is sufficient if calculated in the category of all topological spaces. Perhaps the reader would see this right away.

Simplicially, M is the barycentric subdivision of Λ, i.e. the preorder of chains in Λ, and the geometric realisation of Λ is the interval. It is easy to see that the map $M \to \Lambda$ captures the "combinatorics" implicit in the definition of normality: a space X is normal (T_4 but not necessarily T_1) iff $\varnothing \to X \rightthreetimes M \to \Lambda$: indeed, to give a map $X \to \Lambda$ is to give two disjoint closed subsets of X (the preimages of the two closed points of Λ), and to give a factorisation $X \to \Lambda$ is to give their disjoint neighbourhoods (the preimages of the open subsets of M separating the preimages of the two closed points of Λ). Instead of $M \to \Lambda$, one may consider the more complicated map implicit in the definition of hereditary normal (separation axiom T_5, see Fig.3), see the proof of Lemma 2.3.1 for a discussion. Seeing that the map $M \to \Lambda$ captures the "combinatorics" implicit in the proof of Tietze extension theorem and, arguably, the notion of contractability, is slightly less obvious, see the proof of Lemma 2.3.1(2).

Everything in this note is very elementary: a reader is likely to improve upon our claims, and any proofs can be given as exercise to any student familiar with the terminology.

2. Observations

A number of basic notions in topology can be concisely defined, often starting from simplest examples, by repeatedly taking the orthogonal with respect to the Quillen lifting property in the category of topological spaces [**1, 2, 3**].

Here is a sample: *connected, compact, and contractible*; see [**2**] for a longer list.

2.1. Connected. — We "generate" several basic notions by merely applying ^{l} and ^{r} to the simplest possible map, the embedding of the empty set into a singleton.

Lemma 2.1.1 ($\varnothing \to \{o\}$). — *In the category of (all) topological spaces,*

r: $\{\varnothing \to \{o\}\}^r$ *is the class of* surjections

rl: $\{\varnothing \to \{o\}\}^{rl}$ *is the class of maps* $A \to A \sqcup D$ *where* D *is discrete*

rllr: $\{\varnothing \to \{o\}\}^{rllr}$ *is the class of maps* $A \to A \sqcup D$

rr: $\{\varnothing \to \{o\}\}^{rr}$ *is the class of* subsets*, i.e. inclusions* $A \hookrightarrow B$ *with topology on* A *induced from* X

lrrrl: $\{\varnothing \to \{o\}\}^{lrrrl}$ *is the class of* quotients*, i.e. the maps* $f : A \to B$ *such that a subset* $U \subset B$ *is open in* B *iff its preimage* $f^{-1}(U) \subset A$ *is open in* A*.*

rll: *A map* $f : A \to B$ *of "nice" spaces belongs to* $\{\varnothing \to \{o\}\}^{rll}$ *iff the induced map* $\pi_0(f) : \pi_0(A) \to \pi_0(B)$ *of connected components is surjective. In particular,*

- *A non-empty topological space* X *is* connected *iff for each, equiv. any, map* $\{o\} \to X$ *from a singleton it holds*

$$\{o\} \to X \in \{\varnothing \to \{o\}\}^{rll}$$

Here in (r) and (rl), $A \sqcup D$ denotes the disconnected union of A and D, i.e. both subsets A and D are closed and open, and the topology on both A and D is induced.

In (rll), by a space being "nice" we mean that it splits into a disconnected union of closed and open connected components.

Proof. — 1. By definition

$$\{\varnothing \to \{o\}\}^r := \left\{ X \xrightarrow{g} Y : \varnothing \to \{o\} \rightthreetimes X \xrightarrow{g} Y \right\}$$

is the class of maps which have the right lifting property with respect to the embedding of the empty subset into a singleton. This lifting property says that any point of Y (the image of $\{o\}$ in Y) has a preimage in X (the image of $\{o\}$ in X), i.e. is surjective.
2. By definition

$$\{\varnothing \to \{o\}\}^{rr} = \left\{ X \xrightarrow{g} Y : f \rightthreetimes g \text{ for any } f \in \{\varnothing \to \{o\}\}^r \right\}$$

is the class of maps which have the right lifting property with respect to any surjection. If map $g : X \to Y$ represents a subset, i.e. $X \subset Y$, the topology on X is induced from Y, and and $f_{|X} = \mathrm{id}_{|X}$, then the image of $B \to Y$ is contained in X, and, as the topology on X is induced, the lifting is continuous. In the opposite direction, take B to be the image of $g : X \to Y$, and A to be the preimage of $g : X \to Y$ with topology induced from Y. Then $f \rightthreetimes g$ lifts iff $g : X \to Y$ represents a subset. Rest is similar. □

2.2. Compact. — Thinking of Quillen orthogonals as a form of negation in category theory (i.e. that taking the orthogonal of a class/property of morphisms is perhaps the simplest way to define a class/property of morphisms *without* the property in a way useful in a diagram chasing computation), we try to define compactness (rather, the class of proper morphisms) by picking a example of a non-proper map. Perhaps the simplest example of a map which is not proper is the embedding of a point as the open point in the two-point space with one point open and one point closed. We denote this map by $\{o\} \longrightarrow \{o{\to}c\}$.

Lemma 2.2.1 ($\{o\} \longrightarrow \{o{\to}c\}$). — *In the category of (all) topological spaces, the class* $\left(\{\{o\} \longrightarrow \{o{\to}c\}\}^r_{<5}\right)^{lr}$ *is a class of proper maps, and*

– *a map of "nice" spaces is proper iff it lies in* $\left(\{\{o\} \longrightarrow \{o{\to}c\}\}^r_{<5}\right)^{lr}$

In particular, a Hausdorff space K is compact iff

$$K \to \{o\} \in \left(\{\{o\} \longrightarrow \{o{\to}c\}\}^r_{<5}\right)^{lr}$$

Here, "nice" may be taken to mean Hausdorff hereditary normal (separation axioms T_1 and T_5), and $\{\{o\} \longrightarrow \{o{\to}c\}\}^r_{<5}$ denotes the subclass of $\{\{o\} \longrightarrow \{o{\to}c\}\}^r$ consisting of maps of spaces with less than 5 points.

Proof. — See §2.2.1 or [**19**, §2.2] for a verbose explanation; here we are brief. First check that a map f of finite spaces is closed, equiv. proper, iff $\{o\} \longrightarrow \{o{\to}c\} \rightthreetimes f$. The definition of being proper via ultrafilters (see Bourbaki [**14**, I§10.2,Th.1(d)], quoted in §3.2) expresses being proper as a lifting property with respect to a class of maps associated with ultrafilters: f is proper iff

$$A \to A \sqcup_{\mathcal{U}} \{\infty\} \rightthreetimes X \xrightarrow{f} Y$$

where the topology on $A \sqcup_{\mathcal{U}} \{\infty\}$ is such that ∞ is closed, $\mathfrak{U}$ is the neighbourhood filter of ∞, and the topology on A is induced [**14**, I§6.5, Def.5, Example]. The definition of this topology is "read off" from the definition of a limit of an ultrafilter: for $X = A$ and $\mathfrak{U}$ an ultrafilter on X, a map $A \sqcup_{\mathfrak{U}} \{\infty\} \to X$ extending the identity on X, is continuous iff it takes ∞ to a limit of $\mathfrak{U}$ in X. These maps belong to $\left(\{\{o\} \longrightarrow \{o{\to}c\}\}^r_{<5}\right)^l$, hence any map in $\left(\{\{o\} \longrightarrow \{o{\to}c\}\}^r_{<5}\right)^{lr}$ is proper.

Smirnov-Vulikh-Taimanov theorem [**15**, 3.2.1,p.136] gives sufficient conditions to extend a map to a compact Hausdorff space, and can be generalised to give the required lifting property. It says that a map to a compact Hausdorff space can be extended to the whole space X from a dense subset A satisfying the (in fact necessary) condition for every pair B_1, B_2 of disjoint closed subsets of A the inverse images $f^{-1}(B_1)$ and $f^{-1}(B_2)$ have disjoint closures in the space X. A verification shows that the following four maps are closed and their left orthogonals define these sufficient conditions on $A \to X$:(7)

$\{a{\leftarrow}u{\rightarrow}b\} \longrightarrow \{a = u = b\}$	$\{a \leftrightarrow b\} \longrightarrow \{a = b\}$	$\{o{\to}c\} \longrightarrow \{o = c\}$	$\{c\} \longrightarrow \{o{\to}c\}$
(disjoint closures)	(injective)	(pullback topology)	(dense image)
$\{a \overset{=}{\leftarrow} u \overset{=}{\rightarrow} b\}$	$\{a \overset{=}{\leftrightarrow} b\}$	$\{o \overset{=}{\rightarrow} c\}$	$\{o {\color{red}\rightarrow c}\}$
`{a<-u->b}-->{a=u=v}`	`{a<->b}-->{a=b}`	`{o->c}-->{o=c}`	`{c}-->{o->c}`

Hence, the Smirnov-Vulikh-Taimanov theorem [**15**, 3.2.1,p.136] implies that a Hausdorff space K is compact iff $K \to \{o\}$ is in

(7)Our notation represents finite topological space as preorders or finite categories with each diagram commuting, and is hopefully self-explanatory; see [**3**] for details. In short, an arrow $o \to c$ indicates that $c \in \mathrm{cl}\, o$, and each point goes to "itself"; the list in $\{..\}$ after the arrow indicates new relations/morphisms added, thus in $\{o \to c\} \longrightarrow \{o = c\}$ the equality indicates that the two points are glued together or that we added an identity morphism between o and c. The notation in the 3rd line informal (red indicates new/added elements), and in the 4th line reminds of a computer syntax.

$$\Big\{\{a{\leftarrow}u{\rightarrow}b\} \longrightarrow \{a=u=b\}, \{a \leftrightarrow b\} \longrightarrow \{a=b\}, \{o{\rightarrow}c\} \longrightarrow \{o=c\}, \{c\} \longrightarrow \{o{\rightarrow}c\}\Big\}^{lr},$$

and the latter is a subclass of $\left(\{\{o\} \longrightarrow \{o{\rightarrow}c\}\}^{r}_{<5}\right)^{lr}$. □

In what way may it be useful to say that these four maps of preorders reveal combinatorics implicit in the notion of compactness ?

Note that for this statement it is important that the category of topological spaces contains spaces associated with ultrafilters that would usually be considered to belong to wild phenomena such as curves cheerfully filling cubes, which are irrelevant from the point of view of the topological intuition of shapes, cf. [**4**, §5, pp.28/29].

Remark 2.2.2. — Other choices of examples lead to modifications of compactness discussed in literature. [**12**]calls a space X E-compact iff it is a closed subset of a power of E (for Hausdorff E a retract is necessarily closed), i.e. $X \to \{o\} \in \{E \to \{o\}, \{c\} \to \{o \to c\}\}^{lr}$. Later [**13**] introduced the notion of *extension closed subspace* and considered the class of extension closed subspaces of a power of a finite space containing those X such that $X \to \{o\} \in (\{\{c\} \to \{o \to c\}\} \cup \{F \to \{o\} : |F| < \infty\})^{lr}$. A retract of a Hausdorff space is necessarily closed, and a retract of an arbitrary space is necessarily extension closed, and this allows to relate results of [**12, 13**] to the weak factorisation systems indicated.

2.3. Contractible.

— A simple example of a map contracting something is provided by by the map $M \to \Lambda$ from a space M with 5 points (two open and three closed), into a space Λ with 3 points (one open and two closed), see Fig. 1: it contracts the V in M to get Λ. Accordingly, we use this example to try and define *contractible* (among "nice" spaces).

Lemma 2.3.1 ($M \to \Lambda$). — *In the category of (all) topological spaces, $\{M \to \Lambda\}^{lr}$ is a class of trivial Serre fibrations, and*

1. *A "nice" space Y is contractible iff*

$$Y \to \{o\} \in \{M \to \Lambda\}^{lr}$$

2. *X is normal (not necessarily Hausdorff) iff $\varnothing \to X \in \{M \to \Lambda\}^{l}$, i.e.*

$$\varnothing \to X \rightthreetimes M \to \Lambda$$

3. *For a map $A \to X$ from a Hausdorff space A to a "nice" (meaning Hausdorff hereditary normal) space X, it represents a closed subset $A \subset X$ iff $A \hookrightarrow X \in \{M \to \Lambda\}^{l}$, i.e.*

$$A \hookrightarrow X \rightthreetimes M \to \Lambda$$

In (1), "nice" may be taken to mean "being a finite CW complex".[(8)] What we need is that Y is a retract of some Euclidean space $\mathbb{R}^n$ iff Y is weakly contractible.

Remark 2.3.2. — It is not hard to see that Lemma 2.3.1(2,3) implies the following. If calculated in a category of piecewise linear maps of finite CW complexes and finite topological spaces (defined appropriately), $\{M \to \Lambda\}^{lr}$ is the class of trivial Serre fibrations. Michael selection theorem ([**6**, Thm.1.2], see §3.1) implies that, if calculated in the category of paracompact spaces of finite Lebesgue dimension, the double Quillen negation $\{M \to \Lambda\}^{lr}$ contains many natural examples of trivial fibrations, for example all locally trivial maps of completely metrisable spaces with locally connected fibres. Note that this category is large enough for the calculation of Lemma 2.2.1 for the triple Quillen negation of compactness to be valid.

Of course, this tempts a conjecture

Conjecture 2.3.3. — *A map of "nice" spaces is a trivial fibration iff it belongs to* $\{M \to \Lambda\}^{lr}$.

Proof. — Recall that

$$\{M \to \Lambda\}^{lr} = \left\{ Y \xrightarrow{p} B \,:\, A \xrightarrow{i} X \rightthreetimes Y \xrightarrow{p} B \text{ whenever } A \xrightarrow{i} X \rightthreetimes M \to \Lambda \right\}$$

Thus, to see that $\{M \to \Lambda\}^{lr}$ is a class of trivial Serre fibrations it is enough to verify that $\mathbb{S}^n \to \mathbb{D}^{n+1} \in \{M \to \Lambda\}^{l}$, where $\mathbb{S}^n \to \mathbb{D}^{n+1}$ denotes the standard embedding of an n-sphere into the $n+1$-ball as the boundary. This is done in (3) using that $\mathbb{D}^{n+1}$ is hereditary normal.(2). To give a map $X \longrightarrow \Lambda$ is to give two disjoint closed subsets of X; to give a lifting to M is to find their disjoint neighbourhoods. (1). It is enough to show that for $Y = [0,1]$: indeed, ^{r}-orthogonals are closed under products and retracts, and any contractible finite CW complex is a retract of some $[0,1]^n$, $n > 0$ [**10**]. The proof for $Y = [0,1]$ we give is the standard proof of the Tietze extension theorem retold in a diagram chasing notation.

Represent the interval $[0,1]$ as a union

$$[0,1] = \{0\} \cup (0,t_1) \cup \{t_1\} \cup (t_1,t_2) \cup \ldots \cup (t_{n-1},1) \cup \{1\}$$

Contract the open intervals to (open) points, and denote the resulting map by $[0,1] \to \Lambda_n$ where $\Lambda_n = \left\{ {}_0 \swarrow t^-_{0,1} \searrow {}_{t_1} \swarrow t^-_{1,2} \searrow {}_{t_2} \swarrow \ldots. \searrow {}_{t_{n-2}} \swarrow t^-_{n-2,n-1} \searrow {}_{t_{n-1}} \swarrow t^-_{n-1,n} \searrow {}_1 \right\}$. Subdividing the open intervals gives maps $\Lambda_{2n} \to \Lambda_n$. The map $\Lambda_2 = M \to \Lambda = \Lambda_1$ corresponds to subdividing a single open interval into two. Use that ^{r}-orthogonals are

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(8) As pointed out by Tyrone Cutler at mathoverflow.net, "finite" is important: Let $C\mathbb{N}$ be the cone over a countably infinite discrete complex (this is a contractible 1-dimensional polyhedron). van Douwen and Pol [van Douwen, Eric K.; Pol, Roman. Countable spaces without extension properties. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 25 (1977), no. 10, 987–991.] have constructed a countable regular T_2 space X (which is thus perfectly normal) and a function $A \to C\mathbb{N}$, defined on a certain closed $A \subset X$, which does not extend over any neighbourhood in X. In particular, the map of countable complexes $C\mathbb{N} \to \{o\}$ is both a Hurewicz fibration and a homotopy equivalence, but is not soft wrt all perfectly normal pairs.

closed under pullbacks to see that $\Lambda_{2n} \to \Lambda_n \in \{M \to \Lambda\}^{lr}$, and that ^{r}-orthogonals are closed under inverse limits to see that $\Lambda_\omega \to \Lambda \in \{M \to \Lambda\}^{lr}$ where $\Lambda_\omega := \lim\limits_{\Lambda_{2n}\to\Lambda_n} \Lambda_{2^n}$ and that ^{r}-orthogonals are closed under composition to see that $\Lambda_\omega \to \Lambda \in \{M \to \Lambda\}^{lr}$, and that ^{r}-orthogonals are closed under composition to see that $\Lambda_\omega \to \{o\} \in \{M \to \Lambda\}^{lr}$ as $\Lambda \to \{o\}$ is a retract of $\Lambda_4 \to \Lambda_2$. Finally, the maps $[0,1] \to \Lambda_n$ induce an embedding $[0,1] \to \Lambda_\omega$ of $[0,1]$ into Λ_ω as a retract, hence, an orthogonals are closed under retract, we get the required result. (3). Pick a map sending X to the open point of Λ, and the separating neighbourhoods of two distinct points of A to the two open points of M. A lifting would provide separating neighbourhoods of their images. Therefore, the map $A \to X$ is injective. To see that it is closed, pick a map sending the whole of A to the closed point in the "middle" of M, and an arbitrary point x of $X - A$ into a closed point of Λ. A lifting would provide neighbourhood of x disjoint from A. To see that the topology on A is induced, pick a map $X \to \Lambda$ sending X to the open point of Λ, and a map $A \to M$ sending an arbitrary open subset U of A into an open point of Λ. A lifting would provide an open subset of X whose intersection with A is U. In the opposite direction, use that a space is hereditary normal iff whenever each of two disjoint subsets can be separated from the other by an open neighbourhood, they have disjoint open neighbourhoods, cf. Fig. 3. We remark that this characterisation is a lifting property. □

Remark 2.3.4. — [**9**, Thm.3.5] almost says that $|\Delta Y| \to Y \in \{|\Delta\Lambda| \to \Lambda\}^{lr}$ where $|\Delta Y|$ denotes the geometric realisation of the simplicial set $n_\leq \mapsto \mathrm{Hom}_\leq(n_\leq, Y)$ of non-decreasing chains in a finite preorder Y.(9) Note that $|\Delta\Lambda| = [0,1]$ and $|\Delta\Lambda| \to \Lambda$ is the map $[0,1] \to \Lambda$ contracting $(0,1)$ to the open point of Λ. This suggests that $||Y|| \to Y \in \{M \to \Lambda\}^{lr}$ where $||Y|| := \lim\limits_{Y^{(n+1)}\to Y^{(n)}} Y^{(n)}$ is the non-Hausdorff geometric realisation of Y, where $Y^{(n+1)}$ is the preorder of increasing chains (the barycentric subdivision) of $Y^{(n)}$, and $Y^{(0)} := Y$. Note $M = \Lambda^{(1)}$.

2.4. A naïve finitist's "computational" model structure. — In a model category, the class of fibrations can sometimes be defined in terms of cofibrations and

(9) [**9**, Thm.3.5] says also that the lifting $X \to |\Delta Y|$ is unique up to homotopy over Y. This is implied by $X \times \{0,1\} \cup A \times [0,1] \to X \times [0,1] \in \{|\Delta\Lambda| \to \Lambda\}^{lr}$.

trivivial fibrations in a diagram chasing manner.[10] This and considerations above suggest the following naive conjecture.[11]

***Conjecture 2.4.1** ($M \to \Lambda$). — A closed model structure on the category of topological spaces is defined as follows:*

(c) $\{M \to \Lambda\}^l$ *is the class of cofibrations.*

(wf) $\{M \to \Lambda\}^{lr}$ *is the class of trivial fibrations.*

(f) *a map $p : Y \to B$ is a fibration iff for any commutative diagram of solid arrows labelled as shown there exist a diagonal arrow $X \to Y$ making the total diagram commutative:*

$$\begin{tikzcd} A & A \arrow[l, "\mathrm{id}"'] \arrow[r] \arrow[d, "(c)"] & Y \arrow[d, "p"] \\ & X \arrow[ul, "(wf)"] \arrow[ur, dotted] \arrow[r] & B \end{tikzcd}$$

(wc) *a map is a weak cofibration iff it is has the right lifting property with respect to all fibrations: (wc):=(f)^l*

(w) *a weak equivalence is the composition of a trivial cofibration with a trivial fibration: (w):=(wc)(wf)*

With these definitions, classes of (trivial) fibrations are subclasses of (trivial) Serre fibrations, and a map of Hausdorff hereditary normal spaces is a cofibration iff it is a closed inclusion; and, for any contractible finite CW complex I, the map $A \times I \xrightarrow{(wf)} A$

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[10] For example, using the following diagram rule, valid in any model category: a map $p : Y \to B$ is a fibration iff

– given a commutatitive diagram of solid arrows labelled as shown, you can always construct the dotted arrow making the whole diagram commutative.

$$\begin{tikzcd} A & A \arrow[l, "\mathrm{id}"'] \arrow[r] \arrow[d, "(c)"] & Y \arrow[d, "(f)"] \\ & X \arrow[ul, "(wf)"] \arrow[ur, dotted] \arrow[r] & B \end{tikzcd}$$

This gives the right definition of a Serre or Hurewicz fibration in the category of topological spaces.

[11] A sketch of a conjecture, rather. The choice of $M \to \Lambda$ is somewhat arbitrary; the conjecture with the more complicated map for separation axiom T_5 (hereditary normal, see Fig. 3) would have been equally well-motivated. We may need to consider the assumptions of Michael continuous selection theory, paracompactness and finite Lebesgue dimention, which can also be represented as lifting properties, cf. Appendix 3.1. To demonstrate the flexibility/vagueness, let us say that we can easily add to the conjecture the assumption of "Lebesgue zero dimension" by considering $\{M \to \Lambda, \{a \leftarrow u, v \to b\} \longrightarrow \{a \leftarrow u = v \to b\}\}$, using that a topological space X is zero-dimensional iff $\varnothing \to X \rightthreetimes \{a \leftarrow u, v \to b\} \longrightarrow \{a \leftarrow u = v \to b\}$ [**5**, Proposition 2(c)]. Of course, this particular example is not what we would want. We may also to consider an analogue of this conjecture in another category such as the of posets, simplicial sets, or the category of simplicial objects in the category of filters [**21**] extending the category of topological spaces, of simplicial sets, and of uniform spaces.

Importantly, (f) can probably be improved, see §2.5 for a discussion.

is a trivial fibration, and map $A \xrightarrow{(wc)} A \times [0,1]$ is a trivial cofibration whenever $A \times [0,1]$ is Hausdorff hereditary normal.

In §2.5 below we show that the lifting property of a microfibration can be formulated as follows, thereby suggesting replacing (f) by

(f)$_{\mathrm{micro}}$ *a map is a fibration iff it admits a decompostion*

$$\cdot \xrightarrow{\{\{o\to a \leftrightarrow b\}\longrightarrow\{o=a\leftrightarrow b\}\}^{l}} \cdot \xrightarrow{\{\{o\to a \leftrightarrow b\}\longrightarrow\{o=a\leftrightarrow b\}\}^{lr},\{M\to\Lambda\}^{lr}} \cdot$$

Formal topological spaces: a model category of diagram chasing constructions. — The language of this conjecture is purely combinatorial. Can we define a model category of "formal" topological spaces ("formal" as in formal power series), i.e. a model category whose objects and arrows belong to a calculus of diagram chasing computations, so to say ? A naïve hope is that the size of spaces appearing in the Quillen orthogonals (negations) representing basic notions of topology [**2, 3**] is small enough (< 7) to make feasible the exponential growth in the computer processing of such a calculus.

The following conjecture would represent a rule in such a diagram chasing calculus of formal topological spaces,[(12)] and perhaps might imply the previous conjecture.

***Conjecture 2.4.2* (M2).** — *For each finite set P of maps of finite spaces, and each string consisting of letters l and r, each map in the category of topological spaces*[(13)] *decomposes as a map in $(P)^{sl}$ followed by a map in $(P)^{slr}$, and as a map in $(P)^{srl}$ followed by a map in $(P)^{sr}$:*

$$\begin{tikzcd} & \cdot \arrow[dr, dotted, "(P)^{slr}"] & \\ \cdot \arrow[ur, dotted, "(P)^{sl}"] \arrow[rr, "\forall"] & & \cdot \end{tikzcd} \qquad \begin{tikzcd} & \cdot \arrow[dr, dotted, "(P)^{sr}"] & \\ \cdot \arrow[ur, dotted, "(P)^{srl}"] \arrow[rr, "\forall"] & & \cdot \end{tikzcd}$$

The reformulations in [**2, 3**] show that this conjecture would imply that each topological space can be "approximated" by a compact space (Stone-Cech compactification, up to separation axioms, see Lemma 3.2); by a space satisfying separation axiom T_0,T_1, T_4, T_5, totally disconnected, extremally disconnected, Lebesgue zero-dimensional, ultranormal, discrete; that it is well-defined to take a connected component; add an open/closed subset to a topology, take a non-Hausdorff mapping cone/cylinder, etc. The decompositions required by the conjecture are reminiscent to cofibration-fibration decompositions required of Axiom M2 of a Quillen model category, and are typically proved by the Quillen's small object argument using set-theoretic assumptions that certain classes of morphisms involved are small, which do not necessarily hold in our case.

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(12) See [**19**, §2.3] for a discussion how to view the axioms of topology as rules of diagram chasing with preorders.

(13) Here it may be important (and, inasmuch, in Conjecture 2.4.1) to consider the category of *all* topological spaces rather than a convenient category of topological spaces, for the orthogonal (Quillen negation) for compactness is a natural example which, in a sense, captures the definition of compactness via ultrafilters, and, accordingly, its calculation requires considering topological spaces associated with ultrafilters. However, note that these spaces are (non-Hausdorff) paracomact of finite Lebesgue dimension, simply because they have very few open covers.

Logical ideas.— Of course, the real temptation is to develop a computer algebra system doing topology using a syntax extending the concise syntax for topology we discuss, and to use it in teaching.

Problem 2.4.3. — *Find a diagram chasing calculation deriving the axioms of a model category from Conjecture M2 and other diagram chasing rules of topology such as* [**19**, 2.3]*,*

Arguably, this might clarify the topological notions/intuition and be an indication that our observations lead to a tame topology of [**4**].

2.5. Vista: the naïve defining lifting property of a fibration.

— We explain the motivation and, to an extent, meaning of the conjectural definition of a model structure we give in §2.4.

2.5.1. The naïve defining lifting property of a fibration. — If all spaces were "nice", we could perhaps define fibrations and trivial fibrations as follows:

(wf) a map $p : Y \to B$ is a trivial fibration iff the lifting property $A \to X \rightthreetimes Y \xrightarrow{p} B$ holds whenever $A \subset X$ is a closed subset of a space X.

(f) a map $p : Y \to B$ is a fibration iff, whenever $A \subset X$ closed, for any lifting problem $A \to X \rightthreetimes Y \xrightarrow{p} B$, there exists a diagonal lifting defined on some open neighborhood of A.(14)

In (wf), we get the definition of trivial Serre fibration if we restrict $A \subset X$ to be cellular inclusions of finite CW complexes, or indeed just the inclusions $\mathbb{S}^n \to \mathbb{B}^{n+1}$ of an n-sphere as the boundary of $n+1$-ball, $n \geq 0$. In (f), the same restriction almost gives the definition of a Serre microfibration.(15)

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(14) Formally in notation, for any commutative square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & Y \arrow[d, "p"] \\ X \arrow[r, "\phi"] & B \end{tikzcd}$$

there is an open $A \subset U \subset X$ and a map $\tilde{f}_U : U \to Y$ such that the diagram

$$\begin{tikzcd} & A \arrow[r, "f"] \arrow[d] \arrow[dl, dotted] & Y \arrow[d, "p"] \\ U \arrow[r, dotted] \arrow[urr, dotted, "\tilde{f}_U"] & X \arrow[r, "\phi_{|U}"] & B \end{tikzcd}$$

commutes.

(15) A Serre microfibration with weakly contractible fibres is necessarily a Serre fibration, see [**18**] and references thein.

We quote [**17**, 5.1.2] for background: "Examples of microfibrations that occur in the general theory are as follows:

(i) If $X \subset Y$ is open then the inclusion map $i : X \to Y$ is a microfibration.

(iv) If M, N are smooth manifolds then a submersion $\rho : M \to N$ is a microfibration. In addition, if M is compact, then ρ is a fibration. In case Y is a manifold then a microfibration $\rho : X \to Y$ is an open map, though not conversely. There are microfibrations $\rho : X \to \mathbb{S}^1$, $X \subset \mathbb{R}^3$ is compact

Michael selection theory (see §3.1) says that we do get the standard notions of a trivial fibration, and of a (micro)fibration, if we take X to vary among paracompact spaces of finite Lebesgue dimension; then it is sufficient for $p : Y \to B$ to be a map of complete metric spaces with uniformly weakly contractible fibres, e.g. locally trivial with weakly contractible fibres, or that for each $\varepsilon > 0$ there is $\delta > 0$ such that, inside each fibre, each sphere of diameter $< \delta$ can be contracted by a homotopy remaining both in the fibre and its ϵ-neighbourhood. iThese assumptions come from Michael continuous selection theory [**6**, Thm.1.2], see §3.1.

We rewrite (wf) and (f) in the diagram chasing manner using Lemma 2.5.1 and the notion of non-Hausdorff mapping cone/cylinder,[(16)]

Lemma 2.5.1. — *In a full subcategory of "nice" topological spaces,*

(wf)′ *a "very nice" map is a trivial fibration iff it belongs to* $\{M \to \Lambda\}^{lr}$

(f)′ *a "very nice" map is a fibration iff the map from its non-Hausdorff mapping cone to the base belongs to* $\{M \to \Lambda\}^{lr}$

$$Y_{p \searrow B} \to B \in \{M \to \Lambda\}^{lr}$$

Here, being "nice" means being (possibly non-Hausdorff) paracompact of finite Lebesgue dimension, and "very nice" means say a map of finite CW complexes or being smooth in a suitable sense (we need something to ensure that a fibration is necessarily a map of complete metrisable spaces with uniformly locally contractible fibres), and $Y_{p \searrow B}$ denotes the non-Hausdorff mapping cone of $Y \xrightarrow{p} B$.

Proof. — Recall that

$$\{M \to \Lambda\}^{lr} = \left\{ Y \xrightarrow{p} B \, : \, A \xrightarrow{i} X \rightthreetimes Y \xrightarrow{p} B \text{ whenever } A \xrightarrow{i} X \rightthreetimes M \to \Lambda \right\}$$

A map to a Hausdorff spaces necessarily glues together points which cannot be separated by neighbourhoods (for their images can if distinct), hence we may assume that both A and X are Hausdorff and by Lemma 2.5.1(3) that $A \xrightarrow{i} X$ is the inclusion of a closed subset. Hence, (f)′ states precisely (f) above, i.e. the conclusion of Michael selection theorem Theorem 3.1.2 for trivial fibrations.

(not a polyhedron), that are not Serre fibrations. Let $\rho : X \to Y$ be a microfibration where X, Y are compact polyhedra and ρ is piecewise linear. It seems reasonable to conjecture that ρ is a Serre fibration. The literature does not seem to address this question."

(16) Intuitively, this is the usual (Hausdorff) mapping cone $Y \times [0,1]/\{(y,1) = p(y)\}$ where we contracted $[0,1)$ of $[0,1]$ to point, thereby replacing $[0.1]$ by the two-point Sierpinski-Kolmogorov space $\{o \to c\}$. Set-theoretically, the *non-Hausdorff mapping cone/cylinder* of a map $p : Y \to B$, denoted by $Y_{p \searrow B}$, is $Y \times \{o \to c\} / (-, c) = p(-)$, i.e. the disjoint union $Y \sqcup B$ equipped with the following topology: an open subset is either an open subset of X, or the union of an open subset of B and its preimage. Diagram-chasingly, a non-Hausdorff mapping cone fits into a $(P)^l (P)^{lr}$ decomposition, cf. Conjecture 2.4.2, for $P := \big\{ \{o \to a \leftrightarrow b\} \longrightarrow \{u = a \leftrightarrow b\} \big\}$ or $P' := P \cup \big\{ \{a \leftrightarrow b\} \to \{a = b\} \big\}$ defining "open subset", as, e.g. $Y \xrightarrow{\{\{o \to a \leftrightarrow b\} \longrightarrow \{o = a \leftrightarrow b\}\}^l} Y_{p \searrow B} \xrightarrow{\{\{o \to a \leftrightarrow b\} \longrightarrow \{o = a \leftrightarrow b\}\}^{lr}} B$

Similarly, (wf)′ is (wf) using the diagram chasing property of the non-Hausdorff mapping cone:

– is to give a map $X \to Y_{p\searrow B}$ is the same as to give a commutative square

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "i"'] & Y \arrow[d, "p"] \\ X \arrow[r, "\phi"] & B \end{tikzcd}$$

for some open subset U of X.

Indeed, this means that the lifting property $A \xrightarrow{i} X \rightthreetimes Y_{p\searrow B} \xrightarrow{p} B$ of item (f)′ holds iff for any open subset U of A and a commutative square

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "i"'] & Y \arrow[d, "p"] \\ X \arrow[r, "\phi"] & B \end{tikzcd}$$

there is an open $U \subset V \subset X$ and a map $\tilde{f}_V : V \to Y$ such that the diagram

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d] & Y \arrow[d, "p"] \\ V \arrow[r, "\phi_{|V}"] \arrow[ur, dotted, "\tilde{f}_V"] & B \end{tikzcd}$$

commutes. This is almost the conclusion of Michael selection theorem Theorem 3.1.2 for fibrations as stated.

Finally, by Lemma 2.5.1(3), $\{M \to \Lambda\}^l$ contains the inclusion $\mathbb{S}^n \to \mathbb{B}^{n+1}$ of an n-sphere as the boundary of $n+1$-ball, $n \geq 0$, and thereby $\{M \to \Lambda\}^{lr}$ is a subclass of trivial Serre fibrations. (f)′ implies that $Y \xrightarrow{p} B$ is a microfibration [**17**, 5.1.2], and, under strong suitable assumptions (e.g., a proper submersion), that it is a fibration. □

2.5.2. The diagram chasing of the non-Hausdorff mapping cone. — However, (f)′ is not quite satisfactory, as it is not in terms of Quillen negation (orthogonals) and does not reveal a combinatorial structure. To reveal the combinatorics of the notion/construction of the non-Hausdorff mapping cone, observe that it fits into[(17)]

$$Y \xrightarrow{\{\{o_{\searrow a} \leftrightarrow b\} \longrightarrow \{o{=}a \leftrightarrow b\}\}^l} Y_{p\searrow B} \xrightarrow{\{\{o_{\searrow a} \leftrightarrow b\} \longrightarrow \{o{=}a \leftrightarrow b\}\}^{lr}} B$$

Indeed, $\{\{o_{\searrow a} \leftrightarrow b\} \longrightarrow \{o{=}a \leftrightarrow b\}\}^l$ is the class of open maps such that the topology on the domain is induced from the target (for T_0 spaces, it means being open subsets),

(17) Recall that we write $X \xrightarrow{P^l} Y$ to mean that the morphism $p : X \to Y$ lies in (equiv., has property) P^l.

and thus the required lifting property is straightforward to check.[18] This shows that (f)′ is equivalent to

(f)″ *a "very nice" map $p : Y \to B$ is a fibration iff it admits a decomposition*

$$Y \xrightarrow{\{\{o\rightarrow a \leftrightarrow b\}\longrightarrow\{o{=}a\leftrightarrow b\}\}^{l}} \cdot \xrightarrow{\{\{o\rightarrow a \leftrightarrow b\}\longrightarrow\{o{=}a\leftrightarrow b\}\}^{lr},\{M\rightarrow\Lambda\}^{lr}} B$$

3. Appendix.

3.1. Appendix. Michael continuous selections. — We sketch the statement of the Michael continuous selections theorem [**6**, Thm.1.2] we use, see also [**5**, **7**].

3.1.1. The statement of [**6**, Theorem 1.2].— Let $(F_x)_{x\in X}$ be a family of non-empty subsets of a topological space Y. Michael selection theory thinks of such a family as a multivalued function $\phi : X \to 2^Y$ and refers to the family as a *carrier*. Michael selection theory gives sufficient conditions for existence of a continuous choice function $f(x) \in F_x, x \in X$. These conditions are satisfied when the family $(F_x)_{x\in X}$ is the family of fibres of a fibration of "nice" spaces. [**7**] considers families of convex subsets of a Banach space but we do not discuss it here.

The family $(F_x)_{x\in X}$ is *lower semi-continuous* iff, whenever $U \subset Y$ is open in Y, the subset $\{x \in X : F_x \cap U \neq \varnothing\}$ is open in X. This subset can be thought of as the preimage of U under the multivalued function $(F_x)_{x\in X}$.

A set F is called locally n-connected (LC^n) if,for every $y \in Y$ and neighborhood U of$y$, there exists a neighborhood V of y such that every continuous image of an m-sphere($m \leq n$) in V is contractible in U; Y is called n-connected (C^n) if every continuous image of an m-sphere ($m \leq n$) in Y is contractible in Y. (Note that since there is no such thing as a (-1)-sphere every Y is (-1)-locally contractible and (-1)-contractible.)

The family $(F_x)_{x\in X}$ is *uniformly locally n-contractible* (*equi* $-$ LC^n) iff, for every $x \in X$ and every $y \in F_x$, and every neighbourhood $U \in y$ of $y \in Y$, there exists a neighbourhood $V \ni y$ of $y \in Y$ such that, for every $F_{x'}, x' \in X$, every continuous image of an m-sphere ($m \leq n$) in $F_{x'} \cap V$ is contractible in $F_{x'} \cap U$. [**6**] uses the convention that each family is uniformly locally -1-contractible "since there is no such thing as a (-1)-sphere". As a diagram $\forall x \in X\, \forall y \in F_x\, \forall U_y \ni y\, \exists V_y \ni y,\, V_y \subset U_y \forall x' \in X$

$$\begin{tikzcd} \mathbb{S}^n \arrow[r, "\forall"] \arrow[d] & F_{x'} \cap V_y \arrow[d] \\ \mathbb{B}^{n+1} \arrow[r, dashed, "\exists"] & F_{x'} \cap U_y \end{tikzcd}$$

By dimension, or dim, we mean the *Lebesgue (covering) dimension*; i.e., $\dim X \leq n$ iff every finite open covering $\mathcal{U}$ of X has a finite, open refinement $\mathcal{V}$ of order $\leq n$ (i.e. every $x \in X$ is in at most $n+1$ elements of $\mathcal{V}$). If $A \subset X$ is closed, then we say that $\dim_X(X - A) \leq n$ if $\dim(C) \leq n$ for every $C \subset X - A$ which is closed in X; for

[18] To see that this double Quillen negation (orthogonal) is rather close to being an equivalent definition, consider the lifting property $Y \to Y_{p\searrow_B} \rightthreetimes Y' \to B$

metric X, this is equivalent to $\dim(X - A) \le n$. For a normal space X, $\dim X \le n$ iff $A \to X \rightthreetimes \mathbb{S}^n \to \{o\}$ for every closed subset $A \subset X$. A space is paracompact iff every open covering has a locally finite open refinement; for a regular space it is equivalent to require only that every open covering has a closure-preserving refinement [**8**, Thm.1]. [19][20]

Theorem 3.1.1 ([**6**, Thm.1.2]). — *Let X be a paracompact Hausdorff space, $A \subset X$ closed with $\dim_X(X - A) \le n+1$, and let $(F_x)_{x \in X}$ be a uniformly locally n-contractible family of non-empty closed subsets of a complete metric space Y.*

Then every continuous choice function on A extends to a continuous choice function on an open neighborhood of A. Moreover, if every $F_x, x \in X$ is n-contractible, then every continuous choice function on A extends to a continuous choice function on the whole of X.

We repeat the conclusion in notation: for every continuous choice function $f : A \to Y$ such that $f(x) \in F_x$ whenever $x \in A$, there is an open neighbourhood $U \subset A$ of A and a continuous choice function $\tilde{f} : U \to Y$ such that $\tilde{f}(u) \in F_u$ whenever $u \in U$, and $f(a) = \tilde{f}(a)$ whenever $a \in A$.

3.1.2. Our application: fibrations and soft maps. — Given a commutative square with a surjective open map $p : Y \to B$ and a subset $A \subset X$

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & Y \arrow[d, "p"] \\ X \arrow[r] & B \end{tikzcd}$$

[19] We combine [**7**, §9] and [**8**] to give background and terminology on coverings:
"A *open covering* of a topological space X is, in [[**7**]], a collection of open subsets of X whose union is X. Its elements need not be open unless that is specifically assumed. A *refinement* of a covering $\mathcal{U}$ is a covering $\mathcal{V}$ such that every $V \in \mathcal{V}$ is a subset of some $U \in \mathcal{U}$. A covering $\mathcal{U}$ is *point-finite* if every $x \in X$ is an element of only finitely many $U \in \mathcal{U}$, it is *locally finite* if every $x \in X$ has a neighbourhood intersecting only finitely many $U \in \mathcal{U}$.
Call a collection $\mathcal{U}$ of subsets of a topological space *closure-preserving* if, for every subcollection $\mathcal{V} \subset \mathcal{U}$ the union of closures is the closure of the union (i.e. $\cup\{\bar{U} : U \in \mathcal{U}\} = [\cup\{U : U \in \mathcal{U}\}]^-$). Any locally finite collection is certainly closure-preserving, but the converse is generally false even for discrete spaces.

[20] These notions can probably be expressed as lifting properties as follows. To give a finite open, resp. closed, covering $\mathcal{U}$ is to give a map $X \longrightarrow \{\mathcal{V} : \varnothing \ne \mathcal{V} \subset \mathcal{U}\}$ where the topology is defined by the order $\mathcal{V}_1 \to \mathcal{V}_2$ iff $\mathcal{V}_1 \supset \mathcal{V}_2$, resp. $\mathcal{V}_1 \subset \mathcal{V}_2$. To give a finite open covering $\mathcal{U}$ of order $\le n$ is to give a map $X \longrightarrow \{\mathcal{V} : \varnothing \ne \mathcal{V} \subset \mathcal{U}, |\mathcal{V}| \le n+1\}$. A finite open covering $\mathcal{U}$ of X has a finite, open refinement $\mathcal{V}$ of order $\le n$ iff $\varnothing \to X \rightthreetimes \{(\mathcal{W}, \mathcal{V}) : \varnothing \ne \mathcal{W} \subset \mathcal{V} \subset \mathcal{U}, |\mathcal{W}| \le n+1\} \to \{\mathcal{V} : \varnothing \ne \mathcal{V} \subset \mathcal{U}\}$ where the topology is generated by the orders $(\mathcal{W}_1, \mathcal{V}_2) \to (\mathcal{W}_2, \mathcal{V}_2)$ iff $\mathcal{W}_1 \supset \mathcal{W}_2$ and $\mathcal{V}_1 \supset \mathcal{V}_2$, and $\mathcal{V}_1 \to \mathcal{V}_2$ iff $\mathcal{V}_1 \supset \mathcal{V}_2$.
To give a point-finite closure-preserving closed covering $\mathcal{U}$ of X is to give a map $X \longrightarrow \{\mathcal{V} : \varnothing \ne \mathcal{V} \subset \mathcal{U}, |\mathcal{V}| < \omega\}$ where the topology is defined by the order $\mathcal{V}_1 \to \mathcal{V}_2$ iff $\mathcal{V}_1 \subset \mathcal{V}_2$(sic!). An open covering $\mathcal{U}$ has a point-finite closure-preserving refinement $\mathcal{V}$ iff $\varnothing \to X \rightthreetimes \{(\mathcal{W}, \mathcal{V}) : \varnothing \ne \mathcal{W} \subset \mathcal{V} \subset \mathcal{U}, |\mathcal{W}| < \omega\} \to \{\mathcal{V} : \varnothing \ne \mathcal{V} \subset \mathcal{U}\}$ where the topology on the domain is defined by order $(\mathcal{W}_1, \mathcal{V}_2) \to (\mathcal{W}_2, \mathcal{V}_2)$ iff $\mathcal{W}_1 \subset \mathcal{W}_2$(sic!) and $\mathcal{V}_1 \supset \mathcal{V}_2$, on the target by the open subsets $\{\mathcal{V} \subset \mathcal{U} : U \in \mathcal{V}\}$, for $U \in \mathcal{U}$.

define a family of subsets of Y by $F_x := p^{-1}(g(x))$, $x \in X$. The map $p : Y \to B$ being surjective implies the subsets are non-empty, and being open implies the family is lower semi-continuous. A continuous choice function is the same as a lifting map $X \to B$. For this family of subsets, [**6**, Thm.1.2] above amounts to the following lifting property.

Theorem 3.1.2 ([**6**, Thm.1.2]). — *Let X be a paracompact Hausdorff space, and Y be a space admitting a complete metric.*

Let $i : A \hookrightarrow X$ be a closed inclusion with $\dim_X(X - A) \leq n + 1$. Let $p : Y \to B$ be an open surjective map with n-contractible uniformly locally n-contractible fibres. Then $A \xrightarrow{i} B \rightthreetimes Y \xrightarrow{p} B$, i.e.

$$\begin{tikzcd} A \arrow[r, "\forall"] \arrow[d, tail, "i"'] & Y \arrow[d, "p"] \\ X \arrow[r, "\forall"] \arrow[ur, dotted, "\exists"] & B \end{tikzcd}$$

Without the assumption that the fibres are n-contractible, we only get that there is an open $A \subset U \subset X$ and a map $\tilde{f}_U : U \to Y$ such that the diagram

$$\begin{tikzcd} & A \arrow[r, "f"] \arrow[d] \arrow[dl, dotted] & Y \arrow[d, "p"] \\ U \arrow[r, dotted] \arrow[urr, dotted, "\tilde{f}_U"] & X \arrow[r, "\phi_{|U}"] & B \end{tikzcd}$$

commutes.

We say (uniformly, locally) weakly contractible to mean (uniformly, locally) n-contractible for every $n \geq 0$. A map $Y \xrightarrow{p} B$ is called *soft for paracompact (normal, etc) pairs* iff for each subset A of a paracompact (normal, etc, resp.) X it holds $A \subset B \rightthreetimes Y \xrightarrow{p} B$. With this terminology, the theorem implies that an open surjective map with n-contractible uniformly locally n-contractible fibres from a completely metrisable space is necessarily soft for paracompact pairs of finite Lebesgue dimension.

3.2. Extending maps to compact spaces. — We explain in more detail the proof in §2.3 of the characterisation of compactness. The reader may find a verbose exposition focusing on logical ideas in [**19**, §2.2].

3.2.1. Compactness via ultrafilters by Bourbaki. — Item `d)` of the following characterisation of proper maps by Bourbaki [**14**] states almost a lifting property. Arguably, this suggests that the ideas/technique of category theory were present in [**14**], although not the notation or language of category theory.

Theorem 1. *Let $f: \mathrm{X} \to \mathrm{Y}$ be a continuous mapping. Then the following four statements are equivalent:*

a) *f is proper.*

b) *f is closed and $\overset{-1}{f}(y)$ is quasi-compact for each $y \in \mathrm{Y}$.*

c) *If $\mathfrak{F}$ is a filter on X and if $y \in \mathrm{Y}$ is a cluster point of $f(\mathfrak{F})$ then there is a cluster point x of $\mathfrak{F}$ such that $f(x) = y$.*

d) *If $\mathfrak{U}$ is an ultrafilter on X and if $y \in \mathrm{Y}$ is a limit point of the ultrafilter base $f(\mathfrak{U})$, then there is a limit point x of $\mathfrak{U}$ such that $f(x) = y$.*

Item d) expresses the following lifting property (almost): $|X| \longrightarrow |X| \sqcup_{\mathfrak{U}} \{\infty\} \rightthreetimes X \xrightarrow{f} Y$ where $|X|$ denotes the set of points of X equipped with discrete topology, and the topology on $|X| \sqcup_{\mathfrak{U}} \{\infty\}$ is such that $\mathfrak{U}$ is the neighbourhood filter of ∞, and the induced topology on subset $|X|$ is discrete [**14**, I§6.5, Def.5, Example].

3.2.2. Extending maps to compact Hausdorff spaces. — The theorem of Vulikh-Smirnov-Taimanov [**15**, 3.2.1,p.136] is stated in the language of lifting properties almost explicitly ("compact" below stands for "compact Hausdorff"):

3.2.1. THEOREM. *Let A be a dense subspace of a topological space X and f a continuous mapping of A to a compact space Y. The mapping f has a continuous extension over X if and only if for every pair B_1, B_2 of disjoint closed subsets of Y the inverse images $f^{-1}(B_1)$ and $f^{-1}(B_2)$ have disjoint closures in the space X.*

Let us transcribe this to the language/notation of finite topological spaces and lifting properties. We are given a dense subspace $A \xrightarrow{i} X$ of a topological space X and a continuous mapping $A \xrightarrow{f} Y$ of A to a [Hausdorff] compact space Y. The mapping f has a continuous extension over X means that the arrow $A \xrightarrow{f} Y$ factors via $A \xrightarrow{i} X$ (cf. Figure 2f). A pair $\boldsymbol{B_1}, \boldsymbol{B_2}$ of disjoint closed subsets of Y is an arrow $Y \longrightarrow \{\boldsymbol{B_1} \leftarrow O \rightarrow \boldsymbol{B_2}\}$ where $\{\boldsymbol{B_1} \leftarrow O \rightarrow \boldsymbol{B_2}\}$ is the space with one open point denoted by O and two closed points denoted by $\boldsymbol{B_1}$ and $\boldsymbol{B_2}$. To say the inverse images $\boldsymbol{f^{-1}(B_1)}$ and $\boldsymbol{f^{-1}(B_2)}$ have disjoint closures in the space X is to say that the composition $A \xrightarrow{f} Y \longrightarrow \{\boldsymbol{B_1} \leftarrow O \rightarrow \boldsymbol{B_2}\}$ factors as $A \xrightarrow{i} X \longrightarrow \{\boldsymbol{B_1} \leftarrow O \rightarrow \boldsymbol{B_2}\}$ (cf. Figure 2g).

Now we need to define the class of dense subspaces. A dense subspace is an injective map with dense image such that the topology on the domain is induced from the target. This suggests we try to define this class by taking left Quillen negations (orthogonals) of the simplest archetypal examples of a map whose image is not dense ($\{U\} \longrightarrow \{U \rightarrow U'\}$), a non-injective map ($\{x \leftrightarrow y\} \longrightarrow \{x = y\}$), and a map such that the topology on the domain is not induced from the target ($\{o \rightarrow c\} \longrightarrow \{o = c\}$).

Doing so leads to the following reformulation.

Theorem 3.2.1. — *Let Y be Hausdorff compact and let $A \xrightarrow{i} X$ satisfy (cf. Figure 2(ijk))*

(i) *(dense)* $A \xrightarrow{i} X \rightthreetimes \{U\} \longrightarrow \{U \rightarrow U'\}$

(ii) *(injective)* $A \xrightarrow{i} X \rightthreetimes \{x \leftrightarrow y\} \longrightarrow \{x = y\}$

(iii) *(induced topology)* $A \xrightarrow{i} X \rightthreetimes \{o \to c\} \longrightarrow \{o = c\}$

Then the properties of $A \xrightarrow{f} Y$ *defined by Figure 2(f) and Figure 2(g) are equivalent.*

This implies that, for Hausdorff compact Y, items 3.2.1(i-iii) and $A \xrightarrow{i} X \rightthreetimes \{\boldsymbol{B_1} \leftarrow O \searrow \boldsymbol{B_2}\} \longrightarrow \{\boldsymbol{B_1} = O = \boldsymbol{B_2}\}$ imply that $A \xrightarrow{i} X \rightthreetimes Y \longrightarrow \{\bullet\}$.

Further, note that if $X = A \sqcup \{\infty\}$ is obtained from A by adjoining a single closed non-open point, then

$$A \xrightarrow{i} X \rightthreetimes \{\boldsymbol{B_1} \leftarrow O \searrow \boldsymbol{B_2}\} \longrightarrow \{\boldsymbol{B_1} = O = \boldsymbol{B_2}\}$$

iff there exists an ultrafilter $\mathfrak{U}$ such that $A \xrightarrow{i} X$ is of form $A \longrightarrow A \sqcup_{\mathfrak{U}} \{\infty\}$.

This implies that maps of form $A \longrightarrow A \sqcup_{\mathfrak{U}} \{\infty\}$ are in P^l and, finally, that a Hausdorff space K is quasi-compact iff $K \longrightarrow \{\bullet\}$ is in P^{lr} where P consists of

$$\begin{array}{ll} \{\boldsymbol{B_1} \leftarrow O \to \boldsymbol{B_2}\} \longrightarrow \{\bullet\} & \{U\} \longrightarrow \{U \searrow U'\} \\ \{x \leftrightarrow y\} \longrightarrow \{x = y\} & \{o \searrow c\} \longrightarrow \{o = c\} \end{array}$$

3.2.3. A logical point of view: the simplest counterexample negated three times.— We took a (the?) simplest possible non-proper map, took Quillen negation thrice (although once passing to the subclass of finite spaces), and got (almost?) the definition of a proper map.

Let us explicitly state the conjecture.

Conjecture $((\{\{o\} \longrightarrow \{o \to c\}\}^r_{<5})^{lr})$. — *In the category of topological spaces, the following Quillen orthogonal (negation) defines the class of proper maps:*

$$(\{\{o\} \longrightarrow \{o \to c\}\}^r_{<5})^{lr}$$

3.3. Appendix. Properties of the empty subspace of a singleton. — We give a list of properties of maps one can define starting with the simplest possible map $\varnothing \to \{o\}$. Note that the notion of connectivity, discreteness, and quotient arises in this way.

[**2**] gives a longer list of notions one can obtain in this way starting from more complicated maps of finite topological spaces, of up to 7 points. Note compactness arises in this way, and also contractible, as we saw above.

We apologize for likely misprints(=mistakes) in this unproofread lemma.

Lemma 3.3.1. — *In the category of (all) topological spaces,*

r=rrl: $(\varnothing \longrightarrow \{o\})^r$ *is the class of surjections*

l: $(\varnothing \longrightarrow \{o\})^l$ *is the class of maps* $A \longrightarrow B$ *where* $A \neq \varnothing$ *or* $A \xrightarrow{\mathrm{id}} B$

rr=rllrr: $(\varnothing \longrightarrow \{o\})^{rr} = \{\{x \leftrightarrow y \to c\} \longrightarrow \{x = y = c\}\}^l = \{\{x \leftrightarrow y \leftarrow c\} \longrightarrow \{x = y = c\}\}^l$ *is the class of subsets, i.e. injective maps* $A \hookrightarrow B$ *where the topology on* A *is induced from* B

rrr: $(\varnothing \longrightarrow \{o\})^{rrr}$ *is the class of maps* $X \to Y$ *such that* $A \subset B \rightthreetimes X \to Y$ *for every subset* A *of* B

lr: $(\varnothing \longrightarrow \{o\})^{lr}$ *is the class of maps* $\varnothing \longrightarrow B$, B *arbitrary, and* $A \xrightarrow{\mathrm{id}} B$

lrr=lrrrllr: $(\varnothing \longrightarrow \{o\})^{lrr}$ *is the class of maps* $A \longrightarrow B$ *which admit a section*

l: $(\varnothing \longrightarrow \{o\})^{l}$ *consists of maps* $f : A \longrightarrow B$ *such that either* $A \neq \varnothing$ *or* $A = B = \varnothing$

ll=rlll: $(\varnothing \longrightarrow \{o\})^{l}$ *consists of isomorphisms*

rl: $(\varnothing \longrightarrow \{o\})^{rl}$ *is the class of maps of form* $A \longrightarrow A \sqcup D$ *where* D *is discrete*

rll=lrrrlll: $(\varnothing \longrightarrow \{o\})^{rll}$ *is the class of maps* $A \to B$ *such that each non-empty closed and open subset of* B *intersects the image of* A*; for "nice" spaces this means that* $\pi_0(A) \to \pi_0(B)$ *is surjective.*

rllr: $(\varnothing \longrightarrow \{o\})^{rllr}$ *is the class of maps* $A \to B$ *such that* $\operatorname{Im} A$ *is the intersection of all open closed subsets containing it*

lrrrll: $(\varnothing \longrightarrow \{o\})^{lrrrll}$ *is the class of maps of form* $A \to A \sqcup B$ *where* $A \sqcup B$ *denotes the disconnected union of* A *and* B.

lrrr=lrrrrl=rllrr: $\{\varnothing \longrightarrow \{o\}\}^{lrrr}$ *is the class of injective maps, i.e. such that* $f(x) \neq f(y)$ *whenever* $x \neq y$*, equiv.* $\{a,b\} \to \{a{=}b\} \rightthreetimes f$

lrrrr: $\{\varnothing \longrightarrow \{o\}\}^{lrrrr}$ *is the class of "coquetients", i.e. surjective maps* $A \to B$ *where the topology on* A *is pulled back from* B

lrrrrr: $\{\varnothing \longrightarrow \{o\}\}^{lrrrrr}$ *is the class of maps* $A \to B$ *such that no fibre has indistinguishable points, i.e.* $\{a \leftrightarrow b\} \to \{a = b\} \rightthreetimes A \to B$

lrrrl: $\{\varnothing \longrightarrow \{o\}\}^{lrrrl}$ *is the class of* quotients*, i.e. the maps* $f : A \to B$ *such that a subset* $U \subset B$ *is open in* B *iff its preimage* $f^{-1}(U) \subset A$ *is open in* A.

Proof. — Each is an easy exercise in diagram chasing and point set topology. Below we give hints. In the notation in the proof below, we always implicitly consider the lifting property $A \to B \rightthreetimes X \to Y$.

r=rrl: each point, i.e. the image of $\{o\}$ in Y, has a preimage in X.

l: there no maps $A \to \varnothing$ for $A \neq$.

rr: $Y := B$, and $X := A \cup (B \setminus \Im A)$ where the topology on X is pulled back along the obvious map to B

rrr: $(\varnothing \longrightarrow \{o\})^{rrr}$ is the class of maps $X \to Y$ such that $A \subset B \rightthreetimes X \to Y$ for every subset A of B

lr: There are no maps to $\varnothing$ so these maps are in the class. lr- and l-class do not intersect, hence the domain cannot be non-empty unless it is an isomorphism (if $X \neq$, take $A := X$ and $B := Y$).

lrr: take $B := X$.

ll: take $X := A \sqcup \{o\}$, $Y := B \sqcup \{o\}$.

rl: $(\varnothing \longrightarrow \{o\})^{rl}$ is the class of maps of form $A \longrightarrow A \sqcup D$ where D is discrete take $Y := B$, $X := A \sqcup |B \setminus \Im A|$ where $|B \setminus \Im A|$ is equipped with discrete topology.

rll: This class is equal to $\{\{a\} \to \{a,b\}\}^{l}$. The preimage of b in B has to a closed and open subset of B and does not intersects the image of A. The map to $\{a,b\}$ fails to lift iff the preimage of b is non-empty.

rllr: take $X := A$ and Y to be the intersection of all open closed subsets of B containing $\operatorname{Im} A$

lrrr: Composition with a section is a lifting whenever the map is injective. To see it is injective, note that $\{a,b\} \to \{a = b\} \rightthreetimes X \to Y$ says that $f(a) \neq f(b)$ whenever $a \neq b$.

lrrl: denotes the disconnected union of A and B. Consider the following lifting properties. and that maps on the right all admit a section:

– $A \to B$ is injective iff $A \to B \rtimes \{a \leftrightarrow b\} \longrightarrow \{a = b\}$
– the topology on A is pulled back from B via $A \to B$ iff
$$A \to B \rtimes \{o \to c\} \longrightarrow \{o = c\}$$
– $A \to B$ is closed and the topology on A is pulled from B iff
$$A \to B \rtimes \{\{x \leftrightarrow y \to c\} \longrightarrow \{x \leftrightarrow y = c\}\}$$
– $A \to B$ is open and the topology on A is pulled from B iff
$$A \to B \rtimes \{\{x \leftrightarrow y \leftarrow o\} \longrightarrow \{x \leftrightarrow y = o\}\}$$

lrrrr: Consider $B := Y$ and $A := X \cup (Y \setminus \Im X)$ where the topology on X is pulled back from its image $\Im X$ in Y

lrrrrr: $\{\varnothing \longrightarrow \{o\}\}^{lrrrrr}$ is the class of maps $A \to B$ such that no fibre has indistinguishable points, i.e. $\{a \leftrightarrow b\} \to \{a = b\} \rtimes A \to B$ In $A \to B$, all points of a fibre are indistinguishable, hence the fibre would have to map to a single points. $\{a \leftrightarrow b\} \to \{a = b\} \rtimes X \to Y$ means no fibre has topologically distinct points.

lrrrl: $\{\varnothing \longrightarrow \{o\}\}^{lrrrl}$ is the class of *quotients*, i.e. the maps $f : A \to B$ such that a subset $U \subset B$ is open in B iff its preimage $f^{-1}(U) \subset A$ is open in A. this is the standard universal property of quotients with respect to injective maps.

□

We were unable to calculate `rrr*`.

In **lrrrl**, we apply Quillen negation 5 times and get a notion that is worthy of having a word introduced in a first year course. Can it be more than 5 ? I.e. can we apply Quillen negation > 5 times to something simple or natural, and still get a meaningful and/or well-known notion ?

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(a)
$$\begin{tikzcd} K \arrow[r, "\mathrm{id}"] \arrow[d] & K \arrow[d] \\ K \cup_{\mathcal{F}} \{\text{“x”}\} \arrow[r] \arrow[ur, dashed, "\text{“x”}\mapsto x"] & \{\bullet\} \end{tikzcd}$$

(b)
$$\begin{tikzcd} K \arrow[r] \arrow[d] & K \arrow[d] \\ K \cup_{\mathcal{F}} \{\text{“x”}\} \arrow[r] \arrow[ur, dashed] & \{\bullet\} \end{tikzcd}$$

(c)
$$\begin{tikzcd} A \arrow[r] \arrow[d] & A \arrow[d] \\ A \cup_{\mathcal{F}} \{\text{“x”}\} \arrow[r] \arrow[ur, dashed] & \{\bullet\} \end{tikzcd}$$

(d)
$$\begin{tikzcd} X \arrow[r, "\mathrm{id}"] \arrow[d] & X \arrow[d, "f"] \\ X \cup_{\mathfrak{U}} \{\text{“x”}\} \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

(e)
$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d, "g"] \\ A \cup_{\mathfrak{U}} \{\text{“x”}\} \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

(f)
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & Y \arrow[d, "g"] \\ X \arrow[r] \arrow[ur, dashed] & \{\bullet\} \end{tikzcd}$$

(g)
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & Y \arrow[r] & \{\boldsymbol{B_1} \leftarrow O \rightarrow \boldsymbol{B_2}\} \\ X \arrow[urr, dashed] & & \end{tikzcd}$$

(h)
$$\begin{tikzcd} A \arrow[r] \arrow[d] & \{\boldsymbol{B_1} \leftarrow O \rightarrow \boldsymbol{B_2}\} \arrow[d] \\ X \arrow[r] \arrow[ur, dashed] & \{\bullet\} \end{tikzcd}$$

(i)
$$\begin{tikzcd} A \arrow[r] \arrow[d] & \{U\} \arrow[d] \\ X \arrow[r] \arrow[ur, dashed] & \{U \searrow U'\} \end{tikzcd}$$

(j)
$$\begin{tikzcd} A \arrow[r] \arrow[d] & \{x \leftrightarrow y\} \arrow[d] \\ X \arrow[r] \arrow[ur, dashed] & \{x = y\} \end{tikzcd}$$

(k)
$$\begin{tikzcd} A \arrow[r] \arrow[d] & \{o \searrow c\} \arrow[d] \\ X \arrow[r] \arrow[ur, dashed] & \{o = c\} \end{tikzcd}$$

(l)
$$\begin{tikzcd} \{o\} \arrow[r] \arrow[d] & X \arrow[d] \\ \{o \searrow c\} \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

FIGURE 2. These are equivalent reformulations of quasi-compactness of spaces and its generalisation to maps, that of properness of maps. (a) the identity map $K \xrightarrow{\mathrm{id}} K$ factors as $K \longrightarrow K \cup_{\mathcal{F}} \{\infty\} \longrightarrow K$ (b) this is also equivalent to K being quasi-compact (we no longer require the arrow $K \longrightarrow K$ to be identity) (c) and in fact quasi-compact spaces are orthogonal to maps associated with ultrafilters (d) $X \xrightarrow{f} Y$ is proper, i.e. d) If $\mathfrak{U}$ is an ultrafilter on X and if $y \in Y$ is a limit point of the ultrafilter base $f(U)$, then there is a limit point x of $\mathfrak{U}$ such that $f(x) = y$. [Bourbaki, General Topology, I§10.2,Th.1(d)] (e) this is also equivalent to $X \xrightarrow{f} Y$ is proper, i.e. this holds for each ultrafilter $\mathfrak{U}$ on each space A (f) The mapping f has a continuous extension over X (h) for every pair B_1, B_2 of disjoint closed subsets of Y the inverse images $f^{-1}(B_1)$ and $f^{-1}(B_2)$ have disjoint closures in the space X (i) the image of A is dense in B (j) the map $A \longrightarrow B$ is injective (k) the topology on A is induced from B (l) for X and Y finite, this means that the map $X \longrightarrow Y$ is closed, or, equivalently, proper

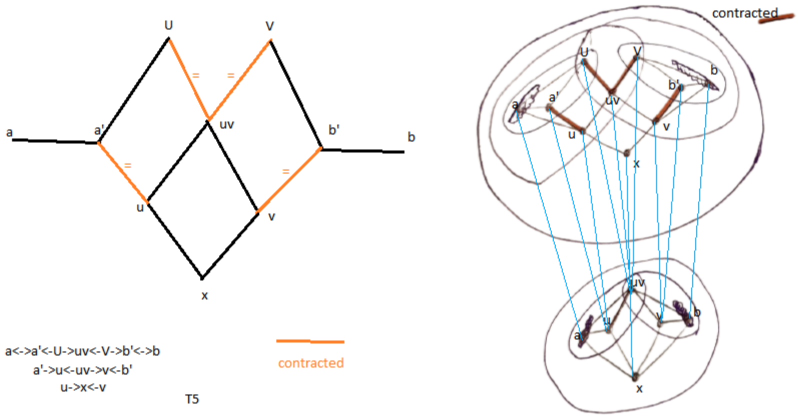

FIGURE 3. Separation Axiom T_5: Hereditary normal.
T_5 says that each subspace is normal (T_4), and can be expressed as follows: a space is hereditary normal iff whenever each of two disjoint subsets can be separated from the other by an open neighbourhood, they have disjoint open neighbourhoods. This figure pictures this as a lifting property of $\varnothing \to X$ with respect to

$$\{a \leftrightarrow a' \leftarrow U \to uv \leftarrow V \to b' \leftrightarrow b, a' \to u \to x \leftarrow v \leftarrow b', u \leftarrow uv \to v\}$$
$$\Downarrow$$
$$\{a \leftrightarrow a' \leftarrow U = uv = V \to b' \leftrightarrow b, a' = u \to x \leftarrow v = b', u \leftarrow uv \to v\}$$

In natural language, this parses as: two arbitrary disjoint subsets are the preimages of a and b, their neihbourhoods are the preimages of open neighbourhoods $\{a \leftrightarrow a' = u \leftarrow U = uv\}$ and $\{b \leftrightarrow b' = v \leftarrow V = uv\}$ of these points, and their disjoint neighbourhoods are the preimages of open neighbourhoods $\{a \leftrightarrow a' \leftarrow U\}$ and $\{b \leftrightarrow b' \leftarrow V\}$.
The preorder of the domain is pictured on the left, and on the right are decompositions of a space into open subsets induced by maps to the domain and the target.