Discerning the indiscernible
in *Higher Topos Theory* by Jacob Lurie
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The purpose of this short note is to observe that the same simplicial diagram defines the notion of an idempotent of an ∞-category in [**LurieHTT**, Def.4.4.5.4(i)], and a model-theoretic notion of an indiscernible sequence, or rather a complete Ehrenfeucht-Mostowski type. We also observe some similarities between the simplicial diagrams expressing the notion of lifting idempotents in an ∞-category [**LurieHTT**, Counterexample 4.4.5.19, Proposition 4.4.5.20(1)], and the notion of non-dividing in model theory.

We also show a simplicial diagram expressing the point of view behind a definition of a definable type by Hrushovski in [**HrBerk**, 1.1], cf. [**TZ**, Ex.8.8.3], namely that invariant and definable types are viewed as operations on formulas or types.

Keywords: simplicial sets; classification theory in model theory. *MSC*: 18N50, 03C45.

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In a verbose manner simplicial reformulations of basic notions in model theory are discussed in [**G**] where the reader may find further references and acknowledgements.

1. Simplicial reformulations of basic notions in model theory

We start by explaining the definition of idempotents in [**LurieHTT**, 4.4.5] important for us: an *idempotent* of a simplicial set $S_\bullet$ (recall that by [**LurieHTT**, Def.1.1.2] an ∞-category is a simplicial set) is a map from from the nerve of the category with one object and one non-identity idempotent morphism:

$$\left({}^{\mathrm{id}}\circlearrowright \bullet \circlearrowleft^{e=e\circ e}\right)_\bullet \longrightarrow S_\bullet$$

Then we proceed to rephrase simplicially the notion of a *type of a first-order theory* in model theory: an n-type over a set A is an $n-1$-simplex of the simplicial quotient of a (large enough) model $\mathcal{M} \supset A$ by the action of its automorphism group fixing A point-wise:

$$M_\bullet/\mathrm{Aut}^T(M)\ ,\ M_\bullet/\mathrm{Aut}^T(M)\ (\bullet_1 \longrightarrow \ldots \longrightarrow \bullet_n) := M^n/\mathrm{Aut}^T(M/A)$$

To clarify the notion of the simplicial quotient of a set by a group action, we remark that the simplicial quotient of a group acting on itself by multiplication is its simplicial classifying space

$$G_\bullet/G = \mathbb{B}_\bullet G = \left(\cdot \circlearrowleft^G\right)_\bullet$$

Finally, we note that a type over a set $A \subset M$ of parameters is a simplicial homotopy contracting A within $M_\bullet/\mathrm{Aut}^T(M)$, in a certain precise sense: namely, it is given by a simplicial diagram Eq. (1) defining contractability for singular simplicial sets associated with topological spaces.

We use this reformulation in the next section to introduce the notion of a type over a **space** of parameters. It allows us to reformulate the characterisation of non-dividing in terms of indiscernible sequences [**TZ**, 7.1.5] as saying that

– a type $p(x/b)$ does not divide iff it extends to a type over each indiscernible sequence containing b

i.e. for each indiscernible sequence I containing b, there is a type $p(x/I)$ over *parameter space* I such that $p(x/b) \subset p(x/I)$.

The notion of a type over a space of parameters also allows us to give a precise simplicial rendering to the point of view behind the definition of a definable type in [**HrBerk**, 1.1], cf. [**TZ**, Ex.8.8.3], namely that an invariant and definable types are viewed as operations on formulas or types.

1.1. Idempotents, simplicially.— Following [**LurieHTT**, Def.4.4.5.2,p.304], "let $\mathcal{R}$ denote the category consisting of two objects X and Y, with morphism sets given by

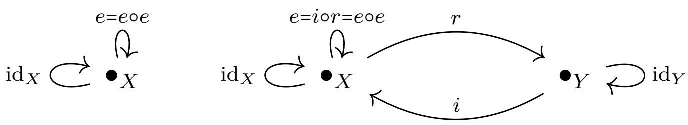
$$\mathrm{Hom}_{\mathcal{R}}(X,X) = \{\mathrm{id}_X, e\} \quad \mathrm{Hom}_{\mathcal{R}}(X,Y) = \{r\}$$
$$\mathrm{Hom}_{\mathcal{R}}(Y,X) = \{i\} \quad \mathrm{Hom}_{\mathcal{R}}(Y,Y) = \{\mathrm{id}_Y\}$$

and composition law determined by $r \circ i = \mathrm{id}_Y$ and $i \circ r = e$ (from which it follows that $r \circ e = r$, $e \circ i = i$, and $e \circ e = e$). We let $\mathrm{Idem}^+_\bullet$ denote the nerve of the category $\mathcal{R}$, and we let $\mathrm{Idem}_\bullet$ denote the full subcategory (i.e. the simplicial subset) of $\mathrm{Idem}^+_\bullet$ spanned by the object X." The ∞-category $\mathrm{Idem}_\bullet$ is the nerve of the full subcategory $\mathcal{I}$ of $\mathcal{R}$ spanned by the object X.

[**LurieHTT**, Remark 4.4.5.3,p.304] gives a combinatorial characterisation of the simplicial set $\mathrm{Idem}_\bullet$ in [**LurieHTT**, Remark 4.4.5.3,p.304], which we detail in Proposition 1.1:

> **Remark 4.4.5.3.** The ∞-category Idem can be described as the classifying space of the two-element monoid $\{\mathrm{id}, e\}$, whose multiplication is given by $e^2 = e$. It contains exactly one nondegenerate simplex of each dimension, and the collection of nondegenerate simplices of Idem is closed under face maps. In fact, Idem is characterized up to (unique) isomorphism by these properties.

The following pictures represents the categories $\mathcal{I} \subset \mathcal{R}$:



category $\mathcal{I}$ category $\mathcal{R}$

The definition above and [**LurieHTT**, Remark 4.4.5.3,p.304] mean that

$$\mathrm{Idem}_\bullet,\ \ \mathrm{Idem}^+_\bullet : \Delta^{\mathrm{op}} \longrightarrow \mathrm{Sets}$$
$$\mathrm{Idem}_\bullet(n^{\leqslant}) := \mathrm{Functor}\,(n^{\leqslant}, \mathcal{I}), \quad n > 0$$
$$\mathrm{Idem}^+_\bullet(n^{\leqslant}) := \mathrm{Functor}\,(n^{\leqslant}, \mathcal{R}), \quad n > 0$$

where $\mathcal{I} \subset \mathcal{R}$ denotes the full subcategory of $\mathcal{R}$ consisting of the object X and morphisms $\mathrm{id}_X, e : X \longrightarrow X$, and $n^{\leqslant}$ denotes the finite linear order with n elements or, equivalently, the category $\bullet_1 \longrightarrow \bullet_2 \longrightarrow \dots \longrightarrow \bullet_n$ with n objects and $n-1$ non-identity morphisms.

Let us rewrite these equations in a more graphic notation:

$$\mathrm{Idem}_\bullet,\ \ \mathrm{Idem}^+_\bullet : \Delta^{\mathrm{op}} \longrightarrow \mathrm{Sets}$$

$$\mathrm{Idem}_\bullet(\bullet_1 \to \dots \to \bullet_n) := \mathrm{Functor}\left(\bullet_1 \to \dots \to \bullet_n, {}^{\mathrm{id}}\circlearrowleft \bullet \circlearrowright^{e=e\circ e}\right), \quad n > 0$$

$$\mathrm{Idem}^+_\bullet(\bullet_1 \to \dots \to \bullet_n) := \mathrm{Functor}\left(\bullet_1 \to \dots \to \bullet_n, \begin{matrix} {}^{\mathrm{id}}\circlearrowleft \\ \bullet_c \\ {}_{e=i\circ r}\circlearrowright \end{matrix} \underset{i}{\overset{r}{\rightleftarrows}} \bullet_b \circlearrowleft^{\mathrm{id}}\right), \quad n > 0$$

1.2. Idempotents, combinatorially. — The simplicial set $\mathrm{Idem}_\bullet$ can be viewed as a "simplicial quotient" $\mathbb{Q}^{\leqslant}_\bullet/\mathrm{Aut}^{<}(\mathbb{Q})$ of the countable dense linear order by its order preserving automorphisms. Model theoretically this is almost the same as to say that it is the simplicial set of quantifier-free types of the theory of linear order, except that we consider only types of tuples $(q_1, \dots, q_n)$ where $q_1 \leqslant \dots \leqslant q_n$. This is seen in the explicit description of $\mathrm{Idem}_\bullet$ given in the proposition below.

Proposition 1.1. — *In each dimension $n \geqslant 0$ there is a unique non-degenerate simplex in $\mathrm{Idem}_\bullet((n+1)^{\leqslant})$ of dimension n. Moreover, each face of any non-degenerate simplex is also non-degenerate.*

There are three equivalent descriptions of an n-simplex of $\mathrm{Idem}_\bullet$:

(1) An n-simplex $s \in \mathrm{Idem}_\bullet((n+1)^{\leqslant})$ is a chain of form

$$x_0 \underset{\mathrm{id}}{\overset{e}{\longrightarrow}} x_1 \underset{\mathrm{id}}{\overset{e}{\longrightarrow}} \dots \underset{\mathrm{id}}{\overset{e}{\longrightarrow}} x_n$$

where each arrow is labelled by either the identity id or the idempotent e.

(2) An n-simplex $s \in \mathrm{Idem}_\bullet((n+1)^{\leqslant})$ is a quantifier-free type of a linearly ordered tuple in the language of order relation $<$, i.e. is form

$$x_0 \underset{=}{\overset{<}{\to}} x_1 \underset{=}{\overset{<}{\to}} \dots \underset{=}{\overset{<}{\to}} x_n$$

where each arrow is replaced by either the equality $=$ or $<$.

(2) An n-simplex $s \in \mathrm{Idem}_\bullet((n+1)^{\leqslant})$ is an orbit of an increasing tuple $q_0 \leqslant q_1 \leqslant \dots \leqslant q_n$ under the diagonal action of the group of order-preserving automorphisms $\mathrm{Aut}^{<}(\mathbb{Q})$ of the countable dense linear order $(\mathbb{Q}, <)$

$$\sigma : \mathbb{Q}^{n+1} \longrightarrow \mathbb{Q}^{n+1}, (q_0, \dots, q_n) \longmapsto (\sigma(q_0), \dots, \sigma(q_n))$$

Proof. — The first part is stated in [**LurieHTT**, Remark 4.4.5.3]. Items (1)-(3) are immediate by the definition. □

1.3. Idempotents, model theoretically. — A model theoretic description is that $\mathrm{Idem}_\bullet(n^{\leqslant})$ is the set of quantifier-free n-types of the non-decreasing n-tuples $x_1 \leqslant \dots \leqslant x_n$ in the language of linear order $\leqslant$. From this point of view it would be natural to denote $\mathrm{Idem}_\bullet$ by either $\mathbb{Q}^{\leqslant}_\bullet/\mathrm{Aut}_{<}(\mathbb{Q})$ or $\mathbb{Q}^{\leqslant}_\bullet/\approx^{\mathrm{qf}}$ reminding of the space of quantifier-free types of linearly ordered tuples in the theory of a dense linear order.

1.4. Types as orbits, for non-model theorist. — The group $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$ of field automorphisms of the field $\mathbb{C}$ of complex numbers is huge and unmanageable, yet the orbits of its action on say $\mathbb{C}\times\ldots\times\mathbb{C}$ admit a very useful syntactic description (namely, as irreducible subvarieties: the orbit of a point is the set of generic points of the least irreducible variety containing it and defined over $\mathbb{Q}$). Similarly, in model theory a *type* is a syntactic notion which in a good situation is equivalent to the the notion of an orbit of the automorphism group $\mathrm{Aut}_L(M)$ acting diagonally on Cartesian powers $M \times M \times \ldots \times M$ of a set M. The automorphism group $\mathrm{Aut}_L(M)$ consists of self-bijections of set M fixing set-wise several distinguished subsets (*predicates*) $L = \{P_i\}_i$ of Cartesian powers of M, where $P_i \subset M^{n_i}$, $n_i \in \mathbb{N}$.

1.5. The simplicial quotient by a group action. — Let G be a group acting on a set M. Then the group G acts naturally on the simplicial set $M_\bullet$, $M_\bullet(n^\leqslant) := M^n$, by the diagonal action $g(a_1,\ldots,a_n) := (ga_1,\ldots,ga_n)$ for $g \in G$, $(a_1,\ldots,a_n) \in M^n$. Define *the simplicial quotient* $M_\bullet/G$ by $M_\bullet/G\,(n^\leqslant) := M^n/G$ for $n \geqslant 1$. Note that the simplicial classifying space $\mathbb{B}G_\bullet$ of a group G is the simplicial quotient of the group acting on itself by multiplication, and is the nerve of the category with one object and its morphism set being G:

$$G_\bullet/G = \mathbb{B}_\bullet G = \left(\cdot \circlearrowleft^{G}\right)_\bullet$$

1.6. The simplicial classifying space $\mathbb{B}_\bullet T$ of a first order theory T. — Let $\mathcal{M}$ be a monster model of a first order theory T, and let $\mathrm{Aut}(\mathcal{M})$ be its automorphism group. Recall that $\mathrm{Aut}(\mathcal{M})$ is defined to be the group of self-bijections of M fixing point-wise certain distinguished subsets of finite Cartesian powers of M forming the *language* of theory T.

Define *the simplicial classifying space* $\mathbb{B}_\bullet T : \Delta^{\mathrm{op}} \longrightarrow \mathrm{Sets}$ of a first order theory T to be the simplicial quotient $\mathcal{M}_\bullet/\mathrm{Aut}(\mathcal{M})$. Whenever $\mathcal{M}$ is sufficiently saturated, the quotient $\mathbb{B}_\bullet T(n^\leqslant) = \mathcal{M}^n/\mathrm{Aut}(\mathcal{M})$ is the set of n*-types* of the theory T, and thus does not depend on the choice of a sufficiently saturated model $\mathcal{M}$. For this reason we also call $\mathbb{B}_\bullet T$ as *the simplicial space of types* of the theory T.

1.7. Types, simplicially. — For a set A let $|A|_\bullet$ denote the simplicial set represented by the set A:

$$|A|_\bullet : \Delta^{\mathrm{op}} \longrightarrow \mathrm{Sets}, \quad |A|_\bullet(n^\leqslant) := \mathrm{Hom}_{\mathrm{Sets}}(n, A) = A^n$$

An inclusion $A \subset M$ into a model of T defines a map

$$|A|_\bullet \longrightarrow M_\bullet/\mathrm{Aut}^T(M)\ , \qquad (a_1,\ldots,a_n) \longmapsto \mathrm{tp}(a_1,\ldots,a_n) \in M_\bullet/\mathrm{Aut}^T(M)\ (n^\leqslant)$$

of simplicial sets, which sometimes in model theory is called *the diagram of* A. The inclusion $A \subset M$ also defines the simplicial sets of types $M_\bullet/\mathrm{Aut}^T(M)\ (A)$ over A in the obvious way, i.e. $M_\bullet/\mathrm{Aut}^T(M)\ (A)(n^\leqslant) := \mathcal{M}^n/\mathrm{Aut}(\mathcal{M}/A)$ where $\mathrm{Aut}(\mathcal{M}/A)$ consists of automorphism of $\mathcal{M}$ fixing A set-wise.

Let $[+1] : \Delta^{\mathrm{op}} \longrightarrow \Delta^{\mathrm{op}}, n^\leqslant \longmapsto (n+1)^\leqslant$ denote the decalage (shift) endomorphism of the category Δ^{op} of finite non-empty linear orders. In a verbose manner the reformulations below are explained in [**G**].

Proposition 1.2. — *Let* $M_\bullet/\mathrm{Aut}^T(M)$ *be the simplicial set of types of a theory* T *defined as above. Let* $A \subset M$ *be a non-empty small subset of a sufficiently saturated model* M *of the theory* T. *The following data are the same.*

(i) a 1-type $p(x/A) \in M/\mathrm{Aut}^T(M)_1\,(A)$, *or, same in other notation, a 1-type* $p(x/A) \in M_\bullet/\mathrm{Aut}^T(M)\;(A)(1^{\leqslant})$
(ii) an orbit $p \in M/\mathrm{Aut}(M/A)$ *of the subgroup* $\mathrm{Aut}(M/A)$ *fixing A point-wise*
(iii) a commutative triangle

(1)
$$\begin{array}{ccc} & & M_\bullet/\mathrm{Aut}^T(M) \circ [+1] \\ & \nearrow^{p_\bullet(x/A)} & \downarrow{\scriptstyle pr_{1,2,\dots}} \\ |A|_\bullet & \xrightarrow{\quad A\subset M \quad} & M_\bullet/\mathrm{Aut}^T(M) \end{array}$$

where the map $|A|_\bullet \longrightarrow M_\bullet/\mathrm{Aut}^T(M)$ *is induced by the inclusion* $A \subset M$.

Proof. — Indeed, to give a map $|A|_\bullet \longrightarrow M_\bullet/\mathrm{Aut}^T(M) \circ [+1]$ is the same as to specify a $n+1$-type $p_{a_1,\dots,a_n}(x, a_1, \dots, a_n)$ for each tuple $a_1, .., a_n \in A$ for each $n \geqslant 1$ satisfying certain conditions. The simplicial set $|A|_\bullet$ is connected and therefore by functoriality this family of types is directed and therefore consistent; thus the union of these types is a 1-type over A. A detailed proof is in [**G**, 2.1-2]. □

Proposition 1.3. — *Let* $M_\bullet/\mathrm{Aut}^T(M)$ *be the simplicial set of types of a theory* T *defined as above. The following data are the same.*
(i) a 2-type $p(x,y) \in M/\mathrm{Aut}^T(M)_2$, *or, same in other notation, a 2-type* $p(x,y) \in M_\bullet/\mathrm{Aut}^T(M)\;(1^{\leqslant})$
(ii) an orbit $p \in M \times M/\mathrm{Aut}(M/A)$ *of the subgroup* $\mathrm{Aut}(M/A)$ *fixing A point-wise*
(iii) a map $p_\bullet : \Delta^0_\bullet \longrightarrow M_\bullet/\mathrm{Aut}^T(M) \circ [+1]$, *or, same in other notation, a map* $p_\bullet : 1^{\leqslant}_\bullet \longrightarrow M_\bullet/\mathrm{Aut}^T(M) \circ [+1]$.
(iv) a map $p_\bullet : \Delta^1_\bullet \longrightarrow M_\bullet/\mathrm{Aut}^T(M)$, *or, in other notation, a map* $p_\bullet : 2^{\leqslant}_\bullet \longrightarrow M_\bullet/\mathrm{Aut}^T(M)$.

Proof. — (i)⇔(iv): Indeed, to give a map $\Delta^{k+1}_\bullet \longrightarrow M_\bullet/\mathrm{Aut}^T(M)$ is the same as to give a simplex $s \in M/\mathrm{Aut}^T(M)_k$.
(i)⇔(ii)⇔(iii): Essentially this is Proposition 1.2 for $A := \{b\}$ a singleton. □

In (ii), $\mathrm{tp}(y) \subset p(x,y)$ is represented by the map $\Delta^0_\bullet \xrightarrow{p_\bullet} M_\bullet/\mathrm{Aut}^T(M) \circ [+1] \longrightarrow M_\bullet/\mathrm{Aut}^T(M)$ "forgetting the variable x".

2. Types over *spaces* over parameters

Notice that the simplicial reformulation suggests a definition of a type over a *space* of parameters, and not just a set with no structure; the simplicial intuition is that a type over a space of parameters is a homotopy contracting the space A.

Considering a type over the "space" $M_\bullet/\mathrm{Aut}^T(M)$ itself leads to a well-known notion of a *global type invariant over the empty set*, see Proposition 2.3, which, from the simplicial point of view is a section $M_\bullet/\mathrm{Aut}^T(M) \longrightarrow M_\bullet/\mathrm{Aut}^T(M) \circ [+1]$ of the obvious projection $M_\bullet/\mathrm{Aut}^T(M) \circ [+1] \longrightarrow M_\bullet/\mathrm{Aut}^T(M)$ forgetting the 0-th face. In a sense this is a rendering precisely of the point of view on invariant and definable types, as certain operations on formulas or types, whereas the original definition involved a choice of model and picking out the type out of all types over that model by some property.[1]

A type *over an indiscernible sequence* is a (usual) type over the set of its elements containing the statement saying that the sequence remains indiscernible over the variable.

2.1. Ehrenfeucht-Mostowski types as types over *an indiscernible* sequence. — Now consider $\mathrm{Idem}_\bullet$ instead of $|A|_\bullet$ as "the space of parameters" in Proposition 1.3. We shall see that a 1-type *over an indiscernible sequence* (viewed as a space) is a 1-type implying that the sequence remains indiscernible over the variable.

Note that [**LurieHTT**, Def.4.4.5.4(1),p.304] calls a simplicial map $\mathrm{Idem}_\bullet \longrightarrow \mathcal{C}_\bullet$ an *idempotent of an ∞-category $\mathcal{C}_\bullet$*. However, note that usually simplicial sets of types associated with first-order theories are not ∞-categories, as they do not satisfy the lifting properties required in [**LurieHTT**, Def.1.1.2.4]. We shall discuss this in more detail in §3.

Proposition 2.1. — *Let $M_\bullet/\mathrm{Aut}^T(M)$ be as above.*
(i) To give a map $\left({}^{\mathrm{id}}\circlearrowright \bullet \circlearrowleft^{e=e\circ e}\right)_\bullet = \mathrm{Idem}_\bullet \longrightarrow M_\bullet/\mathrm{Aut}^T(M)$ is the same as to give the complete Ehrenfeucht-Mostowski type [**TZ**, Def.5.1.2,p.64] *of a sequence* indiscernible over the empty set.
(ii) To give a type $p(x/I)$ over an indiscernible sequence $I \subset M$ saying that I remains indiscernible over x*, is the same as to give a commutative triangle*

$$\begin{array}{ccc} I^{\leqslant} & \xrightarrow{\quad p_\bullet(x/I) \quad} & M_\bullet/\mathrm{Aut}^T(M) \circ [+1] \\ \big\downarrow{\scriptstyle \mathrm{tp}_\bullet} & \nearrow{\scriptstyle p_\bullet(x/I)} & \big\downarrow{\scriptstyle pr_{1,2,\ldots}} \\ \left({}^{\mathrm{id}}\circlearrowright \bullet \circlearrowleft^{e=e\circ e}\right)_\bullet = \mathrm{Idem}_\bullet & \xrightarrow{\quad I\subset M \quad} & M_\bullet/\mathrm{Aut}^T(M) \end{array}$$

where the map $\mathrm{Idem}_\bullet \longrightarrow M_\bullet/\mathrm{Aut}^T(M)$ corresponds to the complete Ehrenfeucht-Mostowski type of the indiscernible sequence I.

Proof. — (i) Indeed, $\mathrm{Idem}_\bullet$ has a unique non-degenerate simplex of each dimension, and thus such a map specifies a unique type $p_n(x_1, ..., x_n)$ for each n. In each dimension each face of the non-degenerate simplex is also non-degenerate. The faces of the non-degenerate simplex $p_n(x_1, ..., x_n)$ are the subtypes $p_m(x_{i_1}, .., x_{i_m})$ for $1 \leqslant i_1 \leqslant ... \leqslant i_m \leqslant n$, and therefore are the same. This is the defining property of the complete Ehrenfeucht-Mostowski type of an indiscernible sequence [**TZ**, Def.5.1,2,p.64]. (ii) Follows from similar considerations. □

2.2. Non-dividing as extending types over indiscernible sequences or as lifting idempotents of an ∞-category. — The terminology of *a type over a space of parameters* and of *idempotent of an ∞-category* [**LurieHTT**, Def.4.4.5.4(1), Proposition 4.4.5.20(1)] lets one reformulate the notion of non-dividing of types in model theory in three ways appealing to different intuitions but expressing the same mathematical fact (namely, each of them is a way to rephrase the characterisation of non-dividing in terms of indiscernible sequences [**TZ**, Cor.7.1.5]).

Namely, the following are equivalent:

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[1] We thank Ehud Hrushovski for explaining this point of view implicit in the definition of definable types in [**HrBerk**, 1.1].

(1) a type $p(x/b)$ does not divide over the empty set.
(2) the type $p(x/b)$ is such that for each indiscernible I sequence containing b there is a realisation $a \vDash p(x/b)$ of the type $p(x/b)$ such that I is indiscernible over a [**TZ**, Cor.7.1.5].
(3) a type $p(x/b)$ extends to a type over any indiscernible sequence containing b.
(4) the map $\Delta^0 \xrightarrow{p(x/b)} M_\bullet/\mathrm{Aut}^T(M) \circ [+1]$ is such that any idempotent $\mathrm{Idem}_\bullet \longrightarrow M_\bullet/\mathrm{Aut}^T(M)$ consistent with it can be lifted to an idempotent of $M_\bullet/\mathrm{Aut}^T(M) \circ [+1]$ consistent with it, see Eq. (2).

However, note that $M_\bullet/\mathrm{Aut}^T(M) \circ [+1]$ is usually not an ∞-category as this simplicial set does not satisfy the lifting properties required in [**LurieHTT**, Def.1.1.2]. We discuss the reformulation (3) in detail in §3.

Proposition 2.2. — *Let* $M_\bullet/\mathrm{Aut}^T(M)$ *be the simplicial set of types of a first order theory* T.
(i) A map $p_\bullet : \left({}^{\mathrm{id}}\circlearrowleft \bullet \circlearrowright^{e=e\circ e}\right)_\bullet = \mathrm{Idem}_\bullet \longrightarrow M_\bullet/\mathrm{Aut}^T(M)$ *of simplicial sets is the same as a complete Ehrenfeucht-Mostowski type of an indiscernible sequence* $p(x_1, ..., x_n, ...)$.
(ii) A type $p(x, b)$ does not divide over the empty set *iff*

$$\begin{array}{ccc} \{\bullet\}_\bullet = \Delta^0_\bullet & \xrightarrow{\quad p_\bullet(x,b) \quad} & M_\bullet/\mathrm{Aut}^T(M) \circ [+1] \\ \downarrow & \nearrow \exists & \downarrow \\ \left({}^{\mathrm{id}}\circlearrowleft \bullet \circlearrowright^{e=e\circ e}\right)_\bullet = \mathrm{Idem}_\bullet & \xrightarrow{\quad \forall \quad} & M_\bullet/\mathrm{Aut}^T(M) \end{array} \tag{2}$$

Proof. — (i) The unique type $p_n(x_1, .., x_n)$ of an n-tuple in the indiscernible sequence $(x_i)_{i\in\omega}$ is the image of the unique non-degenerate simplex in $\mathrm{Idem}_\bullet(n^\leqslant)$. As each k-face of the unique non-degenerate n-simplex also the unique non-degenerate k-simplex, all the sub-types of $p_n(x_1, ..., x_n)$ corresponding to k-subtuples of the n-tuple $x_1....x_n$, coincide. Thereby $p_n(x_1, ..., x_n)$, $n > 0$, form the type of an indiscernible sequence. (ii) This is equivalent to the characterisation of non-dividing in terms of indiscernible sequences [**TZ**, Cor.7.1.5]. □

2.3. An invariant global type as a type over the space $M_\bullet/\mathrm{Aut}^T(M)$ of types of the theory. — In the proposition above we consider Eq. (1) with "the space of parameters" $M_\bullet/\mathrm{Aut}^T(M)$ instead of $|A|_\bullet$. In terminology of [**?**, (0.12),p.64], the following data are the same:

- a homotopy equivalence with a constant complex
- a split augumented complex (equipeed with a distinguished splitting)
- a contractible complex with a distinguished contraction

Using this we interpret the tensor product of types as a composition of simplicial homotopies, and then give a somewhat messy simplicial rendering of the construction of a Morley sequence of an invariant type. In §4.3 we conjecturally reformulate the notion of q-dividing for an invariant type q as a compatibility condition on two simplicial maps.

2.3.1. An invariant type is a section $M_\bullet/\mathrm{Aut}^T(M) \longrightarrow M_\bullet/\mathrm{Aut}^T(M) \circ [+1]$ *of* $M_\bullet/\mathrm{Aut}^T(M) \circ [+1] \longrightarrow M_\bullet/\mathrm{Aut}^T(M)$. — First we reformulate the definition of an invariant type.

Proposition 2.3. — *Let* $M_\bullet/\mathrm{Aut}^T(M)$ *be as above. The following data are the same.*
(i) a global type p *invariant over the empty set*
(ii) a section $p_\bullet : M_\bullet/\mathrm{Aut}^T(M) \longrightarrow M_\bullet/\mathrm{Aut}^T(M) \circ [+1]$ *of the obvious projection* $M_\bullet/\mathrm{Aut}^T(M) \circ [+1] \longrightarrow M_\bullet/\mathrm{Aut}^T(M)$.
(iii) the section $p_\bullet : M_\bullet/\mathrm{Aut}^T(M) \longrightarrow M_\bullet/\mathrm{Aut}^T(M) \circ [+1]$ *is continuous in the Stone topology iff the type* p *is definable over the empty set.*

Proof. — This is the content of [**TZ**, Exercise 8.3.3], cf. the definition of definable type in [**HrBerk**, 1.1,p.22]. See [**G**, §1.2,2.1-2] for a verbose explanation. □

2.3.2. The tensor product of invariant types as composition of homotopies. — For global types $p, q : M_\bullet/\mathrm{Aut}^T(M) \longrightarrow M_\bullet/\mathrm{Aut}^T(M) \circ [+1]$ invariant over the empty set the composition

$$p \otimes q : M_\bullet/\mathrm{Aut}^T(M) \xrightarrow{q_\bullet} M_\bullet/\mathrm{Aut}^T(M) \circ [+1] \xrightarrow{p_\bullet[+1]} M_\bullet/\mathrm{Aut}^T(M) \circ [+2]$$

is an invariant 2-type known in model theory as *the product* $p \otimes q$ *of invariant types* [**SimonNIP**, 2.2.1]. In fact, the tensor product $p \otimes q$ is defined for an invariant type $p : M_\bullet/\mathrm{Aut}^T(M) \longrightarrow M_\bullet/\mathrm{Aut}^T(M) \circ [+1]$ and an arbitrary type $q : A_\bullet \longrightarrow M_\bullet/\mathrm{Aut}^T(M) \circ [+1]$ over an arbitrary space $A_\bullet$:

$$p \otimes q : A_\bullet \xrightarrow{q_\bullet} M_\bullet/\mathrm{Aut}^T(M) \circ [+1] \xrightarrow{p_\bullet[+1]} M_\bullet/\mathrm{Aut}^T(M) \circ [+2]$$

2.3.3. An indiscernible Morley sequence, simplicially. — A *Morley sequence of an invariant type* $p(x)$ is a (necessarily indiscernible) sequence $(a_i)_{i\in\mathbb{N}}$ such that a_{n+1} realises the type $(p_\bullet(n^\leqslant))(\mathrm{tp}(a_1, ..., a_n))$. A Morley sequence determines a map $\mathrm{Idem}_\bullet \longrightarrow M_\bullet/\mathrm{Aut}^T(M)$ by sending the unique non-degenerate n-simplex of $\mathrm{Idem}_\bullet$ into the type $p^{\otimes n}$.

A more simplicial description is as follows. Consider the category of augmented simplicial sets. Then in this case $M_\bullet/\mathrm{Aut}^T(M)\ (\varnothing) = \{p_\varnothing\}$ is a singleton. Let $p_\varnothing \in M_\bullet/\mathrm{Aut}^T(M)\ (\varnothing)$ denote its only element. Consider the composition

$$p_\bullet^{\otimes n} : M_\bullet/\mathrm{Aut}^T(M) \xrightarrow{p_\bullet} M_\bullet/\mathrm{Aut}^T(M) \circ [+1] \xrightarrow{p_\bullet[+1]} ... \xrightarrow{p_\bullet[+n]} M_\bullet/\mathrm{Aut}^T(M) \circ [+n]$$

$$p_\varnothing^{\otimes n} : M/\mathrm{Aut}^T(M)_\varnothing \xrightarrow{p_{-1}=p_\varnothing} M/\mathrm{Aut}^T(M)_0 = M_\bullet/\mathrm{Aut}^T(M)\circ[+1] \xrightarrow{p_0=p_\varnothing[+1]} ... \xrightarrow{p_{n-1}=p_\varnothing[+n]} M_\bullet/\mathrm{Aut}^T(M)\circ[+n]$$

$$p_\varnothing \longmapsto p_n \in M_\bullet/\mathrm{Aut}^T(M) \circ [+n] \Longleftrightarrow \Delta_\bullet^n \xrightarrow{p_{n\bullet}} M_\bullet/\mathrm{Aut}^T(M)$$

and apply $p_\bullet^{\otimes n}(\varnothing^\leqslant) : M_\bullet/\mathrm{Aut}^T(M)\ (\varnothing^\leqslant) \longrightarrow M_\bullet/\mathrm{Aut}^T(M)\circ[+n](\varnothing^\leqslant) = M_\bullet/\mathrm{Aut}^T(M)\ (n^\leqslant)$ to $p_\varnothing \in M_\bullet/\mathrm{Aut}^T(M)\ (\varnothing^\leqslant)$. This gives for each $n \in \mathbb{N}$ an n-simplex $p_n \in M_\bullet/\mathrm{Aut}^T(M)\ ((n+1)^\leqslant)$, i.e. a map $p_{n\bullet} : \Delta_\bullet^{n+1} \longrightarrow M_\bullet/\mathrm{Aut}^T(M)$.

View here Δ^{n+1} as the non-degenerate $n+1$-simplex of $\mathrm{Idem}_\bullet$. Functoriality of $p_\bullet : M_\bullet/\mathrm{Aut}^T(M) \longrightarrow M_\bullet/\mathrm{Aut}^T(M) \circ [+1]$ implies that together this gives a well-defined simplicial map $\mathrm{Idem}_\bullet \longrightarrow M_\bullet/\mathrm{Aut}^T(M)$ by sending the unique non-degenerate n-simplex of $\mathrm{Idem}_\bullet$ into the type $p^{\otimes n}$.

2.4. Stable theory, simplicially. — An important class of well-understood first order theories, called *stable* theories, with behaviour reminiscent of algebraic geometry, admits a number of different characterisations [**TZ**, 5.2,Ch.8]. We rewrite simplicially two of them below.

Note that defining definable types requires us to equip simplicial sets with a structure of topological nature. This seems necessary: in [**Z1, Z2**] we give similar reformulations of several Shelah's dividing lines NOP, NSOP, NSOP$_i$, NTP, NTP$_i$, NATP, using a category of simpicial sets equipped with a structure of topological nature allowing to make precise the phrase "a property holds for all n-simplicies small enough" for each $n \geqslant 0$. In this category each of these properties can be expressed as Quillen lifting properties of $M_\bullet/\mathrm{Aut}^T(M) \longrightarrow \Delta^0$.

Proposition 2.4. — *Let $M_\bullet/\mathrm{Aut}^T(M)$ be the simplicial set of types of a first order theory T. Let $\mathrm{Sym}\,\mathbb{Q}$ be the group of permutations of the set $\mathbb{Q}$ of rational numbers. Let $\left({}^{\mathrm{id}}\circlearrowleft \bullet \circlearrowright^{e=e\circ e}\right)_\bullet = \mathrm{Idem}_\bullet = \mathbb{Q}^{\leqslant}_\bullet/\mathrm{Aut}^{\leqslant}(\mathbb{Q}) \longrightarrow |\mathbb{Q}|_\bullet/\mathrm{Sym}\,\mathbb{Q}$ be the obvious map induced by the inclusion $\mathbb{Q}^{\leqslant}_\bullet \hookrightarrow |\mathbb{Q}|_\bullet$.*

The following are equivalent:

(1) the theory T is stable.

(2) each indiscernible sequence is set indiscernible.

(3) for each $n \geqslant 0$ it holds

$$\begin{array}{ccc}
(\cdot_0 \longrightarrow \dots \longrightarrow \cdot_n)_\bullet \times \left({}^{\mathrm{id}}\circlearrowleft \bullet \circlearrowright^{e=e\circ e}\right)_\bullet = \Delta^n_\bullet \times \mathrm{Idem}_\bullet = \Delta^n_\bullet \times \mathbb{Q}^{\leqslant}_\bullet/\mathrm{Aut}^{\leqslant}(\mathbb{Q}) & \xrightarrow{\quad\forall\quad} & M_\bullet/\mathrm{Aut}^T(M) \\
\downarrow & \overset{\exists}{\dashrightarrow} & \downarrow \\
\Delta^n_\bullet \times |\mathbb{Q}|_\bullet/\mathrm{Sym}\,\mathbb{Q} = \mathbb{B}_\bullet \mathrm{Sym}\,\mathbb{Q} & \xrightarrow{\quad\forall\quad} & \{\bullet\}_\bullet = \Delta^0_\bullet
\end{array}$$

(4) each 1-type over the model M is definable over M

(5) in the diagram below there is a lifting continuous in the Stone topology

$$\begin{array}{ccc}
|M|_\bullet & \xrightarrow{\quad\forall\quad} & M_\bullet/\mathrm{Aut}^T(M) \circ [+1] \\
\downarrow & \overset{\exists \text{ continuous}}{\dashrightarrow} & \downarrow \\
M_\bullet/\mathrm{Aut}^T(M) & \xrightarrow{\quad\mathrm{id}\quad} & M_\bullet/\mathrm{Aut}^T(M)
\end{array} \tag{3}$$

Proof. — (1)$\Longleftrightarrow$(2)$\Longleftrightarrow$(4): this is standard [**TZ**, Lemma 9.1.1; Exercise 9.1.5; Theorem 8.3.2]. (4)$\Longleftrightarrow$(5): this is Proposition 2.3(iii). (2)$\Longleftrightarrow$(3): Recall that in sSets the direct product is left adjoint to the internal hom, i.e. we may say that a map $f : A \times B \longrightarrow C$ is the same as a map $A \longrightarrow \underline{\mathrm{Hom}}_{\mathrm{sSets}}(B, C)$. Thus by Proposition 2.2(i) the top horizontal arrow represents an indiscernible sequence of n-tuples. The left vertical arrow glues together simplicies corresponding to permutations of variables, and therefore the diagonal arrow says this indiscernible sequence is set indiscernible. □

3. Idempotents as indiscernibles

We explain in §3.1 that there exists a notion of *an idempotent of a simplicial set*, which specializes to the Lurie's "idempotent" when one restricts to ∞-categories, and specializes to a complete "Ehrenfeucht-Mostowski type" when one restricts to the simplicial set (space) of types of a theory. In §3.2 we explain that the notions of *lifting idempotents* [**LurieHTT**, Proposition 4.4.5.20(1)] and *non-dividing of types* in model theory [**TZ**, Cor.7.1.5] are defined by similar simplicial diagrams.

3.1. Idempotents as indiscernibles. — Recall that by [**LurieHTT**, Def.1.1.2.4,p.14] an ∞-category is a simplicial set with certain lifting properties. Note that the simplicial set of types of the theory of a triangle-free random graph fails these lifting properties, as was pointed out to me by Martin Bays. We quote [**LurieHTT**, Def.4.4.5.4,p.304]:

> **Definition 4.4.5.4.** Let $\mathcal{C}$ be an ∞-category.
>
> (1) An *idempotent* in $\mathcal{C}$ is a functor of ∞-categories $\mathrm{Idem} \to \mathcal{C}$. We will refer to $\mathrm{Fun}(\mathrm{Idem}, \mathcal{C})$ as the *∞-category of idempotents in* $\mathcal{C}$.
>
> (2) A *weak retraction diagram* in $\mathcal{C}$ is a map of simplicial sets $\mathrm{Ret} \to \mathcal{C}$. We will refer to $\mathrm{Fun}(\mathrm{Ret}, \mathcal{C})$ as the *∞-category of weak retraction diagrams in* $\mathcal{C}$.
>
> (3) A *strong retraction diagram* in $\mathcal{C}$ is a functor of ∞-categories $\mathrm{Idem}^+ \to \mathcal{C}$. We will refer to $\mathrm{Fun}(\mathrm{Idem}^+, \mathcal{C})$ as the *∞-category of strong retraction diagrams in* $\mathcal{C}$.

Proposition 2.2 says that in model theory, an idempotent of the simplicial set $M_\bullet/\mathrm{Aut}^T(M)$ is the same as a complete Ehrenfeucht-Mostowski type on indiscernible sequence, and gives a characterisation of non-dividing using the category $\mathrm{Idem}_\bullet$ somewhat similar to [**LurieHTT**, Proposition 4.4.5.20(1),p.310], as we explain next.

3.2. Non-dividing and [**LurieHTT**, Counterexample 4.4.5.19, Proposition 4.4.5.20(1)]**?**— We now explain that [**LurieHTT**, Proposition 4.4.5.20(1)] mentions a property somewhat reminiscent of the simplicial diagram in Proposition 2.2(ii) defining non-dividing in model theory [**TZ**, Cor.7.1.5].

Recall that by [**LurieHTT**, Def.1.1.2.4,p.14] an ∞-category is a simplicial set with certain lifting properties. Recall that an *object* of a ∞-category $\mathcal{C}_\bullet$ is defined to be a 0-*simplex* $X \in \mathcal{C}_0 = \mathcal{C}_\bullet(1^<)$, or, equivalently, a map $\Delta^0_\bullet \longrightarrow \mathcal{C}_\bullet$ of simplicial sets. Recall that an *morphism* of a ∞-category $\mathcal{C}_\bullet$ is defined to be a 1-*simplex* $X \in \mathcal{C}_1 = \mathcal{C}_\bullet(2^<)$, or, equivalently, a map $\Delta^1_\bullet \longrightarrow \mathcal{C}_\bullet$ of simplicial sets. The ideology is that the composition of morphisms is well-defined only up to homotopy, and simplicies $\mathcal{C}_n, n > 1$, of higher dimension allow you to keep track of these homotopies needed to e.g. to make the composition associative.

Thus, "a morphism $e : X \longrightarrow X$ in an ∞-category $\mathcal{C}_\bullet$" is a 1-simplex $e' \in \mathcal{C}_1 = \mathcal{C}_\bullet(2^<)$ such that its both faces coincide. Equivalently, it is a map $e_\bullet : \Delta^1_\bullet \longrightarrow \mathcal{C}_\bullet$ such that for the both inclusion maps $\Delta^0_\bullet \longrightarrow \Delta^1_\bullet$ the compositions $\Delta^0_\bullet \longrightarrow \Delta^1_\bullet \xrightarrow{e_\bullet} \mathcal{C}_\bullet$ coincide.

For an ∞-category or a simplicial set $\mathcal{C}_\bullet$ [**LurieHTT**, 1.2.3] defines its homotopy category $h\mathcal{C}_\bullet$. We quote [**LurieHTT**, Proposition 4.4.5.20(1),p.310]:

> **Proposition 4.4.5.20.** *Let $\mathcal{C}$ be an ∞-category, let X be an object of $\mathcal{C}$, and let $e : X \to X$ be a morphism. The following conditions are equivalent:*
>
> (1) *The morphism e can be lifted to an idempotent in the ∞-category $\mathcal{C}$. That is, the map $\Delta^1 \to \mathcal{C}$ determined by e extends to a map* Idem $\to \mathcal{C}$, *where* Idem *is the ∞-category of Definition 4.4.5.2.*
>
> (2) *Let $\mathcal{E}$ denote the fundamental groupoid of the space $\mathrm{Map}_{\mathcal{C}}(X, X)$, so that composition of morphisms determines a composition product $\circ : \mathcal{E} \times \mathcal{E} \to \mathcal{E}$. Let $[e]$ denote the object of $\mathcal{E}$ corresponding to the morphism e. Then there exists an isomorphism $h : [e] \to [e] \circ [e]$ in $\mathcal{E}$ such that the diagram σ :*
>
> $$\begin{array}{ccc} & [e]\circ[e] & \\ {\scriptstyle h\circ \mathrm{id}_{[e]}} \swarrow & & \searrow {\scriptstyle \mathrm{id}_{[e]}\circ h} \\ ([e]\circ[e])\circ[e] & \longrightarrow & [e]\circ([e]\circ[e]) \end{array}$$
>
> *commutes up to homotopy.*

The words of item (1) describe the following diagram reminiscent of the diagram in Proposition 2.2(ii) defining non-dividing:

$$\begin{array}{ccc} \Delta^1_\bullet & \xrightarrow{e} & \mathcal{C}_\bullet \\ \downarrow & \nearrow_{\exists} & \\ \mathrm{Idem}_\bullet & & \end{array}$$

The words preceeding [**LurieHTT**, Counterexample 4.4.5.19] quoted below, describe the following diagram

> Let $\mathcal{C}$ be an ∞-category and let F : Idem $\to \mathcal{C}$ be an idempotent in $\mathcal{C}$. Then F determines an idempotent in the homotopy category h$\mathcal{C}$, which we can identify with an object $X \in \mathcal{C}$ and a morphism $e : X \to X$ such that e^2 is homotopic to e. Beware that the converse is false: in general, an idempotent in the homotopy category h$\mathcal{C}$ cannot necessarily be lifted to an idempotent in $\mathcal{C}$:

$$\mathcal{I} \xrightarrow{\quad e \quad} h\mathcal{C}_\bullet$$

lifts to

$$\mathrm{Idem}_\bullet \longrightarrow \mathcal{C}_\bullet$$

It is tempting to think that the following diagram is implicit in the words preceeding [**LurieHTT**, Counterexample 4.4.5.19] quoted above. However, I am not sure that the map $\mathcal{C}_\bullet \longrightarrow h\mathcal{C}_\bullet$ is well-defined. In this form it is somewhat more like the diagram of Proposition 2.2(ii) defining non-dividing:

$$\begin{array}{ccc} \Delta^1_\bullet & \xrightarrow{\quad e \quad} & \mathcal{C}_\bullet \\ \downarrow & \exists\text{: not necessarily} & \downarrow ?? \\ \mathrm{Idem}_\bullet & \xrightarrow{\quad \forall \quad} & h\mathcal{C}_\bullet \end{array}$$

3.3. Non-forking and $\mathrm{Idem}^+_\bullet$. — In the print edition Lurie used the category $\mathrm{Idem}_\bullet$. After some inaccuracies were pointed out by Gal Dor [**GalDor**], he replaced by the category $\mathrm{Idem}^+_\bullet$ which remembers more and captures the notion of a retraction. Thus we wonder what happens if we replace $\mathrm{Idem}_\bullet$ by $\mathrm{Idem}^+_\bullet$ in the simplicial reformulation of dividing in Eq. (2), and whether we get a better behaved model theoretic notion, e.g. non-forking. We are not able to do this. In Proposition we show that a category related to $\mathrm{Idem}^+_\bullet$ defines a property of types stronger than non-forking.

Let us say a couple of words about non-forking. In model theory, a key distinction between forking and dividing is that by a tautological argument each type $p(x/b)$ not forking over a set C, admits an extension $q(x/bD) \supset p(x/b)$ to an arbitrary set D of parameters such that $q(x/bD)$ does not fork over C. This fails for dividing, and in fact a type does not fork iff it admits an extension $q(x/bD) \supset p(x/b)$ to an arbitrary set D of parameters such that $q(x/bD)$ does not divide over C.

One wonders if this has to do with the inaccuracies in the first edition of [**LurieHTT**] dealing with $\mathrm{Idem}_\bullet$ rather than $\mathrm{Idem}^+_\bullet$.

Let the functor $\bullet_X \longrightarrow \mathcal{R}^n$ be the push-out of n copies of the functor

Let $\mathrm{Idem}^{n,+}_\bullet$ denote the nerve of the category $\mathcal{R}^n$, i.e. for any $i \in \mathbb{N}$

$$\mathrm{Idem}^{n,+}_\bullet(\bullet_1 \to \ldots \to \bullet_i) := \mathrm{Hom}_{Cats}(\bullet_1 \to \ldots \to \bullet_i, \mathcal{R}^n)$$

Proposition 3.1. — *Let* $M_\bullet/\mathrm{Aut}^T(M)$ *be the simplicial set of types of a first-order theory* T^{eq}*. A type* $p(x,b)$ does not fork over the empty set *if for each* $n > 0$ *either of the following equivalent conditions hold:*

(1) for any finitely many sequences $(c^i_s)_{i\in\omega}$, $0 < s < r$, *indiscernible over* b, *there is a realisation* $a \models p(x,b)$ *of* $p(x,b)$ *making each of* $(c^i_s)_{i\in\omega}$, $0 < s < r$, *indiscernible over* ab.

(2)

$$\begin{array}{ccc} \Delta^0_\bullet & \xrightarrow{\quad p_\bullet(x,b) \quad} & M_\bullet/\mathrm{Aut}^T(M) \circ [+1] \\ \downarrow & \exists & \downarrow \\ \mathrm{Idem}^{n,+}_\bullet & \xrightarrow{\quad \forall \quad} & M_\bullet/\mathrm{Aut}^T(M) \end{array}$$

Proof. — Here we use that in T^{eq} there is little distinction between a tuple and an element.

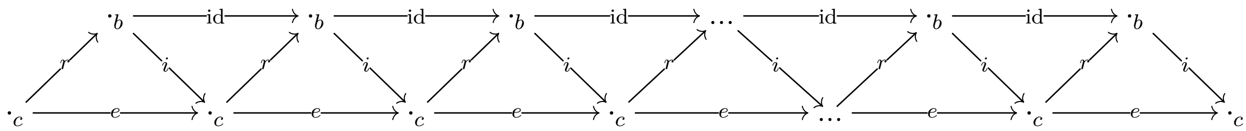
(1) $\Longrightarrow$ (non-forking): First note that for a sequence c^i_s, $i \in \omega$, indiscernible over b,

there is a sequence $b^i c_s^i$, $b^0 = b$, $i \in \omega$, indiscernible over the empty set, and in the proof we may consider the latter sequence.

By the definition [**TZ**, Def.7.1.7,p.111] of forking there are finitely many formulas $\phi_l(x,b,c_l)$, $0 < l < r$, such that $p(x,b) \vDash \bigvee_{0<l<r} \phi_l(x,b,c_l)$ where each $\phi_l(x,b,c_l)$ divides over b. For each $0 < l < r$ pick an indiscernible sequence witnessing dividing, i.e. a sequence c_l^i indiscernible over bc_l which is not indiscernible over each realisation of $\phi_l(x,b,c_l)$. Each realisation a of $p(x,b)$ realises some $\phi_l(x,b,c_l)$, hence the sequence c_l^i is not indiscernible over a, yet indiscernible over b.

Note that we never use that the sequence $(c_l^i)_i$ were indiscernible over c_l; Remark 3.2 shows how this lead to a counterexample showing that (1) is not equivalent to non-forking.

(1) $\iff$ (2): Reading the diagram (2) gives precisely (1). Let us say a couple of words on how to parse the diagram (2). Recall that a simplex in $\mathrm{Idem}^+_\bullet$ is a path though the labelled graph depicted in the diagram. Taking a boundary of a simplex corresponds to replacing a sub-path by the arrow labelled by the composition of the labels of the sub-path. Thus, each simplex in $\mathrm{Idem}^+_\bullet$ is a boundary of



A simplicial map $\mathrm{Idem}_\bullet \longrightarrow M_\bullet/\mathrm{Aut}^T(M)$ sends each such simplex into a type with variables indexed by the vertices of the simplex. The identities between the boundaries of the simplices correspond to inclusions and equalities of the types. We hope these explanations enable the reader to parse the diagram and get (1). □

Remark 3.2. — The theory of a two-graph [**BJM**, 3.4] is a counterexample showing that (1) ⇔ (2) is not equivalent to non-forking.[(2)] Namely, the counterexample is the unique 1-type $p(x)$ over the empty set. The theory of a two-graph is simple, and therefore $p(x)$ does not fork over the empty set. There is a unique 2-type of two distinct elements, and, moreover, for each $a \neq b$ there are two distinct indiscernible sequences $a, b, c_1, c_2, \ldots$ and $a, b, d_1, d_2, \ldots$ such that $R(c_1, c_2, c_3)$ and $\neg R(d_1, d_2, d_3)$. A counting argument shows that for each x at least one of them is indiscernible over x, as follows. Assume not. If $\neg R(x,a,b)$ then by indiscernability $\{x,a,b,c_1\}$ contains precisely one edge $R(a,b,c_1)$, but it has to be even by the defining property of a two-graph. If $R(x,a,b)$, then by indiscernability $\{x,a,b,d_1\}$ contains precisely 3 edges, which is also odd. It would tempting to mistakenly assume that the formula $R(x,a,b) \vee \neg R(x,a,b)$ witnesses forking of $p(x)$ because the two indiscernible sequences witness dividing of $R(x,a,b)$ and $\neg R(x,a,b)$. The mistake is that to witness dividing, the indiscernible sequence has to be a sequence of 2-tuples representing *both* a and b. I cannot find a simplicial reformulation taking care of this.

4. Dividing and forking as lifting idempotents to ∞-category, simplicially

As mentioned earlier, the simplicial diagram Eq. 2 in Proposition 2.2(ii) defines non-dividing in model theory, as is reminiscent of the diagram defining the notion of lifting idempotents.

[(2)] We thank Ehud Hrushovski and Itay Kaplan for this counterexample.

Here we give cumbersome versions of Eq. 2 dealing with types with several variables over finite sets. We hope that a simplicially minded reader may recognise and improve some of these diagrams.

In §4.3 we rewrite simplicially the definition of q-dividing as a compatibility condition between types.

4.1. Dividing as lifting idempotents, simplicially. — We now state the simplicial characterisation of dividing in Proposition 2.2 for an arbitrary l-type over finite sets dividing over a finite set.

Proposition 4.1. — *Let $M_\bullet/\mathrm{Aut}^T(M)$ be the simplicial set of types of a first order theory T. (i) An idempotent $p_\bullet : \mathrm{Idem}_\bullet \longrightarrow M_\bullet/\mathrm{Aut}^T(M) \circ [+m]$ of the simplicial set $M_\bullet/\mathrm{Aut}^T(M) \circ [+m]$ is the same as the complete E-M type of a sequence $p(x_1, ..., x_i, x_{i+1}, ...)$ indiscernible over an m-tuple.*

(ii) A l-type $p(x_1, ..., x_l, b_1, ..., b_m)$, $b \in M^m$, does not divide over an n-tuple $\bar c$ *iff*

$$\begin{array}{ccc}
(\cdot_1 \to \dots \to \cdot_m)_\bullet = m^{\leqslant}_\bullet = \Delta^{m-1}_\bullet & \xrightarrow{\quad p_\bullet(\bar x, \bar b) \quad} & M_\bullet/\mathrm{Aut}^T(M) \circ [+l+n] \\
\downarrow & \overset{\exists}{\nearrow} & \downarrow \\
(\cdot_1 \to \dots \to \cdot_m)_\bullet = m^{\leqslant}_\bullet = \Delta^{m-1}_\bullet \times \left({}^{\mathrm{id}}\circlearrowleft \bullet \circlearrowright^{e=e\circ e}\right)_\bullet = \mathrm{Idem}_\bullet & \xrightarrow{\quad \forall \quad} & M_\bullet/\mathrm{Aut}^T(M) \circ [+n]
\end{array}$$

Proof. — (i) The unique type $p_i(\bar b_1, .., \bar b_i)$ of an i-tuple of m-tuples in the $\bar c$-indiscernible sequence $(\bar b_i)_{i\in\omega}$ of m-tuples is the image of the product of $\mathrm{id}_{m^{\leqslant}\longrightarrow m^{\leqslant}} \times \mathrm{Idem}_{i-1}$ where $\mathrm{id}_{m^{\leqslant}\longrightarrow^{\leqslant}}$ is the unique non-degenerate $m-1$-simplex of $\Delta^{l-1}_\bullet$, and $\mathrm{I}dem_n$ denotes the unique non-degenerate $n-1$-simplex in

$$\mathrm{Idem}_\bullet(n^{\leqslant}) = \mathrm{Hom}_{Cat}(\bullet_1 \longrightarrow \bullet_2 \longrightarrow \dots \longrightarrow \bullet_n \ , \ {}^{\mathrm{id}}\circlearrowleft \bullet \circlearrowright^{e=e\circ e}),$$

i.e. the map which sends each morphism $\bullet_k \longrightarrow \bullet_{k+1}$, $1 \leqslant k < i$, to the idempotent e. As each k-face of the unique non-degenerate n-simplex also the unique non-degenerate k-simplex, all the sub-types of $p_i(x_1, ..., x_i)$ corresponding to k-subtuples of the i-tuple $x_1....x_i$, coincide. Thereby $p_i(x_1, ..., x_i)$, $i > 0$, form a complete E-M type of an indiscernible sequence. (ii) This is equivalent to the characterisation of non-dividing in terms of indiscernible sequences [**TZ**, Cor.7.1.5]. □

4.2. Forking as lifting split idempotents, simplicially?— In the print edition Lurie used the category $\mathrm{Idem}_\bullet$. After some inaccuracies were pointed out by Gal Dor [**GalDor**], he replaced by the category $\mathrm{Idem}^+_\bullet$ which remembers more and captures the notion of a retraction.

We saw a simplicial reformulation (2) uses the ∞-category $\mathrm{Idem}_\bullet$. Lurie found it necessary to replace ∞-category $\mathrm{Idem}_\bullet$ by "a better behaved" $\mathrm{Idem}^+_\bullet$, and so we ask whether replacing $\mathrm{Idem}_\bullet$ in (2) leads to non-forking, arguably "a better behaved" version of non-dividing. In the two propositions below we decypher in model theory language two modifications of (2) we get in this way.

4.2.1. Dividing vs. lifting split idempotents. — The Proposition here considers the inclusion

Proposition 4.2. — *Let $M_\bullet/\mathrm{Aut}^T(M)$ be the simplicial set of types of a first-order theory T. Let $M \vDash T$ be a model.*

For a type $p(x, b_1, ..., b_m)$, and $b_1, .., b_m, d_1, ..., d_n \in M$ we have the following implications:

$$(1) \iff (2) \implies p(x, b_1, ..., b_m) \text{ does not divide over set } \{d_1, ..., d_n\}$$

where (1) and (2) are

(1) For each sequence $(\bar{c}^i)_{i\in\omega}$ of m-tuples indiscernible over $\{b_1, ..., b_m, d_1, ..., d_n\}$, there is a realisation $(a_1, ..., a_l) \vDash p(x_1, ..., x_l, b_1, ..., b_m)$ of $p(x_1, ..., x_l, b_1, ..., b_m)$ making $(\bar{c}^i)_{i\in\omega}$ indiscernible over $a_1, ..., a_l, b_1, ..., b_m, d_1, ..., d_n \in M$.

(2)

$$\begin{array}{ccc}
(\bullet_{b_1} \longrightarrow ... \longrightarrow \bullet_{b_m})_\bullet = m_\bullet^{\leqslant} = \Delta_\bullet^{m-1} & \xrightarrow{p_\bullet(x_1,...,x_l,b_1,...,b_m)} & M_\bullet/\mathrm{Aut}^T(M) \circ [+l+n] \\
\downarrow & \nearrow \exists & \downarrow \\
(\bullet_1 \longrightarrow ... \longrightarrow \bullet_m)_\bullet \times \left(\mathrm{id} \circlearrowleft \bullet_c \underset{i}{\overset{r}{\rightleftarrows}} \bullet_b \circlearrowleft \mathrm{id},\ e = i \circ r \right)_\bullet = \Delta_\bullet^{m-1} \times \mathrm{Idem}_\bullet^+ & \xrightarrow{\forall} & M_\bullet/\mathrm{Aut}^T(M) \circ [+n]
\end{array}$$

Proof. — (1) $\iff$ (2): Reading the diagram (2) gives precisely (1). See the proof of Proposition 3.1. □

4.2.2. Dividing vs. lifting split idempotents. — The Proposition here considers the other inclusion

$$\mathrm{id}_Y \circlearrowright \bullet_Y \qquad \Longrightarrow \qquad \mathrm{id}_X \circlearrowright \underset{e = i \circ r = e \circ e \ \circlearrowleft}{\bullet_X} \underset{i}{\overset{r}{\rightleftarrows}} \bullet_Y \circlearrowleft \mathrm{id}_Y$$

Proposition 4.3. — *Let $M_\bullet/\mathrm{Aut}^T(M)$ be the simplicial set of types of a first-order theory T. Let $M \vDash T$ be a model.*

For a type $p(x, b_1, ..., b_m)$, and $b_1, .., b_m, d_1, ..., d_n \in M$ we have the following implications:

$$(1) \iff (2) \implies p(x, b_1, ..., b_m) \text{ does not divide over set } \{d_1, ..., d_n\}$$

where (1) and (2) are

(1) For each m-tuple $c_1, ..., c_m$ and for each sequence $(\bar{b}^i)_{i\in\omega}$ of m-tuples indiscernible over $\{c_1, ..., c_m, d_1, ..., d_n\}$, $\bar{b}^0 = (b_1, ..., b_m)$, there is a realisation $(a_1, ..., a_l) \vDash p(x_1, ..., x_l, b_1, ..., b_m)$ of $p(x_1, ..., x_l, b_1, ..., b_m)$ making $(\bar{b}^i)_{i\in\omega}$ indiscernible over $a_1, ..., a_l, c_1, ..., c_m, d_1, ..., d_n \in M$.

(2)

$$\begin{array}{ccc}
(\bullet_{c_1} \longrightarrow ... \longrightarrow \bullet_{c_m})_\bullet = m_\bullet^{\leqslant} = \Delta_\bullet^{m-1} & \xrightarrow{p_\bullet(x_1,...,x_l,b_1,...,b_m)} & M_\bullet/\mathrm{Aut}^T(M) \circ [+l+n] \\
\downarrow & \nearrow \exists & \downarrow \\
(\bullet_1 \longrightarrow ... \longrightarrow \bullet_m)_\bullet \times \left(\mathrm{id} \circlearrowleft \bullet_b \underset{i}{\overset{r}{\rightleftarrows}} \bullet_c \circlearrowleft \mathrm{id},\ e = i \circ r \right)_\bullet = \Delta_\bullet^{m-1} \times \mathrm{Idem}_\bullet^+ & \xrightarrow{\forall} & M_\bullet/\mathrm{Aut}^T(M) \circ [+n]
\end{array}$$

Proof. — (1) $\iff$ (2): Similarly to the previous Proposition, reading the diagram (2) gives precisely (1). □

4.2.3. Forking vs. lifting split idempotents. — Let the functor $\bullet_X \longrightarrow \mathcal{R}^n$ be the push-out of n copies of the functor

$$\mathrm{id}_X \circlearrowright \bullet_X \quad\Longrightarrow\quad \mathrm{id}_X \circlearrowright \bullet_X \overset{e=i\circ r=e\circ e}{\circlearrowleft} \underset{i}{\overset{r}{\rightleftarrows}} \bullet_Y \circlearrowleft \mathrm{id}_Y$$

Let $\mathcal{R}^n_\bullet$ denote the nerve of the category $\mathcal{R}^n$, i.e. for any $i \in \mathbb{N}$

$$\mathcal{R}^n_\bullet(\bullet_1 \to \ldots \to \bullet_i) := \mathrm{Hom}_{Cats}(\bullet_1 \to \ldots \to \bullet_i, \mathcal{R}^n)$$

Proposition 4.4. — *Let $M_\bullet/\mathrm{Aut}^T(M)$ be the simplicial set of types of a first-order theory T. Let $M \models T$ be a model.*

For an l-type $p(x_1, \ldots, x_l, b_1, \ldots, b_m)$, $b_1, \ldots, b_m \in M$ we have that $(1) \Longleftrightarrow (2)$. Furthermore, if T has elimination of imaginaries, we have the following implication:

$$(1) \Longleftrightarrow (2) \implies p(x_1, \ldots, x_l, b_1, \ldots, b_m) \textit{ does not fork over } \varnothing$$

where (1) and (2) are

(1) For each m-tuple $(c^j_1, \ldots, c^j_m)$, $0 < j \leqslant N$, and each sequences $(\bar{b}^i_j)_{i\in\omega}$, $0 < j \leqslant N$, of m-tuples indiscernible over $c^j_1, \ldots, c^j_m, d_1, \ldots, d_n\}$, $\bar{b}^0_j = (b_1, \ldots, b_m)$, $0 < j \leqslant N$, there is a realisation $(a_1, \ldots, a_l) \models p(x_1, \ldots, x_l, b_1, \ldots, b_m)$ of $p(x_1, \ldots, x_l, b_1, \ldots, b_m)$ making for each $0 < j \leqslant N$ the sequence $(\bar{b}^i_j)_{i\in\omega}$ indiscernible over $a_1, \ldots, a_l, c^j_1, \ldots, c^j_m, d_1, \ldots, d_n \in M$.

(2) For each $N > 0$

$$\begin{array}{ccc}
(\bullet_1 \to \ldots \to \bullet_m)_\bullet = m^{\leqslant}_\bullet = \Delta^{m-1}_\bullet & \xrightarrow{\quad p_\bullet(x_1,\ldots,x_l,b_1,\ldots,b_m) \quad} & M_\bullet/\mathrm{Aut}^T(M) \circ [+l+m] \\
\downarrow & \overset{\exists}{\nearrow} & \downarrow \\
(\bullet_1 \to \ldots \to \bullet_m)_\bullet = m^{\leqslant}_\bullet = \Delta^{m-1}_\bullet \times \mathcal{R}^N_\bullet & \xrightarrow{\quad \forall \quad} & M_\bullet/\mathrm{Aut}^T(M) \circ [+n]
\end{array}$$

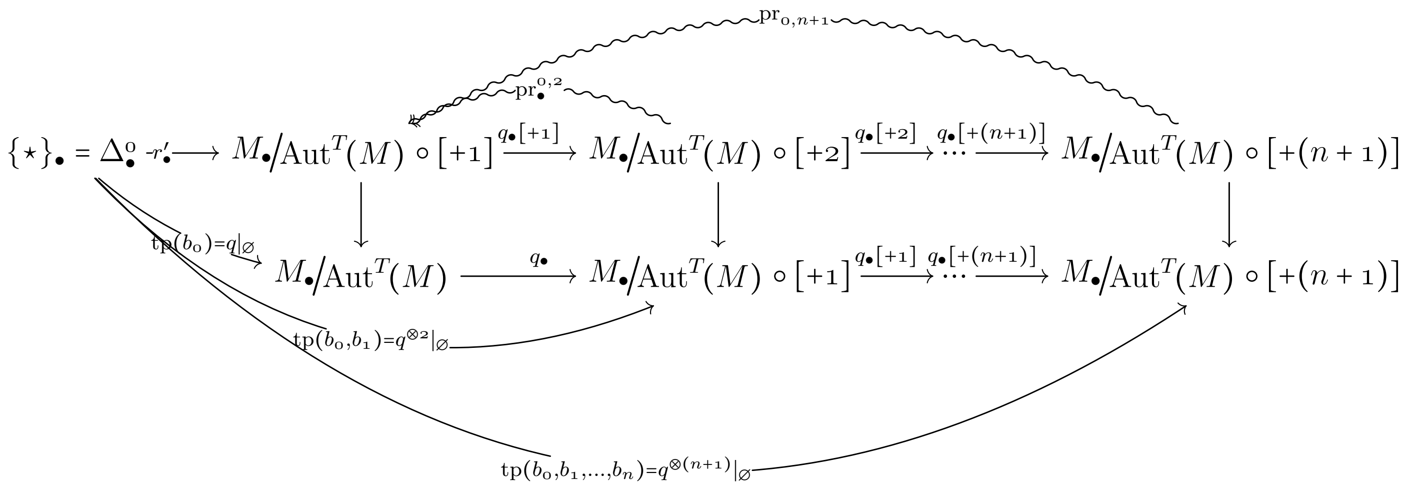
Proof. — $(1) \Longleftrightarrow (2)$: Reading the simplicial diagram in (2) is similar to that of Proposition 4.2.

$(1) \implies$ non-forking: We use elimination of imaginaries only to be able to ignore the distinction between elements and tuples. By the definition [**TZ**, Def.7.1.7,p.111] of forking there are finitely many formulas $\phi_l(x, b, c_l)$, $0 < l < r$, such that $p(x, b) \models \bigvee_{0<l<r} \phi_l(x, b, c_l)$ where each $\phi_l(x, b, c_l)$ divides over b. For each $0 < l < r$ pick an indiscernible sequence witnessing dividing, i.e. a sequence c^i_l indiscernible over b which is not indiscernible over each realisation of $\phi_l(x, b, c_l)$. Each realisation a of $p(x, b)$ realises some $\phi_l(x, b, c_l)$, hence the sequence c^i_l is not indiscernible over a, yet indiscernible over b. □

4.3. q-dividing as lifting homotopy?— Recall the definition of q-dividing in [**Hr**, 1.5,p.195;2.1,p.196]:

> If $q = q(y)$ is a global type, we say that $r'(x, y)$ q-divides over A if for some n, if $b_i \models q|A(b_0, \ldots, b_{i-1})$ for $i \leq n$, then $\bigcup_{i\leq n} r'(x, b_i)$ is inconsistent. This is equivalent to dividing, with the additional requirement that the indiscernible sequence be q-indiscernible.

Assume that q is a $\varnothing$-invariant global 1-type. The data implicit in the words above are depicted in the following simplicial diagram, as indicated by the labels. Note that the notation $\bigcup_{i\leqslant n} r'(x,b_i)$ implicitly considers (x,b_i) as a subtuple of $(x,b_0,...,b_n)$. This identification is made explicit by the curly arrows $\mathrm{pr}_\bullet^{0,i} : M_\bullet/\mathrm{Aut}^T(M) \circ [+i] \longrightarrow M_\bullet/\mathrm{Aut}^T(M) \circ [+1]$.



It is then tempting to assume that the definition of q-dividing is equivalent to commutativity of all the cycles passing through $\Delta^0_\bullet \xrightarrow{p_\bullet} M_\bullet/\mathrm{Aut}^T(M) \circ [+1]$ in the simplicial diagram above.

The diagram suggests that instead of a few distinguished curly arrows one should consider all the simplicial arrows of $M_\bullet/\mathrm{Aut}^T(M) \circ [+1]$, i.e. fixing the 0-th face. This leads to the following conjectural reformulation of q-dividing for an invariant type q reminiscent of the characterisation in terms of q-invariant sequences:

Conjecture 4.5. — *Let $q(y)$ be a $\varnothing$-invariant 1-type, and let $r'(x,y)$ be a 2-type.*

A type $r'(x,y)$ does not q-divide iff the map $\Delta^0_\bullet \longrightarrow M_\bullet/\mathrm{Aut}^T(M) \circ [+1]$ is compatible with the map $q : M_\bullet/\mathrm{Aut}^T(M) \longrightarrow M_\bullet/\mathrm{Aut}^T(M) \circ [+1]$ and simplicial maps of $M_\bullet/\mathrm{Aut}^T(M) \circ [+1]$ in the following sense: for each $m,n \in \mathbb{N}$ and for each monotone map $\tau : m^{\leqslant} \longrightarrow n^{\leqslant}$ the following diagram commutes:

$$\begin{array}{ccccccccc}
\Delta^0_\bullet & \xrightarrow{r'_\bullet} & M_\bullet/\mathrm{Aut}^T(M) \circ [+1] & \xrightarrow{q_\bullet[+1]} & M_\bullet/\mathrm{Aut}^T(M) \circ [+2] & \xrightarrow{q_\bullet[+2]} \cdots & \xrightarrow{q_\bullet[+n]} & M_\bullet/\mathrm{Aut}^T(M) \circ [+n] \\
\big\downarrow & & & & & & & \big\downarrow{\scriptstyle M_\bullet/\mathrm{Aut}^T(M)\circ[+1](\tau)} \\
\Delta^0_\bullet & \xrightarrow{r'_\bullet} & M_\bullet/\mathrm{Aut}^T(M) \circ [+1] & \xrightarrow{q_\bullet[+1]} & M_\bullet/\mathrm{Aut}^T(M) \circ [+2] & \xrightarrow{q_\bullet[+2]} \cdots & \xrightarrow{q_\bullet[+m]} & M_\bullet/\mathrm{Aut}^T(M) \circ [+m]
\end{array}$$

Note that the composition defines an $(n+1)$-type given a 2-type $r'(x,y)$ and an invariant 1-type $q(y)$.

5. Open questions

It is a question whether our simplcial reformulations warrant any study. Nonetheless, we make two suggestions.

5.1. Forking for types over spaces of parameters. — Replacing $|A|_\bullet$ in Proposition 1.2(ii) by an arbitrary space delivers a notion of a type over an arbitrary space of parameters; for example, replacing $|A|_\bullet$ by $M_\bullet/\mathrm{Aut}^T(M)$ gives the notion of a global invariant type by Proposition 2.3. Replacing $|A|_\bullet$ by $\mathrm{Idem}_\bullet$ gives the notion of a type *over an indiscernible sequence* $(a_i)_{i\in\omega}$: this means a type $p(x/a_1...)$ such that it implies the statement that the sequence $a_1...$ remains indiscernible over x.

The diagrams in Propositions 2.2(ii) and 3.1 suggest it may be possible to define dividing and forking for types over arbitrary spaces of parameters.

Model theory works with simplicial quotients $M_\bullet/\mathrm{Aut}^T(M)$ of very specific actions. Will model-theoretic notions remain meaningful for the simplicial quotient $\mathbb{B}_\bullet G = G_\bullet/G$, ∞-categories, or any other simplicial sets ?

5.2. Simplicial sets equipped with structure of a topological nature. — The reformulation of a definable type in Proposition 2.3(iii) suggests it may be better to consider simplicial sets equipped with an additional structure of topological nature, in that instance topology. Model theoretic examples in [**Z1, Z2**] suggest it may be useful to consider filters as such an additional structure. By a filter on a set we mean a finitely additive 0-1 valued measure. Without justification here we suggest to consider the category of simplicial objects of the following category Φ, which we discuss in forthcoming notes. The idea behind the definition is to think of the category $s\Phi$ of simplicial objects of the category Φ as a category of generalised spaces allowing to give precise meaning to the phrases "a property $P(s)$ holds whenever the simplex s is sufficiently small" and "a property $P(s)$ fails whenever the simplex s is sufficiently large".

Definition 5.1 **(The Chu category of filters)**. — Let Φ be the following category. An object of Φ is a set X equipped with two filters $\mathcal{U}$ and $\mathcal{B}$ on X.

A morphism $f : (X,\mathcal{U},\mathcal{B}) \longrightarrow (X',\mathcal{U}',\mathcal{B}')$ is a map $f : X \longrightarrow X'$ of the underlying sets such that the preimage of an $\mathcal{U}'$-big set is $\mathcal{U}$-big, and the image of an $\mathcal{B}$-small set is $\mathcal{B}'$-small. Let $\Phi_{:}$ be the category Φ where we identify morphisms equal on some neighbourhood, i.e. we consider two morphism $f : (X,\mathcal{U},\mathcal{B}) \longrightarrow (X',\mathcal{U}',\mathcal{B}')$ equal in $\Phi_{:}$ iff there is an $\mathcal{U}$-big subset U of X such that for each $x \in U$ $f(x) = g(x)$.

It is convenient to refer to the $\mathcal{U}$-big sets as *neighbourhoods*, and to $\mathcal{B}$-small sets as *bounded.* Thus we may say that a map is *continuous* iff the preimage of a neighbourhood is a neighbourhood, and is *bounded* iff the image of a bounded subset is bounded. A Φ-morphism is a map of the underlying subsets which is both continuous and bounded.

The $\mathrm{Hom}_\Phi(X, X')$-set can be turned into an object of Φ as follows, by a definition similar to that of compact open topology on the space of functions between topological spaces. A subset of $\mathrm{Hom}_\Phi(X, X')$ or $\mathrm{Hom}_{Sets}(X, X')$ is defined to be a neighbourhood iff it contains $\{f : X \longrightarrow X' : f(K) \subset \delta\}$ for some bounded subset $K \subset X$ and a neighbourhood $\delta \subset X'$. A subset of $\mathrm{Hom}_\Phi(X, X')$ or $\mathrm{Hom}_{Sets}(X, X')$ is defined to be bounded iff it is contained in $\{f : X \longrightarrow X' : f(\varepsilon) \subset K\}$ for some bounded subset $K \subset X$ and a neighbourhood $\varepsilon \subset X$.

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22.2.22 i wasted time, and now doth time waste me;