

EXERCICES DE STYLE: A HOMOTOPY THEORY FOR SET THEORY

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ABSTRACT. We construct a model category (in the sense of Quillen) for set theory, starting from two arbitrary, but natural, conventions. It is the simplest category satisfying our conventions and modelling the notions of finiteness, countability and infinite equi-cardinality. We argue that from the homotopy theoretic point of view our construction is essentially automatic following basic existing methods, and so is (almost all) the verification that the construction works.

We use the posetal model category to introduce homotopy-theoretic intuitions to set theory. Our main observation is that the homotopy invariant version of cardinality is the covering number of Shelah's PCF theory, and that other combinatorial objects, such as Shelah's revised power function - the cardinal function featuring in Shelah's revised GCH theorem — can be obtained using similar tools. We include a small “dictionary” for set theory in QtNaamen, hoping it will help in finding more meaningful homotopy-theoretic intuitions in set theory.

Part 1. The construction

The mischief of it is, nature will have to take its course: every production must resemble its author, and my barren and unpolished understanding can produce nothing but what is very dull, very impertinent, and extravagant beyond imagination. – Miguel de Cervantes Saavedra, Don Quixote

1. INTRODUCTION

Arguably, homology represents one of the major developments of mathematics in the 20th century. However, model theory and set theory are among the few fields of mathematics where homotopy theory has, essentially, never been applied. Indeed, with the exception of o-minimality, where homotopy/homology theories generalizing those arising in real geometry are used on a regular basis, we do not know of any applications of homotopy theory in either fields. In recent years, model theoretic questions arise to which, so it seems, homotopy theoretic tools should be applied.

The immediate motivation for carrying out the present work is a series of works by Zilber, Bays and the first author. These works are concerned with Zilber's program, launched in [22], to apply model theoretic methods to the study of (non compact) complex analytic structures. In his D.Phil [7], the first author, showed that Zilber's main (technical) result in [22] can be naturally translated into familiar algebro-geometric terms. This translation also allowed Gavrilovich to generalize the statement of Zilber's result, casting them in the form of algebro-geometric conjectures. To prove some of these conjectures (modulo necessary corrections), Bays in his thesis ([4], [5]) translated them back into the language of model theory,

where Shelah’s machinery of *excellent classes* had to play a significant role. But in order to get Shelah’s technology into work, non-trivial algebraic and algebraic-geometric information had to be obtained. To algebraic geometers and algebraists, information of the sort required in these proofs is usually given in the language of homology theory. Unfortunately, there is currently no dictionary translating modern homology theory into the language of model theory. Therefore, in order to be applied in this context, the algebraic tools first have to be “translated” into a more classical language. For Bays’ work this translation was readily available, but it may turn out to be a serious obstacle for developing the theory further.

Motivated by these problems, the first author embarked on an attempt to develop at least some parts of the missing toolbox. As a baby version he started playing with the simplest, most naive, construction he could imagine of a homotopy theory akin to the one he was looking for: an almost degenerate homotopy theory for set theory. Quite surprisingly, playing around with this new toy, we — very naturally — reached, from a totally different angle, some set theoretic concepts playing a central role in Shelah’s PCF theory: we recover the covering number of a cardinal λ as the “correct” homotopy theoretic version of cardinality, and — by a slight variation — retrieve Shelah’s revised power function.

The present paper is a concise, publishable, version of a more detailed text available on-line, [8] for the first part of the paper and [9] for the second part.

Structure of the paper. The first part of the paper is dedicated to the construction of a model category, QtNaamen, for set theory modeling the notions of finiteness, countability and infinite equi-cardinality. In the second part of the paper we explore the expressive power of our model category (5.1) which naturally leads us to the study of covering families (5.2) and to the main result of this paper, Theorem 40, where Shelah’s covering number (see below) is interpreted as a derived functor of cardinality. We conclude (5.3-4) with a brief exploration of several variants of our construction and the set theoretic concepts they lead to, namely generalised covering numbers, measurable cardinals and the covering lemmas.

For readers not interested in the details of the construction and the proofs the following could be a useful guide for a shorthand reading. §1.1 is a quick sketch of our construction for a reader familiar with homotopy theory.

§2 describes our point of view on the axioms of a model category as those of a derivation calculus in a labelled category. Definition 3 defines the category StNaamen; Definition 7 defines fibrations, cofibrations and weak equivalences as labels; Proposition 12 spells out the set-theoretic meaning of those definitions. Our main posetal model category QtNaamen is defined in §2.5, Definition 8; Lemma 17 gives several equivalent reformulations. It is worth noting Lemma 29 which seems to require a set-theory argument and for whose proof we could not find a diagram chasing argument. Definition 13 introduces other posetal model categories, QtNaamen $_{\kappa}$, for a regular cardinals κ .

In Part 2, we interpret several set-theoretic invariants in the homotopy language. §5.1, Theorem 40 interprets the covering number $cov(\lambda, \aleph_1, \aleph_1, 2)$ as a homotopy-invariant (slightly generalised) derived functor $\mathbb{L}_c \text{card} : \text{QtNaamen} \rightarrow \text{On}$ of cardinality; §5.2 gives a similar interpretation $\mathbb{L}_c \text{card}(\{\lambda\}) = \lambda^{[\kappa]} := cov(\lambda, \kappa^+, \kappa^+, \kappa)$ to Shelah’s “revisited power function”. Theorem 41 lists known results on these

invariants. In Lemma 44 we show that κ is measurable if and only if the homotopy category of QtNaamen_κ is not dense (as a partial order).

1.1. A homotopy theoretical synopsis. From the homotopy theoretic point of view the present note is, on the technical level, a triviality; the posetal model categories we introduce are defined in Proposition 12 and Definition 13. To a reader with a basic familiarity with category theory and the first few pages of Quillen's book [16] this synopsis should provide a fairly good idea of the contents of this note. Such readers may find it simpler to fill in the details themselves, rather than read our rendering of them. Readers less familiar with these concepts and ideas may find it more convenient to first read the background on model categories provided in the on-line version of this text, before returning to this synopsis.

Imagine a simple minded homotopy theorist, or even better — a homotopy theory android — trying to understand the introductory, set-theoretic, chapter commonly preceding undergraduate texts in topology, say. What is the (simplest) category which could help our homotopy theorist understand at least some of the notions appearing in such a chapter. Sets are default candidates as objects. But what should the arrows be? The membership relation ($a \in A$) is not transitive, and cannot serve as an arrow. Inclusions seem to be the next idea. starting with this simple category, we try to construct a (closed) model category capturing three basic notions of set theory: *finiteness*, *countability* and *equi-cardinality*.

Set theoretically, adding a single element to a single set suggests itself as the least significant operation, and could therefore be declared a weak equivalence. As compositions of weak equivalences are also weak equivalences any arrow $A \rightarrow B$ such that $B \setminus A$ is finite will also be declared a weak equivalence. To keep track of the work done so far those arrows will be labelled (w). Of course, these arrows will not be the only weak equivalences in our category, but ideologically, these are the weak equivalences that will allow us to capture the notion of finiteness.

To try and capture the notion of infinite equi-cardinality, we declare that an arrow $A \rightarrow B$ is a *cofibration* if A and B have the same cardinality. To keep track of this we label such arrows $A \xrightarrow{(c)} B$. Axiom (M2) of model categories requires that — in particular — arrows of the form $A \xrightarrow{(w)} A \cup \{b\}$ (for $\{b\}$ a singleton) decompose as a (weak) cofibration followed by a fibration. For infinite A this means (using induction) that for any finite set $\{\bar{b}\}$ we must declare $A \xrightarrow{(wc)} A \cup \{\bar{b}\}$. A little diagram chasing involving pushouts of (w)-labelled and (c)-labelled arrows shows that, to avoid constructing a trivial model category, the same must be true of any set A (not necessarily infinite), concluding that $A \rightarrow B$ should be labelled (c), if $\text{card } A = \text{card } B$ or both are finite.

Thus if A is finite $\emptyset \xrightarrow{(wc)} A$ (i.e. it is both a weak equivalence and a cofibration). We also know that there must be sets A such that $\emptyset \xrightarrow{(c)} A$ but not $\emptyset \xrightarrow{(wc)} A$, so such sets must be infinite. Since we still have to model the notion of countability, it seems natural to declare $\emptyset \xrightarrow{(c)} A$ if A is countable. It then follows that $A \rightarrow B$ has to be labelled (c) if either $\text{card } A = \text{card } B$ or B is countable.

Here things become trickier, as we are nearing a contradiction: On the one hand, Axiom (M2) assures that, up to weak equivalence, every arrow is a fibration, while — on the other hand — Axiom (M1) requires that any fibration has the left lifting property with respect to weak cofibrations. But no non-identity arrow $A \rightarrow B$

lifts with respect to all arrows of the form $A \xrightarrow{(wc)} A \cup \{b\}$, $b \in B$, and it becomes obvious that our category simply does not have enough arrows (or, objects, which — since arrows between any two objects, when they exist, are unique — is an equivalent statement). To overcome this, we want to formally add as a new arrow the collection of arrows of the form $A \xrightarrow{(wc)} A \cup \{b\}$, $b \in B$, or, equivalently, the collection of (M2)-decompositions $A \xrightarrow{(wc)} A' \rightarrow B$.

In order to solve this problem while keeping the category as simple as possible — i.e. arrows between objects are unique when they exist — we have to introduce as new objects in our category families of sets. In order not to have two kinds of objects, and in order to keep the work already achieved, we identify a set A with the singleton $\{A\}$. As a first approximation we allow all classes as objects. Viewing a class \mathcal{A} as the formal direct limit of (the unique diagram of) all its members, there is little choice but to define $\mathcal{A} \rightarrow \mathcal{B}$ if for all $A \in \mathcal{A}$ there exists $B \in \mathcal{B}$ such that $A \subseteq B$. This is the simplest and most natural definition we could come up with which is compatible with everything considered up to this stage.

Now the process of producing a model category is almost automatic: First, take the structure (on our category) cofibrantly generated by the two classes of arrows explicitly defined above. The resulting structure is not yet a model category — some obvious counterexamples prevent our weak equivalences from satisfying the two-out-of-three axiom of model categories. We remove some objects and morphisms by taking the full subcategory of morphisms lifting to an explicitly defined class of counterexample morphisms, and get an actual model category.

The model category obtained in this way gives a homotopy theoretic interpretation to some basic set theoretic concepts. Most importantly, a set X (viewed as the class $\{X\}$) is countable if and only if $\emptyset \xrightarrow{(c)} \{X\}$, and if this arrow is also a weak equivalence then X is finite; two infinite sets A and B have the same cardinality precisely when $\{A\} \xrightarrow{(c)} \{B\}$. Thus, the task we set to accomplish in the beginning is achieved: we obtained a model category modelling the notions of finiteness, countability and equi-cardinality. But more interesting is the fact that the notion of a covering family of a set A acquires a homotopical interpretation: X is a covering family for A if and only if $X \xrightarrow{(wf)} \{A\}$ is a weak equivalence and a fibration. It is now not hard to recover Shelah's covering numbers — a key notion in PCF theory.

1.2. The exposition. There are two important guidelines to the exposition of this paper. The first is that homotopy theory is best written in homotopy theoretic language. Therefore the paper is written in the language of category theory. Combinatorial properties are, as a rule, transformed into diagrams and proofs are, quite often, translated into (simple) diagram chasing arguments. This is by no means an ideological choice. The standard set theoretic intuition is lost at the early stages of the construction, and we have to stick to Quillen's homotopy theory as a guide. Our choice of language allows us to keep track of this intuition.

The second guideline to the exposition is the realisation that potential readers (plural (!)) of this paper are, probably, set theorists, with little familiarity with category theory, and no familiarity with model categories. Our proofs are, as a rule, more detailed than one would expect in standard papers directed at homotopy theorists. However, space limitations do not allow us to keep the paper fully self-contained for such readers. Thus, we do not include the standard category theoretic

and model category theoretic background, but refer interested readers to the online version, where all necessary definitions and basic constructions are provided.

1.3. A set theoretic disclaimer. Our usage of set theory is naive, and we intentionally ignore set theoretic questions naturally arising in the context of the “category of sets”. Since in the categories we are dealing with arrows are unique (when they exist) the only problem that may arise concerns the nature of the objects in our category. Any standard solution of such problems would address all questions of this nature which may arise in the construction described in this paper. To avoid using large cardinals and stay in ZFC, we could work in NBG, a conservative and equiconsistent extension of ZFC, declare our objects to be classes (definable with parameters), and formulate our claims as properties (in the metatheory) of the NBG formulas defining the partial order and the labels.

A simpler approach would be to fix a strongly inaccessible cardinal κ and identify the objects of our category with $\mathbb{P}(V_\kappa)$. In such a setting the collection of objects of our category (the collection \mathcal{Ob}) can be identified with a subset of $\mathbb{P}(\mathbb{P}(V_\kappa))$. The arrows in our category (\mathcal{Mor}) can then be identified with a definable subset of $\mathcal{Ob} \times \mathcal{Ob}$, and the labelling associated with Quillen’s model categories can be thought of as (definable) unary predicates on the set \mathcal{Mor} .

Having said that, we will from now on ignore all set theoretic questions of this nature, with the conviction that readers concerned with the possibility of set theoretic paradoxes arising as part of the construction can easily fill in all the details in either of above solutions, or any other standard solution they may find more attractive.

1.4. Model categories. There are several axiomatizations of model categories. In the present paper we will be using Quillen’s original version from the first chapter of [16]. We briefly state the axioms for ease of reference. For further discussion see [16] and [8].

We remind that a model category is a category \mathfrak{C} whose arrows can be labelled by any subset of the labels $(c), (f), (w)$. Arrows labelled (c) are called cofibrations, arrows labelled (f) are fibrations, weak equivalences are arrows labelled (w) , and $(wf), (wc)$ -arrows are trivial fibrations and cofibrations respectively. The labelling of arrows should satisfy the following axioms:

(M0): Every (small) diagram has limits and co-limits.

(M1): Trivial fibrations ((wf) -arrows) have the right lifting property with respect to any cofibration and trivial cofibrations ((wc) -arrows) have the left lifting property with respect to any fibration. This will be denoted $(c) \triangleleft (wf)$ and $(wc) \triangleleft (f)$ respectively.

(M2): Any arrow $A \rightarrow B$ decomposes into a trivial cofibration followed by a fibration $A \xrightarrow{(wc)} A_{wc} \xrightarrow{(f)} B$ and to a cofibration followed by a trivial fibration, $A \xrightarrow{(c)} A_{wf} \xrightarrow{(wf)} B$.

(M3): (1) Fibrations and cofibrations are stable under compositions.
 (2) Isomorphisms are fibrations, cofibrations and weak equivalences.
 (3) Fibrations and cofibrations are stable under base change (pull-back) and co-base change (push forward) respectively.

(M4): The pull back of a trivial fibration is a weak equivalence and the push-forward of a trivial cofibration is a weak equivalence.

(M5): (2-of-3) in any triangle, if any two edges are weak equivalences so is the third edge.

(M6): A model category is *closed* if any two of the classes of arrows $(c), (f), (w)$ determine the third.

As it will turn out, the model category QtNaamen that we construct in Part I of the paper will be closed, a useful fact in some of the proofs, but this is not part of the standard definition of model categories.

We spell out some simple facts concerning closed model categories that we will usually use without reference:

- Claim 1.**
- (1) *A model category has initial and terminal objects namely objects \emptyset and \top such that $\emptyset \rightarrow \mathcal{C} \rightarrow \top$ for every object \mathcal{C} .*
 - (2) *A non-degenerate model category (i.e. a model category \mathfrak{C} whose homotopy skeleton is not a singleton with a unique arrow) has non-trivial cofibrant objects, namely there exists an object \mathcal{C} , not isomorphic to \emptyset such that $\emptyset \xrightarrow{(c)} \mathcal{C}$.*
 - (3) *Moreover, in a non-degenerate model category there exist non-trivial cofibrant objects, namely there exists an object \mathcal{C} such that $\emptyset \xrightarrow{(c)} \mathcal{C}$ is not a weak equivalence.*
 - (4) *If all diagrams in the model category commute, then if $\mathcal{C} \rightarrow \mathcal{D}$, $\mathcal{C} \xrightarrow{(c)} \mathcal{E}$ and $\mathcal{D} \rightarrow \mathcal{E}$ then $\mathcal{D} \xrightarrow{(c)} \mathcal{E}$.*
 - (5) *The composition of two (wc)-arrows is a (wc)-arrow.*
 - (6) *If in a labelled category \mathfrak{C} satisfying (M0), (M1) and (M6) all diagrams commute then \mathfrak{C} satisfies (M3) and (M4).*

The proof is standard diagram chasing, and we leave it as an exercise. Readers not familiar with standard homotopy theoretic language may find it more convenient if we remind that:

Definition 1. *Given a model category, \mathfrak{C} , an object \mathcal{C} of \mathfrak{C} is cofibrant if $\emptyset \xrightarrow{(c)} \mathcal{C}$. A cofibrant object is trivial if $\emptyset \xrightarrow{(wc)} \mathcal{C}$.*

2. THE CONSTRUCTION

In this section we describe the construction of a model category for set theory. The construction is carried out in two stages. First we construct a labelled category, StNaamen, and show that it satisfies all the axioms of a model category, except axiom (M5). We describe a counter example showing that, in fact StNaamen fails to satisfy (M5) and use this counter example to motivate the passage to a full sub-category QtNaamen which manages to avoid this family of counter examples. We then proceed to show that QtNaamen is, indeed, a closed model category.

We could, of course, spell out a combinatorial definition of QtNaamen. This would boil down to defining StNaamen using the combinatorial characterisation of Proposition 12 and describing the full sub-category QtNaamen. We feel, however, that such an exposition will miss an essential part of our argument.

First, in our exposition we will try to show that, in a way (which we believe could be formalised, but we do not attempt to do so) QtNaamen is the simplest

model category for set theory, and for that reason – from a homotopy theory point view on set theory – essentially unavoidable.

The second, more important, reason for giving a detailed description of all stages leading to the “right” construction is that in our construction we are aiming to model three basic set theoretic concepts: finiteness, countability and equicardinality. As we show in the second part of the paper, there are many possible variants on the construction, modelling slightly different set theoretic notions, and giving rise other set theoretic “discoveries”. In our original construction we “discover” Shelah’s covering number. Other constructions lead to rediscovering Shelah’s revised power function, measurable cardinals and more. Unravelling the mechanism underlying our construction will allow further investigation into homotopy theoretic definition of important set theoretic notions, and maybe even to the discovery of new such notions.

Finally, in [11] Gromov writes:

“The category/functor modulated structures can not be directly used by ergosystems, e.g., because the morphisms sets between even moderate objects are usually unlistable. But the ideas of the category theory show that there are certain (often non-obvious) rules for generating proper concepts.”

We believe that the present work can shed some light on how “the ideas of category theory” are used to “[automatically] generate proper concepts”. We believe that (significant parts of) our development of the model category theory and our derivation of the set theoretic concepts from it, could be formalised in an (almost) algorithmic way, following some simple rules such as: list the tasks to do and try – greedily – to address them one by one; search for proofs and counter-examples simultaneously. Since proofs — in the context of posetal model categories such as the ones we are working with — amount usually to the existence of a certain arrow, simple “arrow generating rules” such as “use the lifting property”, “take direct and inverse limits” etc. can be used to generate proofs.

As a more detailed example, consider Axiom (M0) and how it can be applied in such an algorithmic approach to the construction:

- (1) Given a (finite) commutative diagram with vertices $\{X_i\}_{i \in I}$ add a new vertex \mathcal{D} and arrows $\{X_i \rightarrow \mathcal{D}\}$, making the whole diagram commute. Mark the new vertex and arrows by a special symbol \odot .
- (2) Given a (finite) commutative diagram with vertices $\{X_i\}_{i \in I}$, \mathcal{D} and D , such that the diagram contains the arrows $\{X_i \rightarrow \mathcal{D}\}_{i \in I}$ marked \odot and the (ordinary) arrows $\{X_i \rightarrow D\}$, construct an arrow $\mathcal{D} \rightarrow D$ making the diagram commute.

Applying (2) above in the case that $D \rightarrow \mathcal{D}$ is an arrow in the diagram, we obtain the uniqueness of the direct limit.

Our exposition is not intended to do any of the above formally or in such detail. Rather, our slow-paced rendering of the work should allow the interested reader to convince herself (or himself) that this can be done, and as a rule we will not dwell on this point. We will only dwell on those parts of the construction where either we were unable to see a simple algorithmic proof, or where a “non-algorithmic” argument could simplify the proof.

2.1. The first round. Our first task is to find the right category which we are then going to label. Our guideline is “try the simplest possible solution first. Correct later”. The simplest candidate for making a category from set theory is, probably, the one whose objects are sets and whose morphisms are inclusions (since the membership relation is not transitive). This category allows to express some basic set theoretic operations. E.g., $Z = X \cup Y$ if and only if the following diagram is true:

$$X \xrightarrow{\circlearrowleft} Z \xleftarrow{\circlearrowleft} Y$$

FIGURE 1

and intersections can be defined by inverting all the arrows in this diagram. Similarly, stating that $A = \emptyset$ amounts to the statement that A is the initial object of this category. The complement of a set (relative to a larger one) can be expressed in a simple diagram:

$$\begin{array}{ccc} \emptyset & \xrightarrow{\circlearrowleft} & X \\ \downarrow \circlearrowleft & & \downarrow \circlearrowleft \\ Y & \xrightarrow{\circlearrowleft} & Z \end{array}$$

FIGURE 2. The set Y is the complement of the set X relative to the set Z if \emptyset is the inverse limit of X and Y and Z is their direct limit.

Thus $A \setminus B$ is the complement (relative to A) of $A \cap B$.

Rather arbitrarily we require that the model category to be constructed should model three basic set theoretic concepts: finiteness, (infinite) countability and (infinite) equi-cardinality. So the next step in the construction is to label the arrows in our category so that (a) the above three concepts are captured by our labelling, and (b) the resulting labelled category is a model category. Regardless of the choice of labelling (wcf)-arrows must be isomorphisms. So an arrow should not be labelled (wcf), unless it is an isomorphism. For simplicity our intention is to try and construct a closed model category. In such a model category once two of the labels have been specified, the third is fully determined. So, in order to stay away from contradictions, we focus on two of the labels, letting them generate the third.

To choose which arrows to label we observe that while \emptyset is a natural object in our category and serves as the initial object, the set theoretic universe V , the (would be) terminal object, is not a set and has to be formally adjoined to the category in order for even Axiom (M0) to hold. Thus, it seems that though there is a natural homotopy theoretic duality between fibrant objects (i.e. objects \mathcal{C} such that $\mathcal{C} \xrightarrow{(f)} \top$, where \top is the terminal object) and cofibrant objects, in the set theoretic context the latter should occur more naturally. Thus, our first, somewhat arbitrary, goal is to define weak equivalences and cofibrations.

Remark 2. *Though the construction of a cofibrantly generated model category seems more natural in the present context, it is plausible that other approaches*

may also work. We do not know whether this is indeed the case, or whether the different possible approaches lead to important set theoretic concepts in the same way that the present construction leads naturally to the definition of the covering number.

In algebraic topology (and in Quillen's model categories) (w) -labelled arrows correspond to morphisms called *weak homotopy equivalences*, and indeed are thought of as some sort of equivalence, as is reflected by Axiom (M5). From the homotopy theorist point of view there is no interesting distinction between two weakly equivalent objects. When are two sets closest? When should we consider two sets "almost identical"? Two sets differing by one element are good candidates for that title. Taking Axiom (M5) into account we define:

Definition 2. Let A, B be sets. We denote

- (1) $A \rightarrow_0 B$ if $A \subseteq B$.
- (2) $A \xrightarrow{(w)_0} B$ if $A \rightarrow B$ and $B \setminus A$ is finite.
- (3) $A \xrightarrow{(c)_0} B$ if $A \rightarrow B$ and either $\text{card } A = \text{card } B$ or B is finite.

Remark 3. The index 0 in the above notation anticipates the failure of the first round of the construction. Since it will play an important role in the final construction, it is useful to keep track of these stages of the construction. To see in what ways our construction so far fails observe:

- (1) It follows from Claim 1 that our model category must have (non-trivial) cofibrant objects (so more (c) -arrows are needed).
- (2) If we aim to construct a closed model category, then labelling $A \xrightarrow{(c)_0} B$ if $A \rightarrow B$ and $\text{card } A = \text{card } B$ would imply that also arrows $A \rightarrow B$ with B finite should be cofibrations. Indeed, there must be an infinite set C such that $\emptyset \xrightarrow{(c)} C$ and clause (4) of Claim 1 assures that if $\emptyset \xrightarrow{(c)} C$ then $\emptyset \xrightarrow{(wc)} \{c\}$ for all $c \in C$. Thus, if $\{a\}$ is any singleton, then $\emptyset \xrightarrow{(c)_0} C \cup \{a\}$, so by (M3) we get $\emptyset \xrightarrow{(c)_0} C \cup \{a\}$, and by the above observation $\emptyset \xrightarrow{(c)_0} \{a\}$. By induction, the same is true for any finite set \bar{a} . It follows that for such a set \bar{a} and any set A the arrow $A \rightarrow_0 A \cup \bar{a}$ lifts with respect to any fibration, and in a closed model category must be a trivial cofibration.
- (3) It follows that all the weak equivalences we defined, are trivial cofibrations.

The above labelling – deficient as it is – captures the notions of finiteness and (infinite) equi-cardinality. So our task now is to model the notion of countability, and turn the resulting labelled category into a model category without losing track of the above notions. The notion of countability should be a proper weakening of the notion of finiteness. So we set

Notation 4. $\emptyset \xrightarrow{(c)_0} C$ if C is countable.

But now we run into trouble. In a model category there should be arrows $A \rightarrow B$ which are not trivial cofibrations. Axiom (M2) requires that there exists an object \mathcal{A}_B such that $A \xrightarrow{(wc)} \mathcal{A}_B \xrightarrow{(f)} B$. But now the arrow $\mathcal{A}_B \xrightarrow{(f)} B$ must have, by Axiom (M1), the right lifting property with respect to any arrow labelled (wc) . In particular, for our labelled category, any finite $b \subseteq B$ should satisfy:

$$\begin{array}{ccc}
A & \longrightarrow & \mathcal{A}_B \\
\downarrow (wc)_0 & \nearrow & \downarrow (f)_0 \\
A \cup \{b\} & \longrightarrow & B
\end{array}$$

implying that if \mathcal{A}_B were an object in our category (a set) then $A \cup b \subseteq \mathcal{A}_B$ for all finite $b \subseteq B$, so that $\mathcal{A}_B = B$, contradicting the assumption that $A \rightarrow B$ is not a weak cofibration. It follows that our category has to be changed. Our ideology of keeping things as simple as possible suggests adding new objects while keeping the posetal structure of the category. So we add a formal object: the collection of all sets $A \cup \{\bar{b}\}$ for $\bar{b} \subseteq B$ finite, which we view as a family of sets. So now we should view our objects as families of sets: the old objects each form a family of one member, The terminal object of the category, V — the universe of set theory, The family of all finite extensions of a set A within a set B (or within V), and possibly more. Again, in order to avoid the problem of choosing which families must be included and which families can be ignored, we choose to add all families of sets. Namely, the objects of our category will be from now on (all) classes of sets.

Notation 5. *The objects of our category will be denoted by calligraphic letters ($\mathcal{A}, \mathcal{B}, \mathcal{X}, \mathcal{Y}, \mathcal{W}$ etc.) Sets will be denoted by capital letters (e.g. $A \in \mathcal{A}$).*

Identifying a set A with the class $\{A\}$, we know what the arrows between such objects should be. But how should arrows between an old object and a new one be defined, or even worse, between two new objects? Our category has to be stable under finite limits. In the category of sets we started with, we showed that the (direct) limit of a diagram containing finitely many sets (possibly, no arrows), is the union of those sets. So it is natural to identify $\{A, B\}$ with the limit of the (arrowless) diagram $\{A\}, \{B\}$. Thus, we define:

Definition 3 (StNaamen). *Let StNaamen be the category whose objects are classes of sets and whose morphisms are given by $\mathcal{X} \rightarrow \mathcal{Y}$ if*

$$(\forall X \in \mathcal{X} \exists Y \in \mathcal{Y})(X \subseteq Y).$$

Remark 6. *Since in StNaamen arrows are unique, any collection of objects of the category describes a unique (necessarily commutative) diagram. Therefore one can talk of (direct or inverse) limits of any (small) collection of objects in StNaamen rather than on the limits of diagrams. We will use this convention freely in what follows.*

2.2. The second round. The labelling of arrows can now proceed almost axiomatically. Keeping the arrows we have already labelled (using the identification of the category of sets we start with with a sub-category of StNaamen), we take the minimal labelling satisfying axioms (M1) and (M2). Again, the resulting labelled category will not be a model category, but it will be quite close to being one.

Definition 4. *An arrow in StNaamen is a fibration (i.e., labelled (f)) if it has the right lifting property with respect to all arrows of the form $\{A\} \xrightarrow{(wc)_0} \{B\}$, i.e., $(f) :=_{\lt} (wc)_0$. An arrow is a trivial cofibration (i.e., labelled (wc)) if it has the left lifting property with respect to all fibrations, i.e., $(wc) := (f)_{\lt}$. See Figure 3.*

Claim 7. *The labelling defined satisfies the first part of Axiom (M1). Labelling any other arrow (wc) or (f) will violate this axiom. All isomorphisms are labelled (wcf).*

The proof is simple, and left as an exercise.

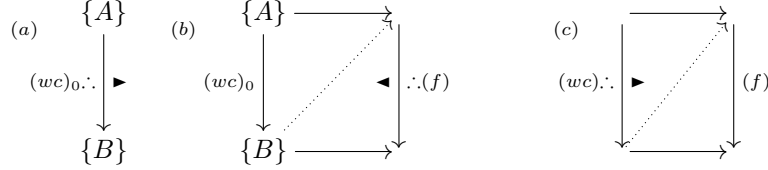


FIGURE 3. The three steps of the labelling:

- (a) The basic $(wc)_0$ arrows: $B \setminus A$ is finite.
- (b) The left hand side arrow is labelled (f) if it has the lifting property with respect to all arrows labelled $(wc)_0$ in step (a).
- (c) the right hand side arrow is labelled (wc) if it has the lifting property with respect to all arrows labelled (f) in step (b).

We proceed in a similar way to describe trivial fibrations:

Definition 5. *An arrow in $StNaamen$ is labelled (wf) if it has the right lifting property with respect to all arrows of the form $\{A\} \xrightarrow{(c)_0} \{B\}$, i.e., $(wf) := \sphericalangle (c)_0$ and $(c) := (wf) \sphericalangle$. See Figure 4.*

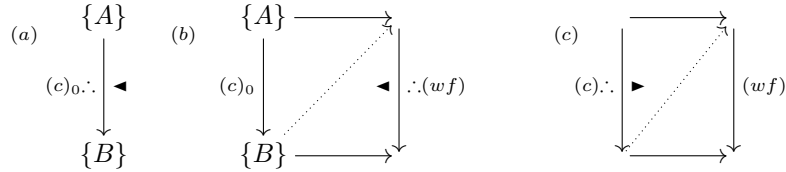


FIGURE 4. The three steps of the labelling:

- (a) The basic $(c)_0$ arrows, $\text{card } B \leq \text{card } A + \aleph_0$.
- (b) The right hand side arrow is labelled (wf) if it has the lifting property with respect to all arrows labelled $(c)_0$ in step (a).
- (c) the left hand side arrow is labelled (c) if it has the lifting property with respect to all arrows labelled (wf) in step (b).

An analogue of Claim 7 is true for the labelling of cofibrations and trivial fibrations, with a similar proof. But we also have to check that the two last steps of the labelling are compatible:

Remark 8. *If an arrow h is labelled (wf) at stage (b) of Figure 4 this means that $(c)_0 \sphericalangle h$, so in particular $(wc)_0 \sphericalangle h$. Thus, the arrow h has already been labelled (f) in Figure 3(b). It follows that if an arrow g was labelled (wc) in Figure 3(c) then $g \sphericalangle (f)$, so in particular $g \sphericalangle (wf)$ and therefore g is also labelled (c) in Figure 4(c).*

Now we proceed to take care of Axiom (M2): This axiom requires that every arrow decomposes into a (c) -arrow followed by an (f) -arrow. Our hope is that if we label (w) those arrows that must be weak equivalences, the axiom will be automatically satisfied. By Axiom (M5) every (w) -arrow can be written as the composition of a (wc) -arrow and a (wf) -arrow, so we define:

Definition 6. *An arrow is a weak equivalence if it is the composition of a (wc) -arrow followed by a (wf) -arrow.*

Before we proceed we need a simple observation following immediately from the construction:

Claim 9. *The following $\forall\exists$ -diagrams are always true:*

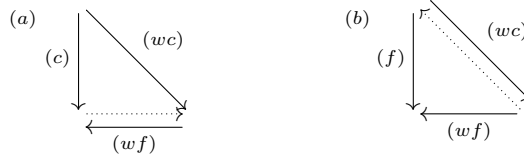


FIGURE 5. The dotted arrows exist whenever the solid ones are labelled as in the figures.

Now, a simple diagram chasing, shows that this step of the construction is consistent:

Claim 10. *If $\mathcal{X} \xrightarrow{(c)} \mathcal{Y}$ (resp. $\mathcal{X} \xrightarrow{(f)} \mathcal{Y}$) and if $\mathcal{X} \rightarrow \mathcal{Y} = \mathcal{X} \xrightarrow{(wc)} \mathcal{Z} \xrightarrow{(wf)} \mathcal{Y}$ then $\mathcal{X} \rightarrow \mathcal{Y}$ was labelled (wc) (resp. (wf)) before the last step of the construction.*

Proof. We have to show that, under the above assumption, if $\mathcal{X} \xrightarrow{(c)} \mathcal{Y}$ then $\mathcal{X} \rightarrow \mathcal{Y} \times \mathcal{V} \xrightarrow{(wf)} \mathcal{W}$ for all arrows $\mathcal{V} \xrightarrow{(wf)} \mathcal{W}$.

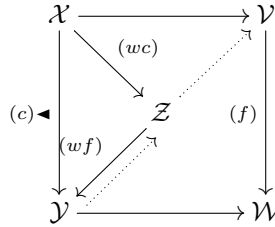


FIGURE 6. Applying the lifting $\mathcal{X} \xrightarrow{(wc)} \mathcal{Z} \times \mathcal{V} \xrightarrow{(f)} \mathcal{W}$ with respect to the arrows $\mathcal{X} \rightarrow \mathcal{V}$ and the composition $\mathcal{Z} \rightarrow \mathcal{V} \rightarrow \mathcal{W}$ produces the arrow $\mathcal{Z} \rightarrow \mathcal{V}$. The arrow $\mathcal{Y} \rightarrow \mathcal{Z}$ is given by Claim 2.2.

Since in our category arrows are unique, the above diagram is commutative. Thus, the composition $\mathcal{Y} \rightarrow \mathcal{Z} \rightarrow \mathcal{V}$ is the desired lifting. The proof of the analogous claim for $\mathcal{X} \xrightarrow{(f)} \mathcal{Y}$ is dual. \square

We sum up the labelling constructed in the present sub-section in the following definition:

Definition 7 (The labelling of StNaamen). (\longrightarrow) : There is a unique arrow $\mathcal{X} \longrightarrow \mathcal{Y}$ iff for every $X \in \mathcal{X}$ there exists $Y \in \mathcal{Y}$ such that $X \subseteq Y$.
(wc)₀: The arrow $\mathcal{A} \longrightarrow \mathcal{B}$ is labelled $(wc)_0$ if and only if $\mathcal{A} = \{A\}$ and $\mathcal{B} = \{B\}$ are sets and $B \setminus A$ is finite.
(f): The arrow $\mathcal{X} \longrightarrow \mathcal{Y}$ is labelled (f) if and only if $(wc)_0 \triangleleft \mathcal{X} \longrightarrow \mathcal{Y}$.
(wf): An arrow $\mathcal{X} \longrightarrow \mathcal{Y}$ is labelled (wf) if and only if $\{A\} \longrightarrow \{B\} \triangleleft \mathcal{X} \longrightarrow \mathcal{Y}$ for all sets A, B such that $\text{card } B \leq \text{card } A + \aleph_0$.
(wc): An arrow $\mathcal{X} \longrightarrow \mathcal{Y}$ is labelled (wc) if and only if $\mathcal{X} \longrightarrow \mathcal{Y} \triangleleft (f)$.
(c): An arrow $\mathcal{X} \longrightarrow \mathcal{Y}$ is labelled (c) if and only if $\mathcal{X} \longrightarrow \mathcal{Y} \triangleleft (wf)$.
(w): An arrow $\mathcal{X} \longrightarrow \mathcal{Y}$ is labelled (w) if and only if it can be written as the composition $\mathcal{X} \xrightarrow{(wc)} \mathcal{Z} \xrightarrow{(wf)} \mathcal{Y}$.

2.3. Almost there. As already mentioned, the above labelling does not make StNaamen into a model category. We give the simple proof (which is lengthy just because we have many things to check) that StNaamen is close to being a model category. There are two key steps to the proof, which will serve us throughout the continuation of the paper. The first is the description of limits in StNaamen. The second, Proposition 12, is the technical heart of the first part of the paper. In this proposition we give a purely combinatorial characterisation of our labelling.

Proposition 11. *The labelling of the category StNaamen given in Definition 7 satisfies Axioms (M0)-(M4).*

Proof. We have already seen (Figure 1) that StNaamen has direct limits, which are given by unions. By duality, inverse limits are given by componentwise intersections. More precisely, given any finite set of objects $\mathcal{A}_1, \dots, \mathcal{A}_n$ in StNaamen their inverse limit is

$$\mathcal{A}_1 \cap \dots \cap \mathcal{A}_n := \bigcup \{A_1 \cap \dots \cap A_n : A_i \in \mathcal{A}_i\}$$

as can be readily checked. Axiom (M1) holds by construction. By construction, also Axiom (M6) is satisfied. By Claim 1 Axioms (M3) and (M4) are automatic.

So it only remains to verify Axiom (M2). Let $\mathcal{X} \longrightarrow \mathcal{Y}$ be any arrow. Let \mathcal{L} be the (direct) limit of all objects, \mathcal{V} , such that $\mathcal{X} \xrightarrow{(wc)} \mathcal{V} \longrightarrow \mathcal{Y}$ (it will follow from Proposition 12 that \mathcal{L} is indeed an object in StNaamen). The corresponding diagram is:

$$\begin{array}{ccc} & \mathcal{L} & \\ & \uparrow & \\ X & \xrightarrow{(wc)} \mathcal{V} & \longrightarrow \mathcal{Y} \end{array}$$

FIGURE 7

Then $\mathcal{X} \longrightarrow \mathcal{Y} = \mathcal{X} \longrightarrow \mathcal{L} \longrightarrow \mathcal{Y}$. So it remains to verify that $\mathcal{X} \xrightarrow{(wc)} \mathcal{L}$ and $\mathcal{L} \xrightarrow{(f)} \mathcal{Y}$. In order to prove the first assertion we need the following observation:

Claim: Let $\{\mathcal{V}_i\}_{i \in I}$ be a class of objects in StNaamen such that $\mathcal{X} \xrightarrow{(wc)} \mathcal{V}_i$ for all $i \in I$. Let \mathcal{L} be the direct limit of $\{\mathcal{V}_i\}_{i \in I}$ (in particular, we assume – as will be proved shortly – that \mathcal{L} is an object of StNaamen). Then $\mathcal{X} \xrightarrow{(wc)} \mathcal{L}$.

Proof. By the universal property of direct limits the following diagram is true:

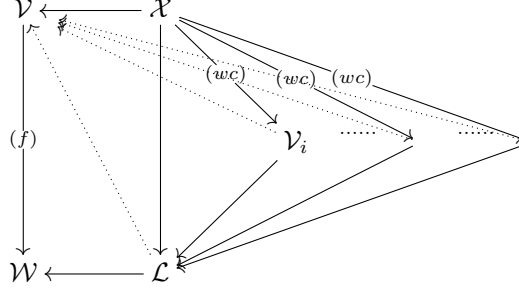


FIGURE 8. The arrows $\mathcal{V}_i \rightarrow \mathcal{V}$ exist by $\mathcal{X} \xrightarrow{(wc)} \mathcal{V}_i \triangleleft \mathcal{V} \xrightarrow{(f)} \mathcal{W}$. So the arrow $\mathcal{L} \rightarrow \mathcal{V}$ exists by virtue of it being the limit of the \mathcal{V}_i .

□_{Claim}

We now have to show that $\mathcal{L} \xrightarrow{(f)} \mathcal{Y}$ which, by Axiom (M6), amounts to $(wc) \triangleleft \mathcal{L} \rightarrow \mathcal{Y}$. Consider the diagram:

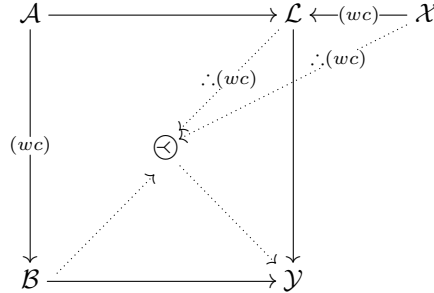


FIGURE 9. Consider the direct limit of \mathcal{L} and \mathcal{B} . The dotted arrows exist by the universal property. The arrow connecting \mathcal{L} with the limit is labelled (wc) by Axiom (M3) and Axiom (M4). The arrow connecting \mathcal{X} to the limit is labelled (wc) as the composition of two (wc) -arrows (see Claim 1).

It follows that $\mathcal{X} \xrightarrow{(wc)} \odot \rightarrow \mathcal{Y}$. By the definition of \mathcal{L} this implies that $\odot \rightarrow \mathcal{L}$. Thus $\odot \cong \mathcal{L}$ and as $\mathcal{B} \rightarrow \odot$ we also have $\mathcal{B} \rightarrow \mathcal{L}$, showing that $\mathcal{A} \xrightarrow{(wc)} \mathcal{B} \triangleleft \mathcal{L} \rightarrow \mathcal{Y}$. Since $\mathcal{A} \xrightarrow{(wc)} \mathcal{B}$ was arbitrary we are done. The other part of Axiom (M2) is proved in a similar way. □

2.4. A set theoretic interlude. In this sub-section we give a set theoretic characterisation of the labelling of arrows in StNaamen . This set theoretic version of the labelling will be important for several reasons. First, it is required in order to verify that our labelling does, as we intended, capture the notions of finiteness, countability and infinite equi-cardinality. It will also allow us to give a set theoretic interpretation of homotopy theoretic concepts, which is – after all – the main idea behind the present construction. Finally, since we cannot expect that all properties of the labelled category StNaamen can be proved from the abstract axioms (M0)-(M6), it will allow us to ultimately prove that (a certain full sub-category of) StNaamen is a model category.

Proposition 12. *The set theoretic interpretation of the labelling of arrows in StNaamen is:*

- (f) an arrow $\mathcal{A} \rightarrow \mathcal{B}$ is labelled (f) if and only if for every $A \in \mathcal{A} \cup \{\emptyset\}$, $B \in \mathcal{B}$ and a finite subset $\{b_1, \dots, b_n\} \subseteq B$ there exists $A' \in \mathcal{A} \cup \{\emptyset\}$ such that $(A \cap B) \cup \{b_1, \dots, b_n\} \subseteq A'$.
- (wf) an arrow $\mathcal{A} \rightarrow \mathcal{B}$ is labelled (wf) if and only if for every $A \in \mathcal{A} \cup \{\emptyset\}$, $B \in \mathcal{B}$ and subset $B' \subseteq B$ such that $\text{card } B' \leq \text{card}(A \cap B) + \aleph_0$, there exists $A' \in \mathcal{A} \cup \{\emptyset\}$ such that $B' \subseteq A'$.
- (wc) an arrow $\mathcal{A} \rightarrow \mathcal{B}$ is labelled (wc) if and only if every $B \in \mathcal{B}$ is contained, up to finitely many elements, in some $A \in \mathcal{A} \cup \{\emptyset\}$ (i.e. $B \setminus A$ is finite for some $A \in \mathcal{A} \cup \{\emptyset\}$).
- (c) an arrow $\mathcal{A} \rightarrow \mathcal{B}$ is labelled (c) if and only if for every $\{B\} \rightarrow \mathcal{B}$ there exists $A \in \mathcal{A} \cup \{\emptyset\}$ such that $A \xrightarrow{\mathcal{B}} B$, where we define $\mathcal{A} \xrightarrow{\mathcal{B}} B$ if there exist $n \in \mathbb{N}$ and $\{B_0, \dots, B_n\} \rightarrow \mathcal{B}$ such that:
 - (a) $\text{card}(A \cap B_0) + \aleph_0 = \text{card } B_0 + \aleph_0$,
 - (b) $\text{card}(B_i \cap B_{i+1}) + \aleph_0 = \text{card } B_{i+1} + \aleph_0$ for all $0 \leq i < n$, and
 - (c) $B = B_n$.
- (w) an arrow $\mathcal{A} \rightarrow \mathcal{B}$ is labelled (w) if and only if for every $A \in \mathcal{A} \cup \{\emptyset\}$, $B \in \mathcal{B}$ and subset $B' \subseteq B$ such that $\text{card } B' \leq \text{card}(A \cap B) + \aleph_0$, there exists $A' \in \mathcal{A} \cup \{\emptyset\}$ such that B' is contained in A' up to finitely many elements.

Proof. (f): By definition (see also Remark 8), an arrow $\mathcal{A} \rightarrow \mathcal{B}$ is labelled

(f) if it has the right lifting property relative to all arrows $\{C\} \xrightarrow{(wc)_0} \{D\}$, i.e. all arrows of the form $\{C\} \rightarrow \{C, \bar{d}\}$ where \bar{d} is a finite set. Thus, if $A \in \mathcal{A}$, $B \in \mathcal{B}$ and $\{b_1, \dots, b_n\} \in B$ then

$$\begin{array}{ccc}
 \{A \cap B\} & \xrightarrow{\quad} & \mathcal{A} \\
 \downarrow (wc)_0 & \nearrow \text{dotted} & \downarrow (f) \\
 \{A \cap B, b_1, \dots, b_n\} & \xrightarrow{\quad} & \mathcal{B}
 \end{array}$$

and by definition this means that there exists $B' \in \mathcal{A}$ such that $(A \cap B) \cup \{b_1, \dots, b_n\} \subseteq B'$.

In the other direction:

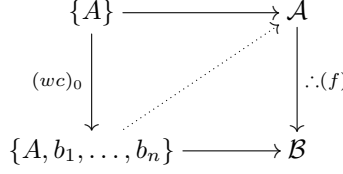


FIGURE 10. The lifting arrow exists by applying the assumption any $A \in \mathcal{A}$ and $\{\{b_1, \dots, b_n\}\} \rightarrow \mathcal{B}$. The conclusion $\mathcal{A} \xrightarrow{(f)} \mathcal{B}$ follows by the construction of Figure 3.

(wf): Exactly the same proof works.

(wc): Given $\mathcal{C} \rightarrow \mathcal{D}$ we let

$$\mathcal{C}_{\mathcal{D}} := \{(C \cap D) \cup \bar{d} : A \in \mathcal{A}, D \in \mathcal{D}, \bar{d} \subseteq D \text{ finite}\}.$$

Then obviously $\mathcal{C} \rightarrow \mathcal{C}_{\mathcal{D}} \rightarrow \mathcal{D}$, and by what we have just shown, $\mathcal{C}_{\mathcal{D}} \xrightarrow{(f)} \mathcal{D}$. Now assume that $\mathcal{C} \rightarrow \mathcal{D} \times \mathcal{A} \xrightarrow{(f)} \mathcal{B}$ for all $A \xrightarrow{(f)} B$. Then, in particular, $\mathcal{C} \rightarrow \mathcal{D} \times \mathcal{C}_{\mathcal{D}} \rightarrow \mathcal{D}$. Hence, as StNaamen has unique arrows $\mathcal{D} \rightarrow \mathcal{C}_{\mathcal{D}} \rightarrow \mathcal{D}$ implies that $\mathcal{C}_{\mathcal{D}} \cong \mathcal{D}$. Now, let $D \in \mathcal{D}$ be any element. Then $D \subseteq C_D$ for some $C_D \in \mathcal{C}_{\mathcal{D}}$. But, by definition, C_D is contained, up to finitely many elements, in some $C \in \mathcal{C}$. This proves that $\mathcal{C} \xrightarrow{(wc)} \mathcal{D}$ satisfies the combinatorial condition of the claim.

In the other direction, if $\mathcal{C} \rightarrow \mathcal{D}$ satisfies the combinatorial condition, then

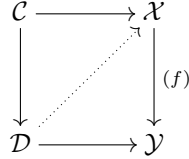


FIGURE 11. Given $D \in \mathcal{D}$ let $D \subseteq Y \in \mathcal{Y}$ and $C \in \mathcal{C}$ such that C contains D up to the finite set $\{d_1, \dots, d_n\}$. Let $C \subseteq X \in \mathcal{X}$, and $X' \in \mathcal{X}$ such that $(X \cap Y) \cup \{d_1, \dots, d_n\} \subseteq X'$, as assures $\mathcal{X} \xrightarrow{(f)} \mathcal{Y}$. So $D \subseteq X'$. Since D was arbitrary, the lifting arrow exists.

By the construction of Figure 3, $\mathcal{C} \xrightarrow{(wc)} \mathcal{D}$, as required.

(c): The proof is quite similar to the classification of (wc) -arrows. Given $\mathcal{A} \rightarrow \mathcal{B}$ let $\bar{\mathcal{B}} := \{B' \subseteq B : B \in \mathcal{B}\}$. Observe that $\mathcal{B} \cong \bar{\mathcal{B}}$. Define

$$\mathcal{A}^{\mathcal{B}} := \{B \in \bar{\mathcal{B}} : A \xrightarrow{\bar{\mathcal{B}}} B, \text{ some } A \in \mathcal{A}\}.$$

As before, $\mathcal{A} \rightarrow \mathcal{A}^{\mathcal{B}} \rightarrow \mathcal{B}$. We claim that $\mathcal{A}^{\mathcal{B}} \xrightarrow{(wf)} \mathcal{B}$. Indeed, let $A_B \in \mathcal{A}^{\mathcal{B}}$, $B \in \mathcal{B}$ and $B' \subseteq B$ be such that $\text{card}(A_B \cap B) + \aleph_0 \geq \text{card } B'$. We know that $A \xrightarrow{\bar{\mathcal{B}}} A_B$ for some $A \in \mathcal{A}$, so there are $B_0, \dots, B_n \in \bar{\mathcal{B}}$ witnessing this, and $B_n = A_B$. Denote $B_{n+1} = (B \cap A_B) \cup B'$. Since,

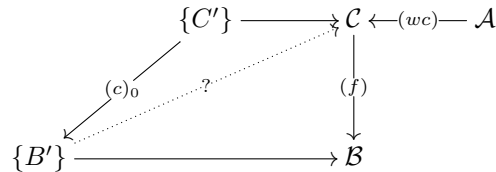
$B_{n+1} \in \bar{\mathcal{B}}$ and $\text{card}(B_{n+1} \cap B_n) + \aleph_0 = \text{card} B_{n+1} + \aleph_0$, we get that A, B_0, \dots, B_{n+1} witness that $A \xrightarrow{\bar{\mathcal{B}}} B_{n+1}$, so $B_{n+1} \in \mathcal{A}^{\mathcal{B}}$. Since $B' \subseteq B_{n+1}$ the combinatorial classification of (wf) -arrows yields $\mathcal{A}^{\mathcal{B}} \xrightarrow{(wf)} \mathcal{B}$.

Now assume that $\mathcal{A} \xrightarrow{(c)} \mathcal{B}$. We have to show that for $\{B\} \rightarrow \mathcal{B}$ there exists $A \in \mathcal{A}$ such that $A \xrightarrow{\mathcal{B}} B$. By the previous paragraph, $\mathcal{A} \xrightarrow{(c)} \mathcal{B} \prec \mathcal{A}^{\mathcal{B}} \xrightarrow{(wf)} \mathcal{B}$ and $\mathcal{B} \rightarrow \mathcal{A}^{\mathcal{B}} \rightarrow \mathcal{B}$. Namely, for every $\{B\} \rightarrow \mathcal{B}$ there is $A_B \in \mathcal{A}^{\mathcal{B}}$ such that $B \subseteq A_B$, and by definition $A \xrightarrow{\bar{\mathcal{B}}} A_B$ for some $A \in \mathcal{A}$ witnessed by $B_0, \dots, B_n \in \bar{\mathcal{B}}$. Note that $B_n = A_B \supseteq B$, so setting $B_{n+1} := B$ we get that $\text{card}(B_{n+1} \cap B_n) = \text{card} B_{n+1}$, implying that $A \xrightarrow{\mathcal{B}} B$.

For the other direction, we have to show that if $\mathcal{A} \rightarrow \mathcal{B}$ satisfies the combinatorial condition, then the arrow is labelled (c) . By (M6) it will suffice to show that if $\mathcal{C} \xrightarrow{(wf)} \mathcal{D}$ and $\mathcal{A} \rightarrow \mathcal{C}, \mathcal{B} \rightarrow \mathcal{D}$ then for every element $B \in \mathcal{B}$ there exists $C \in \mathcal{C}$ such that $B \subseteq C$. We do this for each such B separately by induction on the length of the (shortest possible) chain witnessing that $A \xrightarrow{\mathcal{B}} B$ for some $A \in \mathcal{A}$. For chains of length 0 this is obvious by construction (consider the lifting $\{A\} \xrightarrow{(c)} \{B\} \prec \mathcal{C} \xrightarrow{(wf)} \mathcal{D}$ for any A as above). Now assume that $A \xrightarrow{\mathcal{B}} B$ is witnessed by B_0, \dots, B_{n+1} . Note that $\{B_0, \dots, B_n\}$ witness that $A \xrightarrow{\mathcal{B}} B_n$. Therefore, by induction, there is some $C_n \in \mathcal{C}$ such that $B_n \subseteq C_n$. By definition, there is some $D \in \mathcal{D}$ such that $B_{n+1} \subseteq D$. Applying the combinatorial classification of (wf) -arrows to D, C_n and $B_{n+1} \subseteq D$, the result follows.

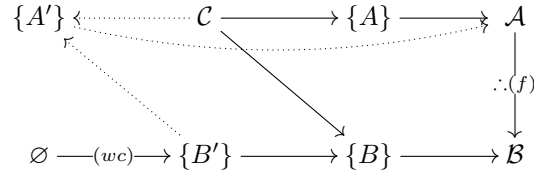
(w): Assume, first, that $\mathcal{A} \xrightarrow{(w)} \mathcal{B}$. By construction, $\mathcal{A} \xrightarrow{(wc)} \mathcal{C} \xrightarrow{(wf)} \mathcal{B}$. Now let $A \in \mathcal{A}, B \in \mathcal{B}$ be any elements and $B' \subseteq B$ such that $\text{card}(A \cap B) + \aleph_0 \geq \text{card} B'$. Let $C \in \mathcal{C}$ such that $A \subseteq C$. So $\text{card}(C \cap B) + \aleph_0 \geq \text{card}(A \cap B) + \aleph_0 \geq \text{card} B'$. By the classification of (wf) -arrows there is $C' \in \mathcal{C}$ such that $B' \subseteq C'$. By the classification of (wc) -arrows there is $A' \in \mathcal{A}$ such that C' is contained in A' up to finitely many elements. So B' is contained in A' , up to finitely many elements as required.

Now assume that $\mathcal{A} \rightarrow \mathcal{B}$ satisfies the combinatorial condition of the claim. We will show that there exists an object \mathcal{C} such that $\mathcal{A} \xrightarrow{(wc)} \mathcal{C} \xrightarrow{(wf)} \mathcal{B}$. By Axiom (M2) we know that $\mathcal{A} \xrightarrow{(wc)} \mathcal{C} \xrightarrow{(f)} \mathcal{B}$. So our goal is to show that $\mathcal{C} \xrightarrow{(wf)} \mathcal{B}$, i.e. — using Axiom (M6) — that it has the lifting property with respect to any (c) -arrow. It will suffice to show (see Figure 4) that $(c)_0 \prec \mathcal{C} \xrightarrow{(f)} \mathcal{B}$. Consider the following diagram:



Let $B \in \mathcal{B}$ be such that $B' \subseteq B$, let $C \in \mathcal{C}$ be such that $C' \subseteq C$, and let $A \in \mathcal{A}$ be such that $C \subseteq A$ up to a finite set (as provided by the combinatorial classification of (wc) -arrows). So $\text{card}(A \cap B) + \aleph_0 = \text{card}(C \cap B) + \aleph_0 \geq \text{card} C' + \aleph_0 = \text{card} B' + \aleph_0$. Applying the assumption we get a set $A' \in \mathcal{A}$ such that B' is contained in A' up to a finite set \bar{b} . So Let $C'' \in \mathcal{C}$ be such that $A' \subseteq C''$. By the classification of (f) -arrows there exists $C''' \in \mathcal{C}$ such that $C''' \supseteq (C'' \cap B) \cup \bar{b}$. But $C'' \cap B \supseteq A' \cap B$, so $(C'' \cap B) \cup \bar{b} \supseteq (A' \cap B) \cup \bar{b} \supseteq B'$. Thus, $C''' \supseteq B'$, as required \square

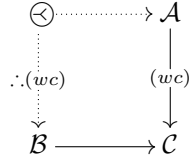
In the proof of Proposition 12 we focused on the simplest and shortest proofs using set theoretic ideas that may not be easy to formalise in an algorithm. It may be worth stressing that the same result could have been obtained using more algorithmic diagram chasing arguments using only some basic properties of infinite cardinality, finite and countable sets. For example, the characterisation of (f) -arrows can be described as follows:



So we only have to verify that an arrow has the lifting property with respect to all $(wc)_0$ -arrows if and only if it satisfies the above diagram, which is an easy argument. Similar arguments could be applied to obtain all other classifications.

We will need later the following:

Corollary 13. *In StNaamen the following diagram is true:*



Proof. Recall that the inverse limit of \mathcal{A}, \mathcal{B} is $\mathcal{A} \cap \mathcal{B}$. Now, if $\{B\} \longrightarrow \mathcal{B} \longrightarrow \mathcal{C}$ there exists $\{C\} \longrightarrow \mathcal{C}$ such that $\{B\} \longrightarrow \{C\}$ and $\{A\} \longrightarrow \mathcal{A}$ such that $C \setminus A$ is finite. Thus, B is contained in A up to finitely many elements. Since $\{B\} \longrightarrow \mathcal{B}$ was arbitrary, by Proposition 12 the corollary follows. \square

2.5. The final round: the model category QtNaamen. In the previous subsections we proved that the labelled category StNaamen satisfies axioms (M0)-(M4) of a closed model category. The reason we have not proved that it also satisfies (M5) is that this is not true: consider the diagram of Figure 12 without the middle arrow. By definition of morphisms in StNaamen the middle arrow exists. By

Proposition 12 it is labelled (c). Considering the left hand side of the diagram, the left and top arrows are trivial fibrations, by Proposition 12, so if (M5) were true, the middle arrow would be a weak equivalence. Combining those two observations, we conclude that if (M5) were true in StNaamen the middle arrow would be labelled (wc). However, by Proposition 12 this would imply that $A \setminus B$ is finite, which need not be the case.

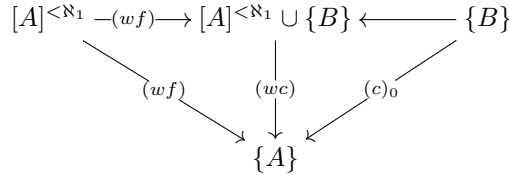


FIGURE 12. For a set A we let $[A]^{<\aleph_1}$ be the set of all countable subsets of A .

So we, somehow, must get rid of (at least) those diagrams of the form

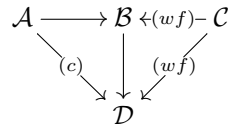


FIGURE 13

where the middle arrow is not a trivial fibration. By considering $\mathcal{B} = \mathcal{D}$ and noting that arrows $\mathcal{A} \xrightarrow{(c)} \mathcal{D}$ and $\mathcal{C} \xrightarrow{(wf)} \mathcal{D}$ may be arbitrary, the reader can easily convince herself that a reasonable way of avoiding such diagrams is to get rid of those arrows which fit as the middle arrow in Figure 13. A homotopy theoretic means of getting rid of arrows is by using the lifting property. The two key observations for achieving this goal are that (a) if an arrow h is an isomorphism, we have no need of getting rid of it and (b) if K is any class of morphisms then $K^\triangleleft \cap K$ contains only isomorphisms. Thus, we let K be the class of all morphisms in StNaamen fitting as the middle arrow in Figure 13. We let K^\triangleleft be the obvious sub-category of StNaamen and \mathfrak{H} the (full) sub-category of K^\triangleleft consisting of those object \mathcal{X} such that both $\emptyset \rightarrow \mathcal{X}$ and $\mathcal{X} \rightarrow \top$ are in K^\triangleleft . This is readily seen to be a full sub-category of StNaamen. This category can be defined as follows:

Definition 8. An object $\mathcal{X} \in \mathcal{ObStNaamen}$ is cute if the following diagram is true:

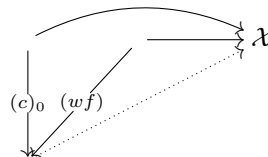


FIGURE 14

We let $QtNaamen$ be the full sub-category of cute objects in StNaamen

Remark 14. It follows right from the definition that for every set A the singleton $\{A\}$ is an object in $QtNaamen$. In particular, all $(c)_0$ -arrows survive the passage to $QtNaamen$.

Remark 15. It is now a easy exercise to verify that $\mathcal{X} \rightarrow \top$ and $\emptyset \rightarrow \mathcal{X}$ are both in K^\wedge if and only if \mathcal{X} is cute.

From the homotopy theoretic viewpoint, the following is a more satisfying definition of $QtNaamen$:

Lemma 16. Let \mathcal{X} be an object of $StNaamen$. Then \mathcal{X} is cute if and only if the following diagram is true for every choice of solid arrows.

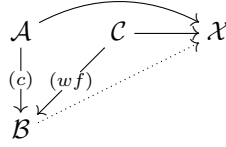


FIGURE 15

Proof. Clearly, if \mathcal{X} satisfies the diagram it must be cute. So we proceed to prove the other direction. We suppose that \mathcal{X} is cute, and verify that it satisfies the above diagram. When given the above diagram we have to find a lifting arrow. It will suffice to show that for every $B \in \mathcal{B}$ there exists an arrow $\{B\} \rightarrow \mathcal{X}$. Fixing such a B and expanding the set theoretic definition of $\mathcal{A} \xrightarrow{(c)} \mathcal{B}$ we chooses A and $\{B_0, \dots, B_n\}$ (with $B_n = B$) witnessing that $A \xrightarrow{\mathcal{B}} B$, getting following diagram:

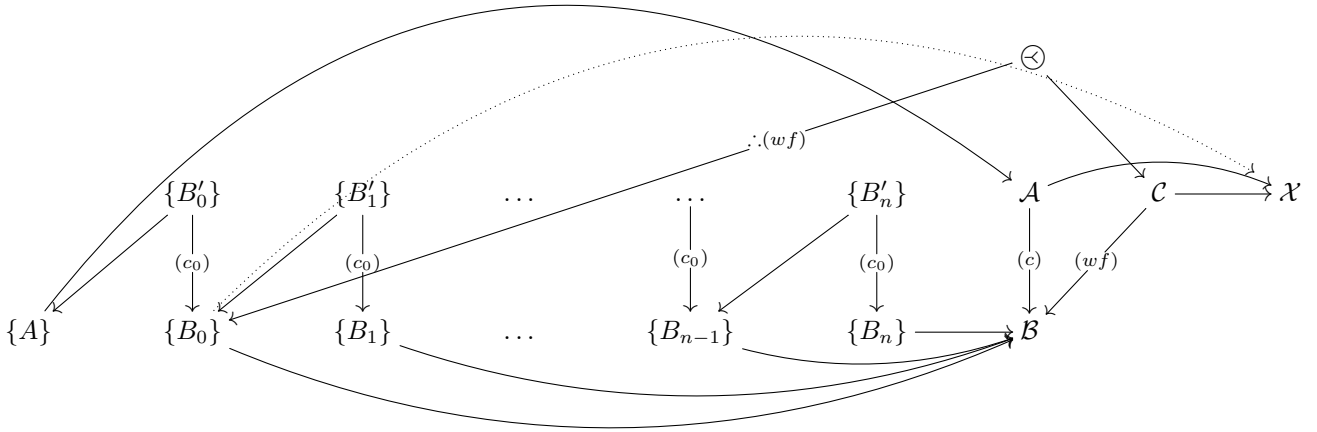


FIGURE 16. To obtain the labelling of the arrow from the limit of $\{B_0\}$ and C to $\{B_0\}$ use Axiom (M3) and Axiom (M4). To obtain the lifting arrow use the assumption with respect to $\{B'_0\} \xrightarrow{(c)_0} \{B_0\}$ and the previously discussed arrow. Observe that $\{B'_0\} \rightarrow \mathcal{X}$ as the composition $\{B'_0\} \rightarrow \{A\} \rightarrow \mathcal{A} \rightarrow \mathcal{X}$.

Observe that if $\mathcal{A} \xrightarrow{(c)} \mathcal{B}$ and $\{B\} \rightarrow \mathcal{B}$ then $\mathcal{A}_B := \mathcal{A} \cup \{B\}$ also satisfies $\mathcal{A}_B \xrightarrow{(c)} \mathcal{B}$. So, using the lifting arrow we found in Figure 16 we can replace \mathcal{A} with \mathcal{A}_B , and now B has a chain of length at most $n - 1$ witnessing that $B_0 \xrightarrow{B} B$. So induction gives the desired result. \square

Summing up the above discussion it may be worth pointing out :

Lemma 17. *Let $\mathcal{X} \in \text{ObStNaamen}$. Then the following conditions are equivalent:*

- (1) $\mathcal{X} \in \text{ObQtNaamen}$.
- (2) If $\mathcal{C} \xrightarrow{(wf)} \mathcal{B}$ then $\mathcal{A} \xrightarrow{(c)} \mathcal{B} \triangleleft \mathcal{X} \rightarrow T$ and if $\mathcal{A} \xrightarrow{(c)} \mathcal{B}$ then $\mathcal{C} \xrightarrow{(wf)} \mathcal{B} \triangleleft \mathcal{X} \rightarrow T$ for all objects \mathcal{A}, \mathcal{B} and \mathcal{C} of StNaamen such that $\mathcal{A}, \mathcal{C} \rightarrow \mathcal{X}$.
- (3) $\bigcup \{X'' : \text{for some } X_0, X_1, X'' \xleftarrow{(wf)} X_0 \rightarrow \mathcal{X} \leftarrow X_1 \xrightarrow{(c)} X''\} \dashrightarrow \mathcal{X}$
- (4) If $\{a\} \rightarrow \mathcal{X}$, $\mathcal{C} \rightarrow X$ and $\{a\} \xrightarrow{(c)} \{b\}$ and $\mathcal{C} \xrightarrow{(wf)} \{b\}$, then $\{b\} \rightarrow \mathcal{X}$.
- (5) For any set A , if $A^{\leq \aleph_0} \rightarrow \mathcal{X}$ then $A^{\leq \text{card}(x \cap A)} \rightarrow \mathcal{X}$ for all $x \in \mathcal{X}$ (where $A^{\leq \lambda} := \{L \subseteq A : \text{card } L \leq \lambda\}$)

We now proceed to show that QtNaamen is, indeed, a model category. First we observe, that there are many cute objects:

Lemma 18. *Assume that \mathcal{C} is a cofibrant object of StNaamen . Then \mathcal{C} is cute.*

Proof. See the following diagrams:

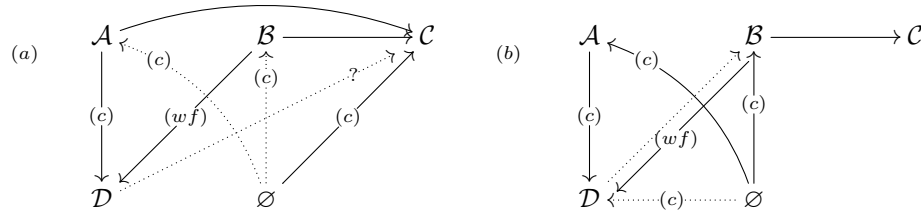


FIGURE 17. (a) The new (c) -arrows exists by Claim 1(4). (b) The arrow $\emptyset \rightarrow \mathcal{D}$ is labelled (c) as a combination of $\emptyset \xrightarrow{(c)} \mathcal{A} \xrightarrow{(c)} \mathcal{D}$. The lifting arrow $\mathcal{D} \rightarrow \mathcal{B}$ exists by $\emptyset \xrightarrow{(c)} \mathcal{D} \triangleleft \mathcal{B} \xrightarrow{(wf)} \mathcal{D}$.

\square

Note that given any object \mathcal{A} in StNaamen by Axiom (M2) we know that $\emptyset \rightarrow \mathcal{A} = \emptyset \xrightarrow{(c)} \mathcal{C} \xrightarrow{(wf)} \mathcal{A}$ for some \mathcal{C} . So any object in StNaamen is weakly equivalent to a cofibrant, hence cute, object. The following will be handy:

Notation 19. *Let \mathcal{X} be any object in StNaamen . We denote \mathcal{X}_c the unique object such that $\emptyset \xrightarrow{(c)} \mathcal{X}_c \xrightarrow{(wf)} \mathcal{X}$.*

How far could QtNaamen be from being a model category? Axioms (M3) and (M4) of model categories are preserved when restricting to an induced subcategory. Since the subcategory is full, the same is also true of Axiom (M1). But the remaining axioms have to be re-verified (since it is not clear, for example, that if $\mathcal{A} \rightarrow \mathcal{B} = \mathcal{A} \xrightarrow{(wc)} \mathcal{C} \xrightarrow{(f)} \mathcal{B}$ and \mathcal{A}, \mathcal{B} are objects in QtNaamen so is \mathcal{C} . Moreover,

though QtNaamen may have limits (Axiom (M0)) it is clear from the example analysed in the end of the previous sub-section that those will not, in general, be the same limits as in StNaamen.

So we start by showing that QtNaamen has limits. As we will see below, the inverse limit (in StNaamen) of cute objects is readily checked to be cute itself, but the same is obviously not true for direct limits. In order to show that QtNaamen has direct limits we will prove a stronger property. We will show that any object \mathcal{X} in StNaamen has a *Qt-fication*, namely, an object $\tilde{\mathcal{X}}$ such that:

- (1) $\tilde{\mathcal{X}}$ is in QtNaamen.
- (2) $\mathcal{X} \longrightarrow \tilde{\mathcal{X}}$.
- (3) If \mathcal{Y} is in QtNaamen and $\mathcal{X} \longrightarrow \mathcal{Y}$ (in StNaamen) then $\tilde{\mathcal{X}} \longrightarrow \mathcal{Y}$.

So $\tilde{\mathcal{X}}$ (if it exists) is the object in QtNaamen *closest* to \mathcal{X} . In particular, this would imply that, given any diagram \mathcal{D} , in QtNaamen if \mathcal{L} is the direct limit of \mathcal{D} (in StNaamen) then $\tilde{\mathcal{L}}$ must be its limit in QtNaamen.

Remark 20. *The third point in the definition of the Qt-fication has a category theoretic interpretation. It can be described as the solution to a lifting property problem: $\mathcal{X} \longrightarrow \tilde{\mathcal{X}} \triangleleft \mathcal{Y} \longrightarrow \mathcal{Z}$ for every arrow $\mathcal{Y} \longrightarrow \mathcal{Z}$ in QtNaamen. To see that this implies (3), take $\mathcal{Z} = \top$ the terminal object: if \mathcal{Y} is in QtNaamen and $\mathcal{X} \longrightarrow \mathcal{Y}$ (in StNaamen) then $\mathcal{Y} \longrightarrow \top$ is in QtNaamen, $\tilde{\mathcal{X}} \longrightarrow \top$ and the lifting property implies $\tilde{\mathcal{X}} \longrightarrow \mathcal{Y}$. The uniqueness of the Qt-fication implies that, indeed, the two conditions are equivalent.*

From the category theoretic point of view, it is clear how to construct the Qt-fication of an object \mathcal{X} . Such an object, \mathcal{X} , is in StNaamen but not in QtNaamen if and only if there are \mathcal{A}, \mathcal{B} and \mathcal{C} as in the solid arrows of Figure 15, but such that the lifting arrow $\mathcal{B} \longrightarrow \mathcal{C}$ does not exist. The object in StNaamen “closest” to \mathcal{X} for which such a lifting arrow exists is the direct limit of \mathcal{X} and \mathcal{B} . Thus, the first candidate for the Qt-fication of \mathcal{X} is the direct limit of all objects \mathcal{B} for which there exist \mathcal{A}, \mathcal{C} such that Figure 15 holds of $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and \mathcal{X} .

Notation 21. *Given \mathcal{X} , an object of StNaamen, denote*

$$\tilde{\mathcal{X}} := \bigcup \{ \mathcal{X}'' : \mathcal{X} \longleftarrow \mathcal{X}_0 \xrightarrow{(c)} \mathcal{X}'' \xleftarrow{(wf)} \mathcal{X}_1 \longrightarrow \mathcal{X} \}.$$

In other words, given \mathcal{X} the object $\tilde{\mathcal{X}}$ is the direct limit of all those objects \mathcal{X}'' for which the following diagram holds:

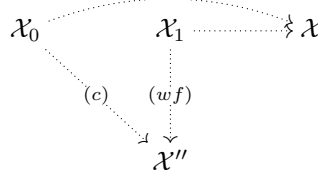


FIGURE 18. The object $\tilde{\mathcal{X}}$ is the direct limit of all \mathcal{X}'' satisfying the diagram.

Remark 22. *It is not hard to check that the cute objects in StNaamen have a set theoretic characterisation in the spirit of Proposition 12. Using such a characterisation, one can then, naturally, arrive at a set theoretic definition of the Qt-fication. Not surprisingly, these two definitions will coincide.*

So it remains to prove:

Lemma 23. *For every \mathcal{X} in $StNaamen$ the object $\tilde{\mathcal{X}}$ is the Qt-fication of \mathcal{X} .*

We have three properties to verify, of which the requirement that $\mathcal{X} \rightarrow \tilde{\mathcal{X}}$ is obvious. So we proceed to proving:

Claim 24. *Let \mathcal{V} be an object in $QtNaamen$. Then for all \mathcal{X} , if $\mathcal{X} \rightarrow \mathcal{V}$ then $\tilde{\mathcal{X}} \rightarrow \mathcal{V}$.*

Proof. Let $\{B\} \in \tilde{\mathcal{X}}$. So the following diagram is true:

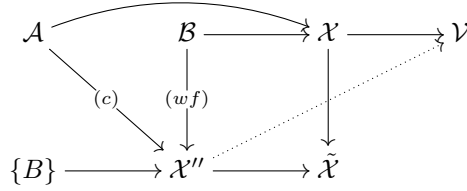


FIGURE 19. The objects $\mathcal{A}, \mathcal{B}, \mathcal{X}''$ (with their associated labelled arrows) exist by the definition of $\tilde{\mathcal{X}}$. The arrow $\mathcal{X}'' \rightarrow \mathcal{V}$ exists because \mathcal{V} is in $QtNaamen$.

□

We will also need the following:

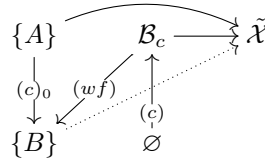
Claim 25. *For any cofibrant object \mathcal{A} in $StNaamen$ we have $\emptyset \xrightarrow{(c)} \mathcal{A} \times \mathcal{X} \rightarrow \tilde{\mathcal{X}}$.*

Proof. Assume first that \mathcal{A} is the singleton $\{A\}$ and $\emptyset \xrightarrow{(c)_0} \{A\}$. By definition of $\tilde{\mathcal{X}}$ the fact that $\{A\} \rightarrow \tilde{\mathcal{X}}$ implies that there exists \mathcal{X}'' such that $\{A\} \rightarrow \mathcal{X}''$ and such that \mathcal{X}'' is as in Figure 18. In particular there exists some \mathcal{X}_1 such that $\mathcal{X}_1 \xrightarrow{(wf)} \mathcal{X}''$. Therefore, we have $\emptyset \rightarrow \{A\} \times \mathcal{X}_1 \rightarrow \mathcal{X}''$. So that $\{A\} \rightarrow \mathcal{X}_1 \rightarrow \mathcal{X}$ is the required arrow. Finally, observe that by the combinatorial classification of cofibrant objects, we know that \mathcal{A} is cofibrant if and only if $\emptyset \xrightarrow{(c)_0} \{A\}$ for all $A \in \mathcal{A}$. So a cofibrant object is a direct limit of sets $\emptyset \xrightarrow{(c)_0} A$, and since the lifting arrow exists for every such A it also exists for the limit. □

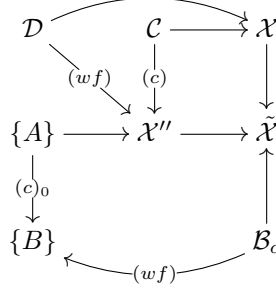
And we conclude the proof of the lemma with:

Claim 26. *For every object \mathcal{X} in $StNaamen$ the object $\tilde{\mathcal{X}}$ is in $QtNaamen$.*

Proof. It is enough to show that for any choice of solid arrows the following diagram is true:



Since $\{A\}$ is a singleton, $\{A\} \rightarrow \tilde{\mathcal{X}}$ implies that there exists $\mathcal{X}'' \rightarrow \tilde{\mathcal{X}}$ such that for some \mathcal{C}, \mathcal{D}



Taking the limit (in StNaamen) of \mathcal{X}'' and $\{B\}$, we obtain:

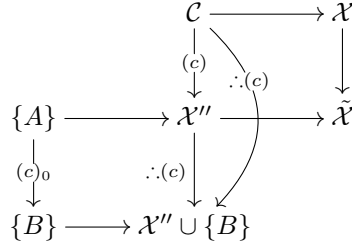


FIGURE 20. The arrow $\mathcal{X}'' \rightarrow \mathcal{X}'' \cup \{B\}$ is labelled (c) by Axiom (M3)(c). The arrow $\mathcal{C} \rightarrow \mathcal{X}'' \cup \{B\}$ is labelled (c) as the composition of two (c) -arrows.

We intend to show that if $\emptyset \xrightarrow{(c)} \mathcal{Y} \xrightarrow{(wf)} \mathcal{X}'' \cup \{B\}$ then $\mathcal{Y} \rightarrow \mathcal{X}$, as — $X \leftarrow \mathcal{C} \xrightarrow{(c)} \mathcal{X}'' \cup \{B\} \xleftarrow{(wf)} Y \rightarrow X$ is a diagram as in Figure 18 — implying that $\mathcal{X}'' \cup \{B\} \rightarrow \tilde{\mathcal{X}}$, and in particular $\{B\} \rightarrow \tilde{\mathcal{X}}$, which is what we want.

By Claim 25, as \mathcal{Y} is cofibrant, it will suffice to show that $\mathcal{Y} \rightarrow \tilde{\mathcal{X}}$. To check this it will suffice that he shows that $\mathcal{Y} \rightarrow \mathcal{D} \cup \mathcal{B}_c$. Since $\mathcal{D} \xrightarrow{(wf)} \mathcal{X}''$, if we let $\emptyset \xrightarrow{(c)} \mathcal{X}_c'' \xrightarrow{(wf)} \mathcal{X}''$ then $\emptyset \xrightarrow{(c)} \mathcal{X}_c'' \times \mathcal{D} \xrightarrow{(wf)} \mathcal{X}''$ implies that $\mathcal{X}_c'' \rightarrow \mathcal{D}$. Thus, in fact, it will suffice to show that $\mathcal{Y} \rightarrow \mathcal{X}_c'' \cup \mathcal{B}_c$, which is immediate from Proposition 12 (from which it follows that $(\mathcal{X}'' \cup \{B\})_c = \mathcal{X}_c'' \cup \mathcal{B}_c$). \square

As discussed above, Lemma 23 implies, in particular, that QtNaamen is closed under direct limits. The following lemma shows that the same is true for inverse limits.

Claim 27. *The category QtNaamen satisfies Axiom (M0).*

Proof. Assume that $\mathcal{X}_1, \dots, \mathcal{X}_n$ are objects in QtNaamen then

$$\bigcap_{i=1}^n \mathcal{X}_i := \left\{ \bigcap_{i=1}^n X_i : X_i \in \mathcal{X}_i \right\}$$

is the inverse limit of the \mathcal{X}_i in QtNaamen and $\tilde{\mathcal{D}}$ is their direct limit in QtNaamen, where

$$\mathcal{D} := \bigcup_{i=1}^n \mathcal{X}_i.$$

Since $\mathcal{I} := \bigcap_{i=1}^n \mathcal{X}_i$ is the inverse limit of the \mathcal{X}_i in StNaamen it will suffice to show, in order to prove the first part of the claim, that it is in QtNaamen. But this is obvious, due to the following diagram:

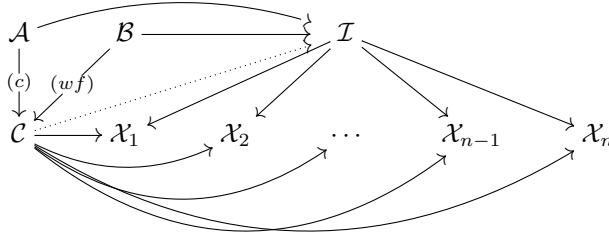


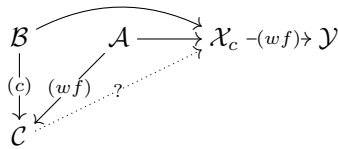
FIGURE 21. The arrows from \mathcal{C} to all the \mathcal{X}_i exists by the definition of QtNaamen. The lifting arrow exists by virtue of \mathcal{I} being the inverse limit of the \mathcal{X}_i in StNaamen.

The second part of the lemma follows from the properties of the Qt-fication. \square

It remains to prove Axioms (M5) and (M2) (and hopefully also (M6)). The first part of Axiom (M2) is easy:

Claim 28. *If $\mathcal{X} \rightarrow \mathcal{Y}$ in QtNaamen and $\mathcal{X} \xrightarrow{(c)} \mathcal{X}_c \xrightarrow{(wf)} \mathcal{Y}$ then \mathcal{X}_c is an object in QtNaamen.*

Proof. We have to check that



But by the definition of QtNaamen, and Axiom (M1) we know:

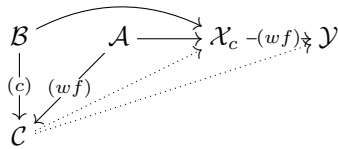
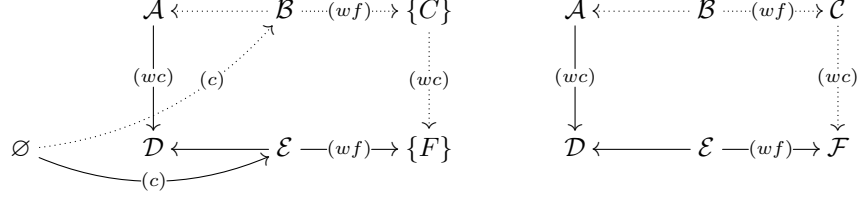


FIGURE 22. The lifting arrow to \mathcal{Y} exists by definition of QtNaamen. The arrow to \mathcal{X}_c is obtained by $\mathcal{B} \xrightarrow{(c)} \mathcal{C} \xrightarrow{(wf)} \mathcal{X}_c \xrightarrow{(wf)} \mathcal{Y}$.

\square

The proof of the second part of Axiom (M2) is harder. This is the only proof in this part of the paper for which we do not have an obvious algorithmic proof. It will be interesting to know whether a simpler proof exists (at least in the sense that it does not use in a significant way infinitary concepts). The key lemma is:

Lemma 29 (A continuity fixed-point argument). *The following diagrams are true:*



Proof. First, we show that the two diagrams are equivalent. Of course, we only have to show that the diagram on the left implies the one on the right. So assume the diagram on the left is true, and we are given \mathcal{A} , \mathcal{D} , \mathcal{E} and \mathcal{F} as in the figure on the right. First, observe that we may assume without loss that $\mathcal{E} \cong \mathcal{F}_c$, so without loss $\emptyset \xrightarrow{(c)} \mathcal{E}$.

We can now apply the diagram on the left as follows: for every $F \in \mathcal{F}$ observe that \mathcal{A} , \mathcal{D} , $\mathcal{F}_c \cap \{F\}$ and $\{F\}$ satisfy the assumptions of the left hand side diagram. So we can find \mathcal{B}_F and $\{C_F\}$ as provided there. Taking $\mathcal{B} := \bigcup \{\mathcal{B}_F : F \in \mathcal{F}\}$ and $\mathcal{C} := \bigcup \{C_F : F \in \mathcal{F}\}$ we get $\emptyset \xrightarrow{(c)} \mathcal{B}$ and by Proposition 12, $\mathcal{B} \xrightarrow{(wf)} \mathcal{C}$, and $\mathcal{C} \xrightarrow{(wc)} \mathcal{F}$ so the diagram on the right is satisfied.

To prove the diagram on the left, first observe that using Proposition 12 it is clear that $\mathcal{A} \xrightarrow{(wc)} \mathcal{D}$ implies that $\mathcal{A}_c \xrightarrow{(wc)} \mathcal{D}_c$. Since $\mathcal{A}_c \rightarrow \mathcal{A}$ it will suffice to prove the diagram with \mathcal{A}_c instead of \mathcal{A} and \mathcal{D}_c instead of \mathcal{D} . So we may assume that \mathcal{A} is a family of countable sets. Thus, combinatorially, the lemma asserts that given a family \mathcal{A} of countable sets and a set F such that for every countable $F_c \subseteq F$ there exists some $A \in \mathcal{A}$ such that $\text{card}(F_c \setminus A) < \aleph_0$ then there exists a sets $C \subseteq F$ such that $\text{card}(F \setminus C) < \aleph_0$ and every countable subset of C is a subset of an element of \mathcal{A} .

Assume by way of contradiction that this is not the case. I.e., we assume that for every finite $\bar{b} \subseteq F$ there exists a countable set $C_{\bar{b}} \subseteq F$ such that $C_{\bar{b}}$ is not contained in any element of \mathcal{A} . Let $C_0 = C_\emptyset$. Define inductively for $i > 0$

$$C_{i+1} = C_i \cup \{C_{\bar{b}} : \bar{b} \in [C_i]^{<\omega}, C_i \setminus \bar{b} \subseteq A_{i,\bar{b}}, \text{ some } A_{i,\bar{b}} \in \mathcal{A}\}$$

Let $C_\omega := \bigcup_{i \in \omega} C_i$. Then C_ω is countable (as the countable union of countable sets). But $C_\omega \setminus \bar{b} \not\subseteq A$ for all finite $\bar{b} \subseteq C_\omega$ and $A \in \mathcal{A}$, a contradiction. Indeed, note that for all \bar{b} as above there exists $n \in \omega$ such that $\bar{b} \subseteq C_n$. So $C_{\bar{b}} \subseteq C_{n+1} \subseteq C_\omega$, with the desired conclusion. \square

For future reference, we point out the following generalization of the fixed point argument, appearing in the last part of the above proof:

Remark 30. *Let κ be a regular cardinal, \mathcal{A} a class of sets of cardinality smaller than κ , and F any set. Assume that for all $\bar{b} \subseteq F$ with $\text{card}(\bar{b}) < \kappa$ there exists $C_{\bar{b}} \subseteq F$ with $\text{card}(C_{\bar{b}}) \leq \kappa$ such that $C_{\bar{b}} \not\subseteq A$ for all $A \in \mathcal{A}$. Then there exists $C \subseteq F$ with $\text{card} C = \kappa$ such that $\text{card}(C \setminus A) = \kappa$ for all $A \in \mathcal{A}$.*

The proof of the above remark goes through precisely as in the above lemma, with a single exception. In the fixed point argument above we define $C_0 = C_\emptyset$ and

$$C_{i+1} = C_i \cup \{C_{\bar{b}} : \bar{b} \in [C_i]^{<\omega}, C_i \setminus \bar{b} \subseteq A_{i,\bar{b}}, \text{ some } A_{i,\bar{b}} \in \mathcal{A}\}.$$

If we tried to repeat the same line of reasoning replacing systematically ω with κ we might run into trouble, since $\text{card } C_0 = \kappa$, but in general, it will no longer be true that $\text{card}([C_0]^{<\kappa}) = \kappa$, and so already C_1 could be too large. Observe, however, that in order to make the argument go through we need not go over all $\bar{b} \in [C_0]^{<\kappa}$. Indeed, all we need is to go over a *covering* family of such subsets. Namely, the argument would go through unaltered if at stage $i+1$ we fixed $\mathcal{C}_i \subseteq [C_i]^{<\kappa}$ such that for $\bar{b} \in [C_i]^{<\kappa}$ there exists $\bar{c} \in \mathcal{C}_i$ with $\bar{b} \subseteq \bar{c}$. Thus, it suffices to show that we can choose \mathcal{C}_i to be of cardinality at most κ . The minimum cardinality of such a family \mathcal{C}_i is precisely $\text{cov}(\kappa, \kappa, \kappa, 2)$ (see [21, §5] for the definition). And the fact that this cardinality is precisely κ for a regular cardinal κ is an immediate consequence of [21] Observation 5.2(2), 5.2(5) and Observation 5.3(2).

The study of further relations between Shelah's covering numbers $\text{cov}(\lambda, \kappa, \theta, \sigma)$ and various constructs arising from the analysis of the model category QtNaamen and its variants is the main theme of the second part of this paper.

We can now return to the proof that QtNaamen is a model category.

Lemma 31. *If $\mathcal{X} \rightarrow \mathcal{Y}$ in QtNaamen and $\mathcal{X} \xrightarrow{(wc)} \mathcal{X}_f \xrightarrow{(f)} \mathcal{Y}$ then \mathcal{X}_f is an object in QtNaamen .*

Proof. Consider the following diagram:

$$\begin{array}{ccccc}
 & & & & \mathcal{X} \\
 & & & & \downarrow \\
 & & & & (wc) \\
 & & & & \downarrow \\
 \{B\} & \xrightarrow{\quad} & \mathcal{A} & \xrightarrow{\quad} & \mathcal{X}_f \\
 \downarrow (c)_0 & \swarrow (wf) & \searrow ? & \downarrow (f) & \downarrow (f) \\
 \{C\} & \xrightarrow{\quad} & & \xrightarrow{\quad} & \mathcal{Y}
 \end{array}$$

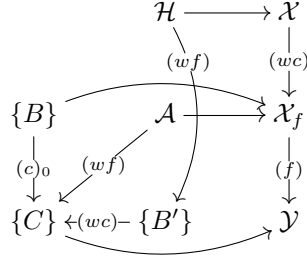
FIGURE 23. The arrow $\{C\} \rightarrow \mathcal{Y}$ exists because $\mathcal{Y} \in \text{ObQtNaamen}$.

Using the arrow $\{C\} \rightarrow \mathcal{Y}$ it will suffice to find an object \mathcal{B}' (in StNaamen) such that $\mathcal{B}' \xrightarrow{(wc)} \{C\}$ and $\mathcal{B}' \rightarrow \mathcal{X}_f$, as then we will obtain the desired conclusion from $\mathcal{B}' \xrightarrow{(wc)} \{C\} \xrightarrow{(f)} \mathcal{Y}$. Indeed, it will be enough to show that \mathcal{B}' we find satisfies $\mathcal{B}' \rightarrow \mathcal{X}$ rather than $\mathcal{B}' \rightarrow \mathcal{X}_f$, which is an advantage, since — towards that end — we can exploit the fact that \mathcal{X} is in QtNaamen .

In the notation of Figure 23, the system

$$\mathcal{X} \xrightarrow{(wc)} \mathcal{X}_f \longleftarrow \mathcal{A} \xrightarrow{(wf)} \{C\}$$

satisfies the assumption of Lemma 29. So we can add some arrows to Figure 23:



Now, if we can find some $\mathcal{I} \longrightarrow \mathcal{X}$ such that $\mathcal{I} \xrightarrow{(c)} \{B'\}$, we will be done. For then, since \mathcal{X} is in QtNaamen, we obtain the lifting arrow $\{B'\} \longrightarrow \mathcal{X}$ that we need. So it remains to find \mathcal{I} . But this is now easy: Let $\{B''\}$ be the inverse limit of \mathcal{X} and $\{B\}$. Then by Corollary 13 $\{B''\} \xrightarrow{(wc)} \{B\}$. By the same argument, setting \mathcal{I} to be the inverse limit of $\{B''\}$ and $\{B'\}$ we have $\mathcal{I} \xrightarrow{(c)} \{B'\}$ and $\mathcal{I} \longrightarrow \{B''\} \longrightarrow \mathcal{X}$, which is what we wanted.

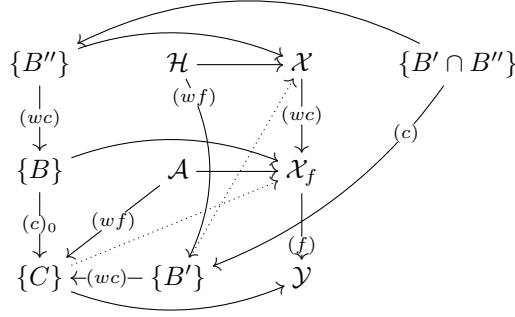


FIGURE 24. The arrow $\{B'\} \longrightarrow \mathcal{X}$ exists because \mathcal{X} is in QtNaamen. The arrow $\{C\} \longrightarrow \mathcal{X}_f$ comes from $\{B'\} \xrightarrow{(wc)} \{C\} \xrightarrow{(f)} \mathcal{X}_f$

□

Now it is easy to verify:

Lemma 32. *The labelled category QtNaamen satisfies Axiom (M6).*

The proof is a diagram chasing argument using (M2) in QtNaamen and (M1) and (M6) in StNaamen which we leave as an exercise to the reader. The remaining task is proving Axiom (M5) for QtNaamen. To explain the ideology of the proof we remind that, in some sense, weak equivalence should be thought of as an equivalence relation. So we are looking for an equivalence relation that may capture the notion of weak equivalence. Recall that by Proposition 12 $\mathcal{X} \xrightarrow{(w)} \mathcal{Y}$ if and only if, for every $X \in \mathcal{X}$ and every $Y' \subseteq Y \in \mathcal{Y}$, if $\text{card } Y' \leq \text{card}(X \cap Y) + \aleph_0$ then Y' is contained, up to finitely many elements in some $X' \in \mathcal{X}$.

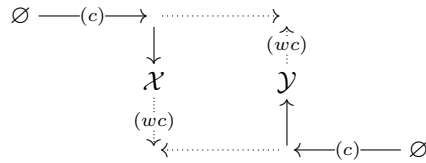
As we already know, this is not an equivalence relation in StNaamen. So we look for another characterisation of weak equivalences which, in QtNaamen, may

turn out to be an equivalence relation. The following, countable version of weak equivalence, is a special case:

Definition 9. Let \mathcal{X}, \mathcal{Y} be objects in *StNaamen*. Denote $\mathcal{X} \sim^w \mathcal{Y}$ if for every countable set L , there exists $X \in \mathcal{X}$ such that $L \setminus X$ is finite if and only if there exists $Y \in \mathcal{Y}$ such that $L \setminus Y$ is finite.

In other words, $\mathcal{X} \sim^w \mathcal{Y}$ if any countable set L is almost covered by an element of \mathcal{X} if and only if it is almost covered by an element of \mathcal{Y} . The following is now obvious:

Claim 33. The relation \sim^w is an equivalence relation. It can be expressed by the following diagram:

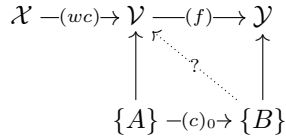


Proof. That \sim^w is an equivalence relation is obvious from the definition. We will show that the figure above is equivalent to $\mathcal{X} \sim^w \mathcal{Y}$. First we show that if \mathcal{X}, \mathcal{Y} satisfy the above figure then they are \sim^w equivalent. So assume that L is a countable set such that $L \subseteq X \cup \ell$ for some $X \in \mathcal{X}$ and finite ℓ . Let $L' := L \setminus \ell$. Then $\emptyset \xrightarrow{(c)} \{L'\} \rightarrow \mathcal{X}$. Then there exists Z such that $\mathcal{Y} \xrightarrow{(wc)} Z \leftarrow \{L'\}$. So there exists $Z \in \mathcal{Z}$ such that $L' \subseteq Z$. By the combinatorial classification of (cw) -arrows there is $Y \in \mathcal{Y}$ such that $Z \subseteq Y \cup \bar{z}$ for some finite \bar{z} . Thus $L \subseteq Y \cup (\ell \cap \bar{z})$, as required. Since the diagram is symmetric the same goes for \mathcal{Y} . The other direction is similar. □

So $\mathcal{X} \sim^w \mathcal{Y}$ is an equivalence relation, and if $\mathcal{X} \rightarrow \mathcal{Y}$ it is a weakening of $\mathcal{X} \xrightarrow{(w)} \mathcal{Y}$. It turns out that, in fact, the two notions are equivalent:

Lemma 34. Assume that $\mathcal{X} \rightarrow \mathcal{Y}$ are objects in *QtNaamen* and that $\mathcal{X} \sim^w \mathcal{Y}$. Then $\mathcal{X} \xrightarrow{(w)} \mathcal{Y}$.

Proof. By Axiom (M2), we know that $\mathcal{X} \xrightarrow{(wc)} \mathcal{V} \xrightarrow{(f)} \mathcal{Y}$. In order to show that $\mathcal{X} \xrightarrow{(w)} \mathcal{Y}$ we have, by the construction of *StNaamen*, to show that $\mathcal{V} \xrightarrow{(wf)} \mathcal{Y}$. By Axiom (M6) the last statement amounts to $(c) \prec \mathcal{V} \rightarrow \mathcal{Y}$. In fact, it will suffice to show that $(c)_0 \prec \mathcal{V} \rightarrow \mathcal{Y}$. So we have to show that



To obtain the next diagram we will use two pieces of information: the first is that \mathcal{V} is in *QtNaamen*, and the second is that $\mathcal{X} \sim^w \mathcal{Y}$. To use the first fact, we

will fit \mathcal{V} into a diagram of the form characterising cute objects; to use the second fact we will try to stick to countable sets. First, using $\mathcal{X} \sim^w \mathcal{Y}$ we have:

$$\begin{array}{ccccc}
 \emptyset & \xrightarrow{(c)} & \mathcal{V} & \xleftarrow{(wc)} & \mathcal{X} \\
 \downarrow (c) & & \uparrow \text{---} & & \downarrow (wc) \\
 \mathcal{A}_c & \xrightarrow{(wf)} & \{A\} & \longrightarrow & \mathcal{V} \\
 \downarrow (c) & & \downarrow (c)_0 & & \downarrow (f) \\
 \mathcal{B}_c & \xrightarrow{(wf)} & \{B\} & \longrightarrow & \mathcal{Y}
 \end{array}$$

FIGURE 25. Decompose $\emptyset \rightarrow \{A\}$ and $\emptyset \rightarrow \{B\}$ by Axiom (M2). Since the composition of two (c) -arrows is a (c) -arrow, and the decomposition provided by Axiom (M2) is unique $\emptyset \xrightarrow{(c)} \mathcal{B}_c \xrightarrow{(wf)} \{B\}$ decomposes into $\emptyset \xrightarrow{(c)} \mathcal{A}_c \xrightarrow{(c)} \mathcal{B}_c \xrightarrow{(wf)} \{B\}$. The dashed arrows exist by the definition of $X \sim^w Y$.

Next, we observe that the configuration $\mathcal{X} \xrightarrow{(wc)} \mathcal{V} \leftarrow \mathcal{B}_c \xrightarrow{(wf)} \{B\}$ fits into the assumptions of Lemma 29. So there is a set B' such that:

$$\begin{array}{ccccc}
 \emptyset & \xrightarrow{(c)} & \mathcal{V} & \xleftarrow{(wc)} & \mathcal{X} \\
 \downarrow (c) & & \nearrow & & \downarrow (wc) \\
 \mathcal{B}'_c & & \{A\} & \longrightarrow & \mathcal{V} \\
 \downarrow (wf) & & \downarrow (c)_0 & & \downarrow (f) \\
 \{B'\} & \xrightarrow{(wc)} & \{B\} & \longrightarrow & \mathcal{Y}
 \end{array}$$

Taking the inverse limit of $\{A\}$ and $\{B'\}$ we get, using the fact that \mathcal{V} is cute:

$$\begin{array}{ccccc}
 \circledast & & \mathcal{X} & & \\
 \downarrow & & \downarrow (wc) & & \\
 \mathcal{B}'_c & \xrightarrow{(c)} & \{A\} & \longrightarrow & \mathcal{V} \\
 \downarrow (wf) & & \downarrow (c)_0 & & \downarrow (f) \\
 \{B'\} & \xrightarrow{(wc)} & \{B\} & \longrightarrow & \mathcal{Y}
 \end{array}$$

FIGURE 26. The arrow $\{B'\} \rightarrow \mathcal{V}$ exists, because \mathcal{V} is in QtNaamen. The arrow $\{B\} \rightarrow \mathcal{V}$ exists by $\{B'\} \xrightarrow{(c)} \{B\} \wedge \mathcal{V} \xrightarrow{(f)} \mathcal{Y}$.

□

Reformulating the last lemma we get:

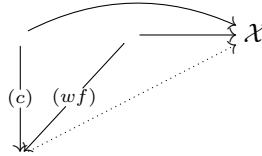
Corollary 35. *The labelled category $QtNaamen$ satisfies Axiom (M5).*

So we have proved:

Theorem 36. *$QtNaamen$ is a closed model category. More precisely, let $StNaamen$ be the category whose objects are classes of sets, and such that for $\mathcal{C}, \mathcal{D} \in ObStNaamen$ there is an arrow $\mathcal{C} \rightarrow \mathcal{D}$ if and only if for all $C \in \mathcal{C}$ there exists $D \in \mathcal{D}$ such that $C \subseteq D$.*

Let \mathfrak{C}_0 be the following colouring of the arrows of $StNaamen$. Label an arrow $\{A\} \xrightarrow{(co)} \{B\}$ if $\text{card } A + \aleph_0 = \text{card } B + \aleph_0$ and $\{A\} \xrightarrow{(wc)} \{B\}$ if $B \setminus A$ is finite. Let $\mathfrak{C}_0 \subseteq \mathfrak{C}$ be the minimal (with respect to inclusion) colouring of $StNaamen$ satisfying Quillen's axioms of model categories (M0), (M1), (M2) and (M6).

Let $QtNaamen$ be the full subcategory of $StNaamen$, whose objects are all those $\mathcal{X} \in ObStNaamen$ such that



Then:

- (1) $QtNaamen$ is a closed model category.
- (2) Every co-cofibrant object of $StNaamen$ is in $QtNaamen$.

Part 2. Set-theoretic invariants

Every man is apt to form his notions of things difficult to be apprehended, or less familiar, from their analogy to things which are more familiar. Thus, if a man bred to the seafaring life, ... should take it into his head to philosophize concerning the faculties of the mind, it cannot be doubted, but he would draw his notions from the fabric of the ship, and would find in the mind, sails, masts, rudder, and compass. – Thomas Reid, “An Inquiry into the Human Mind”, 1764

3. INTRODUCTION

We concluded the previous section with the proof that $QtNaamen$ is a model category for set theory. Our exposition was meant, among others, to convince that it is not unreasonable to think of $QtNaamen$ as the simplest model category for set theory modelling the notions of finiteness, countability and (infinite) equi-cardinality. From the purely category theoretic point of view $QtNaamen$ is extremely simple (arrows are unique whenever they exist, so — e.g. — all diagrams commute), but as a model category the picture is slightly more complicated. On the one hand, most basic tools of model categories (such as the loop and suspension functors) degenerate in $QtNaamen$, but — on the other hand — as a model category $QtNaamen$ does not seem to be such a trivial object (and the homotopy category associated with it is — at least to us — a new set theoretic object).

From the homotopy theoretic point of view, given the axioms of model categories and the set theoretic notions to be modelled the construction of $QtNaamen$ is

almost automatic. Therefore, from that viewpoint QtNaamen should be an almost unavoidable (though somewhat degenerate) object. But, as far as we were able to ascertain, QtNaamen (or any close relative thereof) is not known (under the appropriate translation to set theoretic language, of course) to set theorists. On the face of it, it could be that the reason QtNaamen was not discovered by neither homotopy theorists nor set theorists is that it is too degenerate to be of interest. The aim of this section is to show that this is, maybe, not entirely true. We show that Shelah’s covering numbers — one of the main objects of interest in PCF theory — discovered a century or so after Cantor’s introduction of the notions of countability and cardinality, cannot be missed if one tries to study these notions from the homotopy theoretic point of view. Technically, we prove:

Theorem 37. *Let λ be any cardinal. Then*

$$\mathbb{L}_c\text{card}(\{\lambda\}) = \text{cov}(\lambda, \aleph_1, \aleph_1, 2).$$

where $\mathbb{L}_c\text{card}$ is the cofibrantly replaced left derived functor of the cardinality function (not functor (!)) $\text{card} : \text{QtNaamen} \rightarrow \text{On}^\top$.

The proof of Theorem 37 is, essentially, a triviality, but its formulation — at least for those not fluent in model category jargon — is far from obvious. This part of the paper is dedicated to a large extent to explaining the formulation of Theorem 37, and explaining — given the model category QtNaamen and the cardinality function $\text{card} : \text{QtNaamen} \rightarrow \text{On}^\top$ (where On^\top is the class of ordinal augmented by a terminal object) — how to obtain the functor $\mathbb{L}_c\text{card}$. We then show how other covering numbers (such as Shelah’s revised power function) can be recovered. This and similar constructions are discussed in Section 5.2.

It is intriguing that Shelah, in his book on Cardinal arithmetic, [21], and Kojman, in his survey of Shelah’s PCF theory, [13], use algebraic topology as an analogy to explain the ideology and the usefulness of this theory, Kojman writes, rather directly that “This approach to cardinal arithmetic can be thought of as ‘algebraic set theory’ in analogy to algebraic topology” and Shelah, more by way of example, mentions that: “... for a polyhedron v (number of vertices), e (number of edges) and f (number of faces) are natural measures, whereas $e + v + f$ is not, but from deeper point of view [the homotopy-invariant Euler characteristic] $v - e + f$ runs deeper than all...”. Theorem 37 and its variants can be viewed as consolidating this analogy: they show that (some constructs of) PCF theory have an actual interpretation in terms of algebraic topology. But — to us — it is not clear whether this can be pushed much further, whether this connection with algebraic topology runs any deeper.

Of course, the covering numbers are not the only set theoretic notions that one can recover in QtNaamen . In Proposition 12 we saw that QtNaamen models finiteness, countability and equi-cardinality (at least to some extent). We slightly enlarge the set theoretic dictionary of QtNaamen , giving some natural examples and non-examples (of set theoretic notions that QtNaamen cannot capture — e.g., the power set of a set). Notions such as a cardinal being measurable (Lemma 44) and intriguing possible connections with Jensen’s covering lemma are discussed in Section 5.2.

We conclude the paper with some ideas for further investigation, emanating mainly from problems we identified in our construction: can we overcome the dependence of the derived functor of, say, cardinality on the choice of the model

category (among equivalent model categories), can we find analogues for homotopy theory constructs (homotopy groups, long exact sequences etc.) in QtNaamen despite of it being “degenerate”, can we actually prove set theoretic statements using QtNaamen (and the family of model structures QtNaamen_κ) and not only recover known concepts and definitions?

4. THE EXPRESSIVE POWER OF QTNAAMEN.

As already mentioned several times before, QtNaamen is a very simple model category. Intuitively, QtNaamen should be much simpler than set theory. To formulate this intuition somewhat more precisely, we observe that:

Lemma 38. *Let \mathbf{V} be the universe of set theory and σ a bijective class function on \mathbf{V} . For a class $X \subseteq \mathbf{V}$ let $\tilde{\sigma}(X) = \{\{\sigma(a) : a \in x\} : x \in X\}$. Then $\tilde{\sigma} : \text{QtNaamen} \rightarrow \text{QtNaamen}$ is a bijective functor on QtNaamen. Moreover, $\tilde{\sigma}$ preserves the model structure of QtNaamen.*

Proof. Because σ is a class function, if X is a class so is $\tilde{\sigma}(X)$ (hence, $\tilde{\sigma}$ is indeed a functor from QtNaamen to itself). The only non-trivial part is that $\tilde{\sigma}$ preserves the model structure, which is an immediate corollary of Proposition 12. \square

Observe that in ZFC, given a set S , any $\sigma \in \text{Sym}(S)$ extends to a class-bijection of \mathbf{V} by setting $\sigma(x) = x$ for $x \notin S$. Therefore, the last lemma proves that the model structure on QtNaamen, while it must recognize the subset relation, does not respect — in a strong sense — the membership relation. For example, for a set X the set theoretic operation $X \mapsto \{X\}$ is not respected by QtNaamen, as can be inferred from the existence of an automorphism exchanging $\{\{\emptyset\}\}$ with $\{\{a\}\}$ (for any set a).

Even more trivially, since for any set S we have $\{S\} \longleftrightarrow \mathbb{P}(S)$, we see that QtNaamen cannot distinguish $\{S\}$ from the power set of S . Thus, despite of the fact that QtNaamen was constructed specifically to model the notion of equi-cardinality, it does so with limited success. Moreover, the notion of a set being a singleton is also a notion unknown to QtNaamen, as shows the above example.

In order to extract meaningful information from the model category QtNaamen we can — as is standard in mathematics — beside studying the structure of QtNaamen itself, study functors (and other “natural” set theoretic functions) from QtNaamen to other categories, and vice versa. The next, section, for example, is dedicated to the study of the cardinality function (not functor) $\text{card} : \text{QtNaamen} \rightarrow \text{On}^\top$. In the present section we perform easier computations, showing that by imposing a little extra “natural” set theoretic structure on QtNaamen, more information can be obtained.

4.1. Ordinals. The first example we consider is more easily computed in StNaamen. The computations performed in this sub-section can be readily adapted to QtNaamen (with minor modifications), however, we were not able to find a natural set theoretic interpretation of these computations in the setting of QtNaamen.

Consider Sets^- , the full sub-category of StNaamen, whose objects are precisely those objects of StNaamen which happen to be sets. Consider the class function $S \mapsto S \cup \{S\}$ defined on Sets^- (in fact, restricted to the category Sets^- this is a functor). Indeed, an object of StNaamen is a set precisely if the operation

$S \mapsto S \cup \{S\}$ is defined (in which case, of course, $S \longrightarrow S \cup \{S\}$ is a morphism in StNaamen , and therefore also in Sets^-). Let us label those arrows by (s) .

Observe that the function $S \mapsto S \cup \{S\}$ on Sets^- allows us to define *transitive* sets, namely: a set S is transitive precisely when $S \longrightarrow \{S\}$, or equivalently, if $S \cup \{S\} \longrightarrow S$, i.e., when the arrow $S \xrightarrow{(s)} S \cup \{S\}$ is invertible. Indeed, $S \longrightarrow \{S\}$ if and only if $s \subseteq S$ for all $s \in S$, if and only if S is transitive. Thus, $S \in \text{ObStNaamen}$ is an ordinal if and only if $S \xrightarrow{(s)} S \cup \{S\} \longrightarrow S \longrightarrow \text{On}$, where On is the class of ordinals. We do not know whether the class On , as an object of StNaamen is definable (in some reasonable sense) in StNaamen , even when augmented by the (s) -labelling. We point out however, that at least on the face of it, since the membership relation is not recoverable in StNaamen , isolating the object On in QtNaamen allows us only to identify those objects of QtNaamen all of whose members are ordinals, but not necessarily ordinals themselves.

Note also that our (s) -labelling allows us only to identify arrows $S \longrightarrow S \cup \{S\}$. Given such an arrow, the object $\{S\}$ can be recovered as the complement of S in $S \cup \{S\}$, i.e. it is the unique object whose direct limit (in StNaamen) with $\{S\}$ is $S \cup \{S\}$ and whose inverse limit with S is \emptyset . Of course, the arrow $\{S\} \longrightarrow S$ never exists, as it would imply that $S \subseteq s$ for some $s \in S$, so that $s \in s$ contradicting the regularity axiom of ZFC.

In addition, by Proposition 12, if S is an ordinal then $\emptyset \xrightarrow{(wc)} S$ if and only if $S \leq \aleph_0$ and $\emptyset \xrightarrow{(c)} S$ if and only if $S \leq \aleph_1$. So these two cardinals can also be recovered in StNaamen (with the function $S \mapsto S \cup \{S\}$), as the direct limits of the classes of (wc) and (c) -arrows respectively. This is, of course, not surprising, since StNaamen was constructed to model the notions of finiteness and countability.

Of course, an ordinal α is limit precisely when $\beta \cup \{\beta\} \longrightarrow \beta \cup \{\beta \cup \{\beta\}\} \triangleleft \alpha \longrightarrow \top$ for all $\beta \in \text{On}$. It is a cardinal, precisely when for any ordinal β , if $\{\beta\} \xrightarrow{(c)} \{\alpha\}$ then $\{\alpha\} \xrightarrow{(c)} \{\beta\}$, which can be written as $\emptyset \longrightarrow \{\alpha\} \triangleleft \{\beta\} \xrightarrow{(c)} \{\alpha\}$. Finally, α is a regular cardinal precisely when (it is a cardinal and) $\alpha \xrightarrow{(wf)} \{\alpha\}$. To see this last claim, recall that, by construction, $\alpha \xrightarrow{(wf)} \{\alpha\}$ if and only if $\{A\} \xrightarrow{(c)} \{B\} \triangleleft \alpha \longrightarrow \{\alpha\}$ for all $\{A\} \xrightarrow{(c)} \{B\}$. But for sets, $\{A\} \xrightarrow{(c)} \{B\}$ if and only if $\text{card } A + \aleph_0 = \text{card } B + \aleph_0$. So the lifting property defining the (wf) -arrows assures that for any $B \subseteq \alpha$, if some $A \subseteq B$ of the same (infinite) cardinality satisfies $\{A\} \longrightarrow \alpha$ then $\{B\} \longrightarrow \alpha$. But $\{B\} \longrightarrow \alpha$ implies that there exists $\beta < \alpha$ such that $B \subseteq \beta$, so B is bounded in α . The other direction works in a similar way.

As explained above, the operation $S \mapsto S \cup \{S\}$ is “external” to QtNaamen (or StNaamen). As a side remark to this subsection we point out that some traces of it can be recovered in a more “geometric” way within these labelled categories. Consider, for example, the property $A = \{\{a\}\}$ for some set a . It is easy to see that if A is of this form then $\emptyset \longrightarrow Z \longrightarrow A$ implies that either $\emptyset \cong Z$ or $A \cong Z$. Conversely, if any decomposition $\emptyset \longrightarrow Z \longrightarrow A$ is such that $\emptyset \longrightarrow Z$ is an isomorphism or $Z \longrightarrow A$ is an isomorphism then $A = \{\{a\}\}$ for some set A . So we define,

Definition 10. *An arrow $X \longrightarrow Y$ is indecomposable if whenever $X \longrightarrow Z \longrightarrow Y$ either $X \longrightarrow Z$ is an isomorphism or $Y \longrightarrow Z$ is an isomorphism.*

With this definition the above observation can be stated as: A is of the form $\{\{a\}\}$ for some set a if and only if $\emptyset \rightarrow A$ is indecomposable. Note also that while we do not know whether indecomposability can be expressed as a lifting property it is obviously invariant under graph automorphisms of QtNaamen (or StNaamen), and can therefore be thought of as an intrinsic property of these (labelled) categories.

It follows, for example, that with this in hand the property of A being *isomorphic* to a singleton (i.e., $A \cong \{a\}$ for some set a) can be stated as: for all X, Y if for all $\{\{a\}\}$, $\emptyset \rightarrow \{\{a\}\} \triangleleft X \rightarrow Y$ then $X \rightarrow Y \triangleleft A \rightarrow \top$. It will suffice, of course, to show that this statement is equivalent to $A \cong \{\bigcup A\}$. By definition, $\emptyset \rightarrow \{\{a\}\} \triangleleft X \rightarrow Y$ for all $\{\{a\}\}$ is equivalent to the statement that $\bigcup Y \subseteq \bigcup X$, in particular — for any set A — $\emptyset \rightarrow \{\{a\}\} \triangleleft A \rightarrow \{\bigcup A\}$ for all a . Therefore, if for all X, Y the assumption that $\emptyset \rightarrow \{\{a\}\} \triangleleft X \rightarrow Y$ for all a implies that $X \rightarrow Y \triangleleft A \rightarrow \top$ we can apply this with $A = X$ and $Y = \{\bigcup A\}$ to get $A \rightarrow \{\bigcup A\} \rightarrow A$, as required. The other direction is obvious, since the definition is invariant under changing A with an isomorphic object, and therefore, we may assume that A is a singleton.

4.2. Cofinal and covering families. A class A is \subseteq -cofinal in B if A is a sub-class of B and for all $b \in B$ there exists $a \in A$ such that $b \subseteq a$. In that case we also say that A *covers* B . By definition, this happens precisely when $B \rightarrow A$. Since A is a sub-class of B we automatically get $A \rightarrow B$, so that A is cofinal in B precisely when A is isomorphic to B , which happens if and only if $A \xrightarrow{(cwf)} B$. If B is a set, the cofinality of B is the minimal cardinality of a \subseteq -cofinal subset. In our notation, this can be expressed as:

$$\text{cof}(B, \subseteq) = \min\{\text{card } B' : B' \xrightarrow{(cwf)} B\}.$$

For a class B we have $\emptyset \xrightarrow{(c)} B$ if and only if every element of B is at most countable, and $\emptyset \xrightarrow{(wc)} B$ if and only if every element of B is finite. If S is a set then $\emptyset \xrightarrow{(c)} B \xrightarrow{(wf)} \{S\}$ if and only if B covers $[S]^{\leq \aleph_0}$, i.e., if the set of countable subsets of S is covered by B . In addition $\emptyset \xrightarrow{(c)} [S]^{\leq \aleph_0} \xrightarrow{(f)} \{S\}$ and $\emptyset \xrightarrow{(wc)} [S]^{< \aleph_0} \xrightarrow{(wf)} \{S\}$. Combining all of the above, we get that for a cardinal κ :

$$\begin{aligned} \text{cov}(\kappa, \aleph_1, \aleph_1, 2) &= \min\{\text{card } B' : \emptyset \xrightarrow{(c)} B' \xrightarrow{(wf)} \{\kappa\}\} = \\ &= \inf\{\text{card } B' : \emptyset \xrightarrow{(c)} B' \leftarrow B'' \xrightarrow{(wf)} \{\kappa\}\} \end{aligned}$$

where $\text{cov}(\kappa, \aleph_1, \aleph_1, 2)$ is the minimal cardinality of a family of countable subsets of κ covering $[\kappa]^{\leq \aleph_0}$ (see Subsection 5.1 for more details). This shows that the covering number $\text{cov}(\kappa, \aleph_1, \aleph_1, 2)$ has a simple model categorical interpretation. The main goal of this paper is to show that, in fact, the right-most formula in the above equation is not only a simple translation of this set theoretic notion, but arises naturally from a model categorical study of QtNaamen . It is the *cofibrantly replaced left-derived functor of the cardinality function from QtNaamen to On^\top* . This will be explained in detail in the next section.

4.3. Some non-set theoretic concepts. We conclude with a few simple non-set theoretic statements that can be expressed in QtNaamen . Consider, for example,

\mathcal{N} a monster model of some first order theory T . For a cardinal β let \mathcal{N}_β be the set of all elementary sub-models of \mathcal{N} of cardinality at most β . Then $\mathcal{N}_\beta \xrightarrow{(wf)} \{\mathcal{N}\}$ is the statement that the Lowenheim-Skolem number of T is at most β . Namely, it states that every subset of N of cardinality at most β is contained in a model of size at most β . In particular, if T is countable then $\emptyset \xrightarrow{(c)} \mathcal{N}_{\aleph_0} \xrightarrow{(wf)} \{\mathcal{N}\}$ is the statement that every countable set is contained in a countable model. Of course, the objects \mathcal{N}_β do not seem to be endemic to the model categorical setup.

Recall that if X is a topological space, then a set $A \subseteq X$ is closed if and only if $\text{acc}(A) \subseteq A$, namely, if A contains all its accumulation points. Thus, a topological space can be given (instead of giving a collection of closed sets) by giving, to any subset S of X the collection $\text{acc}(S)$. Therefore, a topological space X gives rise to a functor $\text{acc} : \text{QtNaamen} \rightarrow \text{Sets}^-$ by $S \mapsto \{\text{acc}(S \cap X) : S \in \mathcal{S}\}$, and the topology on X can be recovered from the functor acc only — namely, this gives a purely category theoretic definition of the topology.

5. FUNCTORS AND DERIVED FUNCTORS.

The idea of “forgetting structure” is, of course, a central theme in mathematics. In homotopy theory, it is common to forget information that is not homotopy invariant. In model categories, Quillen’s axiomatization of homotopy theory, homotopy invariance is defined with the help of the *homotopy category* $\text{Ho}\mathfrak{C}$ obtained from the model category \mathfrak{C} by formally inverting all weak equivalence so that weak equivalences, and only weak equivalences, become isomorphisms in $\text{Ho}\mathfrak{C}$. It turns out that constructions performed in the setting of model categories can sometimes be well controlled only by a process of forgetting structure known as the *cofibrant replacement*: first we restrict to the sub-category of cofibrations (i.e., by forgetting all arrows that are not (c) -labelled). Then — if we want the resulting category to have an initial object — we have to restrict further to the category of cofibrant objects (namely those objects X such that $\emptyset \xrightarrow{(c)} X$), and whose only morphisms are cofibrations.

Observe that by axiom (M2), given a model category \mathfrak{C} any object X is isomorphic in the homotopy category $\text{Ho}\mathfrak{C}$ to a cofibrant object, $X_{(wf)}$ such that $\emptyset \xrightarrow{(c)} X_{(wf)} \xrightarrow{(wf)} X$. Thus, from the homotopy category point of view, every object can be replaced with a cofibrant object, a process known as the cofibrant replacement.

Another important means of forgetting structure in model categories is that of *deriving* functors. Given a functor F the (left) derived functor $\mathbb{L}F$ is the homotopy invariant functor “closest” to F (from the left). Here “closest to F ” is interpreted as being universal among the *homotopy invariant* functors such that there exists a natural transformation (also known as morphism of functors) $G \Rightarrow F$, i.e., if $\mathbb{L}F$ is the derived functor of F there exists a natural transformation $\mathbb{L}F \Rightarrow F$ and any natural transformation $G \Rightarrow F$ from a homotopy invariant functor G , factors uniquely via $\mathbb{L}F \Rightarrow F$.

A functor F from a model category \mathfrak{C} is *homotopy invariant* if it factors through $\text{Ho}\mathfrak{C}$. In a posetal model category, i.e., in any category \mathfrak{C} where arrows between any two objects are unique, whenever they exist, the definition of the left derived functor is extremely simple:

Definition 11. Let \mathfrak{C} be a posetal model category, and \mathfrak{C}' any posetal category, $\gamma : \mathfrak{C} \rightarrow \text{Ho}\mathfrak{C}$ the localisation map. Given a functor $F : \mathfrak{C} \rightarrow \mathfrak{C}'$ the left derived functor of F is given by

$$\mathbb{L}^\gamma F(X') = \inf\{F(X) : X' \leq_{\text{Ho}\mathfrak{C}} \gamma(X), X \in \text{Ob}\mathfrak{C}\}.$$

In particular, the left derived functor exists if and only if the right hand side is well-defined.

Observe that the definition of a derived functor $F : \mathfrak{C} \rightarrow \mathfrak{C}'$ depends on \mathfrak{C} being a model category only in as much as the homotopy category $\text{Ho}\mathfrak{C}$ is the category through which we want to factor (an approximation) of F . In general, given a category \mathfrak{D} and a functor $\gamma : \mathfrak{C} \rightarrow \mathfrak{D}$, we can still (left) derive any functor $F : \mathfrak{C} \rightarrow \mathfrak{C}'$ with respect to the functor γ . If all categories involved are posetal, the formula in the above definition still gives the left-derived functor with respect to γ . Thus, for example. if \mathfrak{C}' is well ordered or if \mathfrak{C}' is Dedekind complete then any function $F : \mathfrak{C} \rightarrow \mathfrak{C}'$ can be derived from the left.

Let On^\top be the posetal category of ordinals (i.e., given ordinals α, β there is an arrow $\alpha \rightarrow \beta$ if and only if $\alpha \leq \beta$) augmented by a terminal object \top . The following definition sums up the discussion of the previous paragraphs, with an extra edge:

Definition 12. Let \mathfrak{C} be a posetal model category. For a function $F : \mathfrak{C} \rightarrow \text{On}^\top$ we let the cofibrantly replaced left derived functor of F be:

$$\mathbb{L}_c F(X) = \min \left\{ F(Y) : \begin{array}{ccccccc} & & X_1 & & X_3 & & X_n \rightarrow Y \\ & \nearrow & \leftarrow^{(w)} & \nearrow & \leftarrow^{(w)} & \nearrow & \uparrow^{(c)} \\ X & & X_2 & & \dots & & \perp \end{array} \right\}$$

where the minimum is taken over all finite sequences of the same form.

Observe that given $X, Y \in \text{Ob}\mathfrak{C}$ a sequence of the form

$$(\diamond) \quad X \rightarrow X_1 \xleftarrow{(w)} X_2 \rightarrow X_3 \xleftarrow{(w)} \dots \rightarrow X_n \rightarrow Y$$

exists for some $n \in \mathbb{N}$ if and only if there exists $g \in \text{Mor}\text{Ho}\mathfrak{C}$ such that $X \xrightarrow{g} Y$. Thus, we can write:

$$(\clubsuit) \quad \mathbb{L}_c F(X) = \min\{F(Y) : X \rightarrow_h Y, \perp \xrightarrow{(c)} Y\},$$

where $X \rightarrow_h Y$ means that there is an arrow from X to Y in the homotopy category. Using this notation we immediately see that $\mathbb{L}_c F$ is a homotopy invariant (because it factors through the homotopy category) *functor* (because $X \rightarrow Y$ implies that $\mathbb{L}_c F(X) \geq \mathbb{L}_c F(Y)$) depending only on the values F takes on cofibrant objects.

Note that if $F : \mathfrak{C} \rightarrow \text{On}^\top$ is a functor then for any $X \in \text{Ob}\mathfrak{C}$, letting $\perp \xrightarrow{(c)} X_{(wf)} \xrightarrow{(wf)} X$ we see that $F(X_{(wf)}) \leq F(X)$, but $\mathbb{L}_c F$ is functorial so $\mathbb{L}_c F(X_{(wf)}) \leq \mathbb{L}_c F(X)$. By what we have just said $X_{(wf)} \rightarrow X$ implies that $\mathbb{L}_c F(X_{(wf)}) \geq \mathbb{L}_c F(X)$, so — in the case F is a functor:

$$\mathbb{L}_c F(X) = \min\{F(Y) : X \rightarrow_h Y\} = \mathbb{L}^\gamma \circ \gamma(X).$$

Thus, the cofibrantly replaced left-derived functor generalises the definition of (left) derived functors (but the two definitions need not agree if F is not a functor).

Remark 39. Let \mathfrak{C} , \mathfrak{C}' be equivalent model categories, witnessed by the functors $F : \mathfrak{C} \rightarrow \mathfrak{C}'$ and $G : \mathfrak{C}' \rightarrow \mathfrak{C}$. Assume that $f : \mathfrak{C} \rightarrow \text{On}^\top$ is any function, then there is no reason to expect that $\mathbb{L}_c f(G(Y)) = \mathbb{L}_c(f \circ G)(Y)$. This is, of course, not the case if f is a functor. In other words, the price for deriving arbitrary functions is that the process is not invariant under equivalence of model categories. This is discussed further in Section 6.

5.1. The covering number of \aleph_ω as a value of a derived functor. In this sub-section we prove the main result of this paper, we show that the covering number of \aleph_ω is the value of the cofibrantly replaced left-derived functor of the cardinality function $\text{card} : \text{QtNaamen} \rightarrow \text{On}^\top$. Cardinality is certainly one of the most natural functions anyone studying set theory is bound to run into. Possibly, it is the simplest set theoretic function not arising directly from purely logical operations (in the way the union and intersection operations do). To adapt the notion of cardinality to our setting we define a function $\text{card} : \text{QtNaamen} \rightarrow \text{On}^\top$ such that $X \mapsto \text{card}(X)$ if X is a set and $X \mapsto \top$ otherwise. Observe that cardinality is not a functor on QtNaamen. Indeed $\{X\} \rightarrow \mathcal{P}(X) \rightarrow \{X\}$ but $\text{card}(\{X\}) = 1 < \text{card}(\mathcal{P}(X)) > 1$ for all non-empty X . Similarly,

$$\{\{\bullet_1\}, \{\bullet_1, \bullet_2\}\} \xrightarrow{(wcf)} \{\{\bullet_1, \bullet_2\}\}$$

is an isomorphism but $2 = \text{card}\{\{\bullet_1\}, \{\bullet_1, \bullet_2\}\} > \text{card}\{\{\bullet_1, \bullet_2\}\} = 1$ are non-isomorphic.

However, cardinality is a natural function and the homotopy ideology discussed above suggests (despite of the fact it is not a functor) that we try and find a homotopy invariant approximation to cardinality. As discussed above, any function from a model category to On^\top can be derived. Unfortunately, as we will see later, deriving the cardinality function (according to the formula in Definition 11) gives us an uninteresting result. So we take the cofibrantly replaced left derived functor of cardinality, as in Definition 12. The resulting function, $\mathbb{L}_c \text{card}$, can be viewed, as homotopy theory yoga suggests, as the homotopy invariant version of cardinality.

Interestingly, the homotopy invariant version of cardinality has a purely set theoretic interpretation ($\mathbb{L}_c \text{card}(\{\aleph_\alpha\}) = \text{cov}(\aleph_\alpha, \aleph_1, \aleph_1, 2)$ — where $\text{cov}(\aleph_\alpha, \aleph_1, \aleph_1, 2)$ is the *covering number* to be discussed in detail below). The construction of this function uses fairly little set theory: the only notions needed in an essential way to construct it are $A \subseteq B$, finiteness, countability and infinite equi-cardinality. Thus, $\mathbb{L}_c \text{card}$ will remain meaningful in any set theory where those notions keep their meaning. More importantly, $\mathbb{L}_c \text{card}$ is considerably tamer, say, than the power function, and can be effectively bounded in ZFC (but these are deep results in PCF theory, and we do not claim that they can be identified, let alone proved, using homotopy theoretic tools). For example, Shelah’s famous inequality

$$(\aleph_\alpha)^{\aleph_0} \leq \text{cov}(\aleph_\alpha, \aleph_1, \aleph_1, 2) + 2^{\aleph_0}$$

can be interpreted (paraphrasing Shelah) as a decomposition of $(\aleph_\alpha)^{\aleph_0}$ into a “noise” component (wild and highly independent on ZFC) and a “homotopy invariant” part, which can be well understood within ZFC.

Another curious feature of the function $\mathbb{L}_c \text{card}$ is that it is non-trivial only on singular cardinals. Thus, from a homotopy theoretic view point singular cardinals present themselves almost immediately as a natural object of interest in set theory (compare with, [14], describing the early and spectacular appearance of singular

cardinals on the mathematical stage, and their immediate disappearance for several decades). We now proceed with a detailed exposition of the discussion of the last paragraphs.

By definition, the *covering number*

$$\text{cov}(\lambda, \kappa, \theta, \sigma)$$

is the *least size of a family* $X \subseteq [\lambda]^{<\kappa}$ *of subsets of* λ *of cardinality less than* κ , *such that every subset of* λ *of cardinality less than* θ , *lies in a union of less than* σ *subsets in* X .

Theorem 40. (the covering number as a derived functor). For any cardinal λ

$$\mathbb{L}_c \text{card}(\{\lambda\}) = \text{cov}(\lambda, \aleph_1, \aleph_1, 2)$$

Proof. First, assume that \mathcal{Y} is a covering family for λ witnessing $\text{cov}(\lambda, \aleph_1, \aleph_1, 2) = \kappa$. Then, by definition of the covering number $\emptyset \xrightarrow{(c)} \mathcal{Y}$. We claim that $\mathcal{Y} \xrightarrow{(w)} \{\lambda\}$, which will prove $\mathbb{L}_c \text{card}(\lambda) \leq \kappa$. By Proposition 12 we only have to show that any countable subset of λ is contained in an element of \mathcal{Y} , which is merely the definition of \mathcal{Y} being a covering family. To prove the other inequality, observe that:

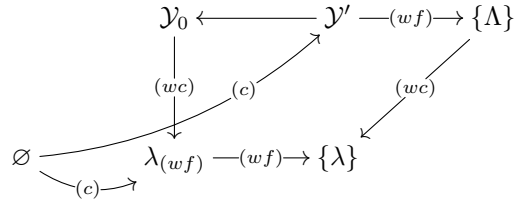
Claim I: If $\mathcal{X} \rightarrow_h \mathcal{Y}$ in QtNaamen , with $\mathcal{X} := \mathcal{X}_0, \mathcal{X}_1, \dots, \mathcal{X}_n =: \mathcal{Y}$ witnessing it (as in (\diamond)) then for every $i \leq n$, every countable subset L with $\{L\} \rightarrow \mathcal{X}$ is contained, up to finitely many elements, in some $\{X\} \rightarrow \mathcal{X}_i$.

Proof. For X_0 there is nothing to prove, and for X_1 this follows from the definition of $\mathcal{X} \rightarrow \mathcal{X}_1$. For $\mathcal{X}_2 \xrightarrow{(w)} \mathcal{X}_1$ this is a special case of Proposition 12, and as the condition is transitive, induction gives this observation. $\square_{\text{Claim I}}$

The proof of the theorem now follows from the following:

Claim II: Let $\emptyset \xrightarrow{(c)} \mathcal{Y}$ be such that $\{L\} \rightarrow^* \mathcal{Y}$ for every countable set, $L \subseteq \lambda$. Then there exists a covering family \mathcal{Z} of λ whose cardinality is at most that of \mathcal{Y} .

Proof. Let \mathcal{Y}_0 be the inverse limit of \mathcal{Y} and $\{\lambda\}$. Then $\text{card } \mathcal{Y}_0 \leq \text{card } \mathcal{Y}$ (by definition of the inverse limit in QtNaamen), and $\emptyset \xrightarrow{(c)} \mathcal{Y}_0$. By assumption, $\emptyset \xrightarrow{(c)} \mathcal{Y}_0 \xrightarrow{(wc)} \lambda_{(wf)}$, where $\emptyset \xrightarrow{(c)} \lambda_{(wf)} \xrightarrow{(wf)} \{\lambda\}$. By Lemma 29 there is a set Λ and some \mathcal{Y}' such that the following diagram is true in StNaamen :



Since QtNaamen is a full sub-category, to show that this diagram is also true in QtNaamen it suffices to verify that all objects in the diagram are objects in

QtNaamen. This amounts to checking that $\{\lambda\}$ and \mathcal{Y}' are in QtNaamen, which is obvious since all singleton sets and all cofibrant objects are.

Let \mathcal{Y}'' be the inverse limit of \mathcal{Y}_0 and $\{\Lambda\}$, by definition this is simply $\{y \cap \Lambda : y \in \mathcal{Y}_0\}$. By definition of the inverse limit we get $\mathcal{Y}' \rightarrow \mathcal{Y}''$. Since $\mathcal{Y}' \xrightarrow{(wf)} \{\Lambda\}$ it follows (e.g., by Proposition 12) that $\mathcal{Y}'' \xrightarrow{(wf)} \{\Lambda\}$. Since all elements in \mathcal{Y}' are countable so are all the elements in \mathcal{Y}'' . By Proposition 12 these two facts together mean precisely that \mathcal{Y}'' is a covering family for Λ .

Finally, since $\{\Lambda\} \xrightarrow{(wc)} \{\lambda\}$ we get (again, using Proposition 12), that $\lambda \setminus \Lambda$ is a finite set, say, C . Let $\mathcal{Z} := \{y \cup C : y \in \mathcal{Y}''\}$. Then \mathcal{Z} is a cofibrant object, and is therefore an object of QtNaamen. All elements in \mathcal{Z} are countable, and every countable subset of λ is contained in an element of \mathcal{Z} . So \mathcal{Z} is a covering family for λ . Observe that $\text{card } \mathcal{Z} \leq \text{card } \mathcal{Y}'' \leq \text{card } \mathcal{Y}_0 = \text{card } \mathcal{Y}$. Thus, \mathcal{Z} witnesses that $\mathbb{L}_c \text{card} \geq \text{cov}(\lambda, \aleph_1, \aleph_1, 2)$. $\square_{\text{Claim II}}$

This completes the proof of the theorem. \square

We conclude with a summary, in our notation, of some of Shelah's results concerning PCF bounds:

Theorem 41 (Shelah). *The following inequalities are true in ZFC:*

(i) *if \aleph_α is regular cardinal, then*

$$\mathbb{L}_c(\{\aleph_\alpha\}) = \mathbb{L}_c(2^{\aleph_\alpha}) = \text{cov}(\aleph_\alpha, \aleph_1, \aleph_1, 2) = \aleph_\alpha$$

(ii) $\mathbb{L}_c(\{\aleph_\omega\}) = \mathbb{L}_c(2^{\aleph_\omega}) = \text{cov}(\aleph_\omega, \aleph_1, \aleph_1, 2) < \aleph_{\omega_4}$

(iii) *If \aleph_δ is a singular cardinal such that $\delta < \aleph_\delta$, then $\mathbb{L}_c(\{\aleph_\delta\}) = \mathbb{L}_c(2^{\aleph_\delta}) = \text{cov}(\aleph_\delta, \text{card } \delta^+, \text{card } \delta^+, 2) < \aleph_{\text{card } \delta^+ + 4}$*

(iv) *(Shelah's Revised GCH). If θ is a strong limit uncountable cardinal, $\lambda \geq \theta$, $\kappa_0 \leq \kappa < \theta$ then $\lambda^{[\kappa]} = \lambda$, where $\lambda^{[\kappa]}$ is Shelah's revised power function (see \dagger below).*

Proof. (i) is immediate by induction; (ii) is a particular case of (iii); (iii) is [1, Theorem 7.2]; (iv) is [1, Theorem 8.1]

Note that we do not say anything about the *fixed points* $\alpha = \aleph_\alpha$ of \aleph_\bullet -function.

5.2. Other model categories and covering numbers. Simple variations on the theme leading us to “rediscover” the covering number $\text{cov}(\lambda, \aleph_1, \aleph_1, 2)$ result in other covering numbers. Since most of the details are quite similar, we will be brief.

For an object A of QtNaamen, let QtNaamen^A be the full sub-category of arrows $A \rightarrow X$ with the induced model structure, i.e., the full subcategory of QtNaamen consisting of those objects X such that $A \rightarrow X$ with the labelling induced from QtNaamen. This is, trivially, a model category. Applying Definition 11 for QtNaamen^A and the function $\text{card} : \text{QtNaamen}^A \rightarrow \text{On}^T$ to obtain the functor

$$\mathbb{L}_c^{\text{Qt}^A} \text{card} : \text{QtNaamen}^A \rightarrow \text{On}^T,$$

the cofibrantly replaced left-derived functor of cardinality (on the model category QtNaamen^A). We obtain:

Theorem 42. *Let $\beta \leq \alpha$ be ordinals. Let $\aleph_\beta^* := [\aleph_\beta]^{<\aleph_\beta}$. Then, with the above notation, if \aleph_β is regular then*

$$\mathbb{L}_c^{Qt\aleph_\beta^*} \text{card}(\{\aleph_\alpha\}) = \text{cov}(\aleph_\alpha, \aleph_\beta, \aleph_\beta, 2).$$

In particular, if $\aleph_\alpha < \aleph_{\aleph_\alpha}$ and $\aleph_\beta = (\text{cof } \aleph_\alpha)^+$ then

$$\mathbb{L}_c^{Qt\aleph_\beta^*} \text{card}(\{\aleph_\alpha\}) = \text{pp}_{\text{cof } \aleph_\alpha}(\aleph_\alpha) = \text{pp}(\aleph_\alpha)$$

is the pseudopower of \aleph_α .

Proof. We remark, first, that the pseudopower $\text{pp}(\aleph_\alpha)$ is defined in [20, Def. 5.1] and Theorem 5.7[ibid.] states that, in our notation, $\text{pp}(\aleph_\alpha) = \text{cov}(\aleph_\alpha, \aleph_\alpha, \text{cof } \aleph_\alpha^+, 2)$. It follows immediately from the definition of QtNaamen that for every set S and every cardinal λ the set $[S]^{<\lambda}$ is an object of QtNaamen. So $\aleph_\beta^* \in \text{ObQtNaamen}$ and the formulation of the theorem makes sense. Now, let $\perp_\beta := \perp^{Qt\aleph_\beta^*}$ be the initial object of $Qt\aleph_\beta^*$, namely $\perp_\beta = \aleph_\beta^*$. Now, assume that \mathcal{Y} is a covering family witnessing that $\text{cov}(\aleph_\alpha, \aleph_\beta, \aleph_\beta, 2) = \kappa$. Then by definition of the covering number $\perp_\beta \rightarrow \mathcal{Y}$. Moreover, by Proposition 12, \mathcal{Y} is a cofibrant object, i.e., $\perp_\beta \xrightarrow{(c)} \mathcal{Y}$ (in the notation of Proposition 12, given any $Y \in \mathcal{Y}$ take $n = 2$, $B_0 \in \perp_\beta$ any element such that $\text{card } B_0 = \text{card } Y$, $B_1 = B_0 \cup Y$ and $B_2 = Y$). By Axiom M4, the product of two weak equivalences is a weak equivalence, and we have that $\{y \cap \aleph_\alpha : y \in \mathcal{Y}\} = \mathcal{Y} \times \{\aleph_\alpha\} \xrightarrow{(w)} \aleph_\alpha$. So, in the notation of (\diamond) , taking $n = 2$, $X = X_1 = \{\aleph_\alpha\}$ and $X_2 = \mathcal{Y} \times \{\aleph_\alpha\}$ we get

$$\{\aleph_\alpha\} \rightarrow \{\aleph_\alpha\} \xleftarrow{(w)} \mathcal{Y} \times \{\aleph_\alpha\} \rightarrow \mathcal{Y} \xleftarrow{(c)} \perp_\beta.$$

Thus, $\mathbb{L}_c^{Qt\aleph_\beta^*} \text{card}(\{\aleph_\alpha\}) \leq \kappa$. So we now turn to the proof of the other inequality. Let $\{\aleph_\alpha\} \rightarrow_h \mathcal{Y}$ for some cofibrant object \mathcal{Y} of minimal cardinality. We first prove:

Claim I Let $L \subseteq \aleph_\alpha$ be any set with $\text{card } L < \aleph_\beta$. Then there exists some $L' \subseteq L$ such that $L \setminus L'$ is finite and such that $\{L'\} \rightarrow \mathcal{Y}$. We denote this property $[\aleph_\alpha]^{<\aleph_\beta} \rightarrow^* \mathcal{Y}$.

Proof. By Claim I of the previous theorem we know that every countable subset of L is contained, up to a finite set, in some element of \mathcal{Y} . That is, if $L_0 \rightarrow L_c$ (where $\emptyset \xrightarrow{(c)} L_c \xrightarrow{(wf)} \{L\}$) then $L_0 \rightarrow^* \mathcal{Y}$. Letting \mathcal{Y}_0 be the inverse limit of \mathcal{Y} and L this means that $\mathcal{Y}_0 \xrightarrow{(wc)} L_c$. By Lemma 29 (or, rather, its proof) L' satisfying the requirements can be found. $\square_{\text{Claim I}}$

To conclude the proof of the theorem we need one additional combinatorial fact, which is, essentially, Lemma 29 (or a degenerate version of the remark following it):

Claim II Assume that $[\aleph_\alpha]^{<\aleph_\beta} \rightarrow^* \mathcal{Y}$. Then there is a finite set B such that $\aleph_\alpha^{<\aleph_\beta} \rightarrow \mathcal{Y}_B$, where \mathcal{Y}_B is the set $\{Y \cup B : Y \in \mathcal{Y}\}$.

So let B be a finite set as in Claim II, then \mathcal{Y}_B covers $\aleph_\alpha^{\aleph_\beta}$, and $\text{card } \mathcal{Y}_B = \text{card } \mathcal{Y}$. Because B is finite and $\mathcal{Y} \in \text{ObQtNaamen}$ it follows immediately from the definition that $\mathcal{Y}_B \in \text{ObQtNaamen}$, with the desired conclusion. \square

The construction of the co-slice category, QtNaamen^A , for an object A is standard in category theory. We proceed now to a slightly different construction, to our taste quite natural from the set theoretic point of view, but not entirely obvious from on the category theoretic side:

Let X be a class of sets, fix a (regular) cardinal κ and denote $\bigcup_{<\kappa} X := \{\bigcup S : S \subseteq X, \text{card } S < \kappa\}$. Call a class X of sets κ -directed if $\bigcup_{<\kappa} X \rightarrow X$, namely if any collection of less than κ members of X has a common upper bound (with respect to \subseteq) in X . Let StNaamen_κ be the full subcategory of StNaamen consisting of κ -directed classes.

Let StNaamen_κ^+ be a category that has the same objects as QtNaamen and $X \rightarrow Y$ in StNaamen_κ^+ if and only if $\bigcup_{<\kappa} X \rightarrow \bigcup_{<\kappa} Y$. Given $X \in \text{ObStNaamen}_\kappa^+$ denote $F(X) := \bigcup_{<\kappa} X$. It is clear that $F : \text{StNaamen}_\kappa^+ \rightarrow \text{StNaamen}_\kappa$ is a functor. Moreover the inclusion mapping $G : \text{StNaamen}_\kappa \rightarrow \text{StNaamen}_\kappa^+$ given by $G(X) = X$ is a functor (as for any $X, Y \in \text{ObStNaamen}_\kappa$ if $X \rightarrow Y$ then $\bigcup_{<\kappa} X \rightarrow \bigcup_{<\kappa} Y$). By definition, for $X \in \text{ObStNaamen}$, $X \leftrightarrow \bigcup_{<\kappa} X$, so the functors F and G show that StNaamen_κ is equivalent to StNaamen_κ^+ .

It is easy to check that for regular κ the category StNaamen_κ equipped with the following labelling satisfies Quillen's axioms (M1)-(M4) and (M6):

- Definition 13.**
- (1) $X \rightarrow Y$ iff $\forall x \in X \exists y \in Y x \subseteq y$
 - (2) $X \xrightarrow{(wc)} Y$ iff $\forall y \in Y \exists x \in X (\text{card } (y \setminus x) < \kappa)$ (and $X \rightarrow Y$)
 - (3) $X \xrightarrow{(c)} Y$ iff $\forall x \in X \exists y \in Y (\text{card } y \leq \text{card } x + \kappa)$ (and $X \rightarrow Y$)
 - (4) $X \xrightarrow{(f)} Y$ iff $\forall x \in X \forall y' \subseteq y \in Y \exists x' \in X (\text{card } y' < \kappa \implies x \cup y' \subseteq x')$ (and $X \rightarrow Y$)
 - (5) $X \xrightarrow{(wf)} Y$ iff $\forall x \in X \forall y' \subseteq y \in Y \exists x' \in X (\text{card } y' \leq \text{card } x + \kappa \implies y' \subseteq x')$ (and $X \rightarrow Y$)
 - (6) $X \xrightarrow{(w)} Y$ iff $\forall x \in X \forall y' \subseteq y \in Y \exists x' \in X \text{card } (y' \setminus x') < \kappa$ (and $X \rightarrow Y$)

Remark 43. Observe that $X \xrightarrow{(wc)} Y$ ($X \xrightarrow{(wf)} Y$) if and only if $X \xrightarrow{(c)} Y$ ($X \xrightarrow{(f)} Y$) and $X \xrightarrow{(w)} Y$. Moreover, $X \xrightarrow{(w)} Y$ if and only if there exists Z such that $X \xrightarrow{(wc)} Z \xrightarrow{(wf)} Y$.

To turn StNaamen_κ into a model category, as with StNaamen , let QtNaamen_κ be the full sub-category of *cute* objects of StNaamen_κ , namely, those objects satisfying the diagram of Figure 15 (with respect to the labelling in the above definition). Now one defines, for $X \in \text{ObStNaamen}_\kappa$, \tilde{X} to be the product of all cute $Y \in \text{ObStNaamen}_\kappa$ such that $X \rightarrow Y$ (we leave it as an exercise to verify that this is indeed an object in StNaamen_κ). It is then easy to verify that \tilde{X} is cute and that if $\emptyset \xrightarrow{(c)} X$ then $X = \tilde{X}$ and that $\{\tilde{S}\} = \{S\}$ for any set S . So QtNaamen_κ satisfies Axiom (M0) (inverse limits are simply products, and the direct limit $\{X_1, \dots, X_k\}$ is simply $\widetilde{\Sigma_{i=1}^k X_i}$, where ΣX_i is the limit of the X_i in StNaamen_κ). That the remaining axioms are satisfied in QtNaamen_κ can be proved precisely as in the first part of this paper, with the obvious adaptations (replacing “countable” there with “of cardinality at most κ ” and “finite” there with “of cardinality smaller than κ ”, and see also Remark 30 for the fixed point argument).

Recall that, as pointed out above, StNaamen_κ^+ is equivalent to StNaamen_κ . This equivalence can be used to label StNaamen_κ^+ uniquely to make the two categories equivalent as labelled categories. Since the definition of QtNaamen_κ is given strictly in terms of the labelling of StNaamen_κ , we obtain a full sub-category, QtNaamen_κ^+ , of StNaamen_κ^+ , equivalent as a labelled category to QtNaamen_κ (QtNaamen_κ^+ is both the image of QtNaamen_κ under the functor mapping StNaamen_κ into StNaamen_κ^+ and the full sub-category of cute objects of StNaamen_κ^+ as a labelled category). Thus, QtNaamen_κ^+ is a model category equivalent to QtNaamen_κ .

As QtNaamen_κ is equivalent (as a model categories) to QtNaamen_κ^+ so are their associated homotopy categories. Computing the homotopy category of QtNaamen_κ is rather simple: objects are $<_\kappa$ -directed classes with arrows $X \longrightarrow Y$ if and only if for all $x \in X$ there exists $y \in Y$ such that $\text{card}(x \setminus y) < \kappa$ (this follows immediately Definition 13 and the fact that $\text{HoQtNaamen}_\kappa^+$ is obtained by inverting all (w) -arrows in QtNaamen_κ^+).

It is now straightforward to verify that the left derived functor of $\text{card} : \text{QtNaamen}_\kappa^+ \longrightarrow \text{On}^\top$ is Shelah's revisited power function:

$$(\dagger) \quad \mathbb{L}_c \text{card}(\{\lambda\}) = \lambda^{[\kappa]} := \text{cov}(\lambda, \kappa^+, \kappa^+, \kappa).$$

Indeed, $\text{cov}(\lambda, \Delta, \theta, \sigma)$ is the least size of a family $X \subseteq [\lambda]^{<\Delta}$, such that every subset of λ of cardinality smaller than θ , lies in a union of less than σ subsets in X . In our notation, taking $\Delta = \kappa^+ = \theta$ and $\sigma = \kappa$, the condition on the family X can be stated as: $X \longrightarrow [\lambda]^{\leq \kappa}$ and $[\lambda]^{\leq \kappa} \longrightarrow \bigcup_{<_\kappa} X$. Now, the first of these conditions is precisely $\emptyset \xrightarrow{(c)} X \longrightarrow \{\lambda\}$, whereas the second condition is $\bigcup_{\leq \kappa} X \longleftarrow Y \xrightarrow{(wf)} \{\lambda\}$ for some Y . But in StNaamen_κ^+ (and therefore in QtNaamen_κ^+), $\bigcup_{<_\kappa} X \longleftrightarrow X$. Therefore, this last condition is equivalent to $X \longleftarrow Y \xrightarrow{(wf)} \{\lambda\}$. Combining everything together we get that $\text{cov}(\lambda, \kappa^+, \kappa^+, \kappa) \geq \mathbb{L}_c \text{card}(\{\lambda\})$. The proof of the other direction is similar (modulo the obvious adaptations) to the proof of the analogous fact in Theorem 40.

The model category QtNaamen_κ^+ allows us to formulate quite easily the notion of the cardinal κ being (non) measurable. Recall that a cardinal κ is *measurable* if it is uncountable and admits a k -complete non-principal ultrafilter, or, equivalently, a 0-1 valued probability countably additive measure such that every subset is measurable. Such an ultrafilter exists on the cardinal $\kappa = \omega$, as any filter is ω -complete. We prove that:

Lemma 44. *The following are equivalent for a regular cardinal $\kappa > \omega$:*

- (1) κ is not measurable.
- (2) For all $X \in \text{ObQtNaamen}_\kappa^+$ if $X \xrightarrow{(i)} \{\kappa\}$ then $X \xrightarrow{(w)} \{\kappa\}$.
- (3) $X \xrightarrow{(i)} Y \xleftarrow{(c)} \perp$ implies $X \xrightarrow{(wc)} Y$.
- (4) In $\text{HoQtNaamen}_\kappa^+$ if $X \xrightarrow{(i)} Y$ then $Y \longrightarrow X$ for all X, Y .

where $X \xrightarrow{(i)} Y$ means that $X \longrightarrow Y$ is an indecomposable arrow.

Proof. First, we observe that if $X \in \text{ObStNaamen}_\kappa^+$ and $\bar{X} = \bigcup_{<_\kappa} X$ then $\bar{X} \longleftrightarrow X$ (in StNaamen_κ^+). In particular, $X \in \text{ObQtNaamen}_\kappa^+$ if and only if \bar{X} is. Note that if $X \in \text{ObStNaamen}_\kappa^+$ and X is a non-empty set then \bar{X} is a κ -complete ideal on $\bigcup X$. Indeed, it is closed under unions of size less than κ , and by definition \bar{X} is downward closed.

Thus, if $X \in \mathcal{ObStNaamen}_\kappa^+$ then the statement $X \longrightarrow \{\kappa\}$ is equivalent to $\bar{X} \longrightarrow \{\kappa\}$ and since $\{\bigcup X\} \in \mathcal{QtNaamen}_\kappa^+$ we get $X \longrightarrow \{\bigcup X\} \longrightarrow \{\kappa\}$. If, in addition, $X \xrightarrow{(i)} \{\kappa\}$ then either $\{\bigcup X\} \leftrightarrow X$ or $\bigcup X = \kappa$. But if $\bigcup X \neq \kappa$ then $X \longrightarrow \{\kappa\}$ is not indecomposable (take $X \cup \{y\}$ for any $y \in \{\kappa\} \setminus \bigcup X$). So $X \xrightarrow{(i)} \{\kappa\}$ is equivalent to \bar{X} being a maximal ideal on κ which is also κ -complete.

It remains, therefore, to ascertain when is such an ideal principal. On the one hand, it is obvious that if X is a maximal principal (κ -complete) ideal on λ then $X \xrightarrow{(wc)} \{\kappa\}$. Now assume that X is a maximal ideal on $\{\kappa\}$ which is κ -complete. Then $X \longrightarrow \{\kappa\}$, which is, by Definition 13, equivalent to $X \xrightarrow{(c)} \{\kappa\}$, and assume that $X \xrightarrow{(w)} \{\kappa\}$. Then $X \xrightarrow{(wc)} \{\kappa\}$, which — by definition — means some $x \in X$ satisfies $\text{card}(\kappa \setminus x) < \kappa$, and since X is k -complete this means that $\bigcup X \neq \{\kappa\}$ (if it were, then already a small union would cover everything). Maximality implies that in that case X is principal.

The above shows the equivalence of (1) and (2) above, as well as (3) \Rightarrow (1). So it remains to prove (2) \Rightarrow (3). Indeed, assume that X, Y are as in (3). We may assume that $X = \bar{X}$. We may also assume that there exists some $y \in Y$ such that $\text{card}(y) = \kappa$ (otherwise $X \xrightarrow{(wc)} Y$ is automatic from Definition 13). So fix any $y \in Y$. It will suffice to show that $X_y := \{x \in X : x \subseteq y\}$ is a maximal k -complete ideal on y . This will be enough since then, by (2) this ideal is principal, and as $y \in Y$ was arbitrary of cardinality κ , we will be done, by Definition 13. So it remains to show that X_y is a maximal ideal on y . That is it is a k -complete ideal is proved exactly as above. So we only have to verify its maximality, which is immediate from the indecomposability of the arrow $X \longrightarrow Y$.

To see the equivalence with (4) we may assume that X and Y are cofibrant. Further, note that $X \xrightarrow{(i)}_h Y$ if there exists a sequence as in (\diamond) in which all but one of the arrows is a (w) -arrow, and this arrow is indecomposable. But, if

$$\perp \xrightarrow{(c)} X \xrightarrow{(w)} X_1 \xleftarrow{(w)} X_2 \xrightarrow{(w)} X_3 \xleftarrow{(w)} \dots X_i \xrightarrow{(i)} X_{i+1} \xrightarrow{(w)} X_n \longrightarrow Y \xleftarrow{(c)} \perp$$

then in HoQtNaamen_κ the object X is isomorphic to X_i and Y is isomorphic to X_{i+1} and X_{i+1} is cofibrant. Thus, (3) \iff (4). \square

We point out that, since $\mathcal{QtNaamen}_\kappa^+$ is a closed model category (i.e., it satisfies axiom (M6)), condition (3) above can be expressed as a lifting property:

$$\perp \xrightarrow{(c)} Y \Rightarrow X \xrightarrow{(i)} Y \triangleleft X' \xrightarrow{(f)} Y'.$$

Thus, the notion of κ being a (non)-measurable cardinal has an essentially model category-theoretic interpretation.

Remark 45. (1) *It is well known (and easy to prove) that if there is no measurable cardinal below κ then κ is measurable if and only if it admits a σ -complete ultrafilter, in other words, a countably additive measure such that every subset has measure either 0 or 1. Thus, the statement “there is no measurable cardinal” is equivalent to the statement “the only indecomposable arrows in $\text{StNaamen}_{\aleph_1}$ are (wc) -arrows”.*

(2) *The previous lemma also gives a definition of strongly compact cardinals. Recall that an uncountable cardinal, κ , is strongly compact if every κ -complete filter can be extended to a κ -complete ultra-filter (so κ -compact*

cardinals are certainly measurable). The proof of (4) of the previous lemma shows that κ is strongly compact if and only if every morphism $X \rightarrow Y$ in $\text{HoQtNaamen}_\kappa^+$ decomposes into $X \rightarrow Y' \xrightarrow{(i)} Y$ with $Y' \rightarrow Y$ not an isomorphism (provided $X \rightarrow Y$ is not an isomorphism).

The following gives another intriguing set theoretic angle to the model category QtNaamen_κ^+ . Let L denote, as usual, Gödel's constructible model of set theory. Recall, e.g., Theorem [12, Theorem 13.9], that L is the least transitive class (i.e., $L \rightarrow \{L\}$) closed under all Gödel operations, and universal in the sense that $X \subseteq L$ implies $X \subseteq Y \in L$. Let StNaamen^L be the full sub-category of StNaamen whose objects are sub-classes of L definable within L . In other words, viewing L as a model of ZFC and ignoring the ambient universe V , we let ObStNaamen^L be all sub-classes of L . Note that as L itself is a class element in ObStNaamen^L is, in fact an object of StNaamen .

Obviously, by the remark concluding the previous paragraph, since $L \models \text{ZFC}$, we can perform the construction of the first part of the paper in StNaamen^L . However, since notions of countability and (infinite) equicardinality are not absolute, the labelling obtained in this way will, in general, not coincide with the labelling induced on StNaamen^L from StNaamen . Indeed, the labelling induced on StNaamen^L from StNaamen will not (in general) satisfy axiom (M2) (whereas, the the labelling following the construction does).

In the above, it seems clear that the labelling induced on StNaamen^L from StNaamen is not the "right" one. The situation is less clear when trying to give sub-categories of StNaamen^L a model structure. As mentioned above, carrying out the construction we can obtain QtNaamen^L and the associated model categories $\text{QtNaamen}_\kappa^+(L)$ (where κ is a cardinal in L). But it seems that under certain set-theoretic assumptions other model structures can also be constructed.

Let κ be a (regular) cardinal in V . Let $\text{QtNaamen}_\kappa^+(L)$ be the full labelled sub-category of QtNaamen_κ^+ whose objects are also in ObStNaamen^L . It is not hard to check that $\text{QtNaamen}_\kappa^+(L)$ is closed under (small) limits, and that, being a full sub-category of QtNaamen_κ^+ it also satisfies (M1) and (M3)-(M6) (though (M6) requires a small calculation). Recall that the (M2)-decomposition of each arrow in QtNaamen_κ^+ is unique (up to QtNaamen_κ^+ -isomorphism), so in order to check whether $\text{QtNaamen}_\kappa^+(L)$ satisfies (M2) we have to show that if either $X \xrightarrow{(c)} Z \xrightarrow{(wf)} Y$ or $X \xrightarrow{(wc)} Z \xrightarrow{(f)} Y$ with $X, Y \in \text{ObQtNaamen}_\kappa^+(L)$ then Z is isomorphic to an element in $\text{ObQtNaamen}_\kappa^+(L)$. Now, recall that, by definition Z is the class of all $z \subseteq y \in Y$ such that $\text{card } z \leq \text{card } x + \kappa$ for some $x \in X$, or Z is the class of all $z \subseteq y \in Y$ such that $\text{card}(z \setminus x) < \kappa$ for some $x \in X$. Observe that, since X and Y are definable within L so is the class $Z_L := Z \cap L$ of all constructible members of Z . Thus, it will suffice to show that $Z_L \longleftrightarrow Z$. Of course, in general, there is no reason for this to be true. But if $\kappa > \aleph_1$ then this statement is equivalent to the conclusion of Jensen's covering lemma (for κ).

Thus, e.g., if $0^\#$ does not exist and $\kappa > \aleph_1$ then $\text{QtNaamen}_\kappa^+(L)$ is a model category. It is an easy exercise to check that there is no cardinal $\lambda \in L$ such that $\text{QtNaamen}_\kappa^+(L)$ is precisely the model category $\text{QtNaamen}_\lambda^+$ constructed within L . We do not know whether $\text{QtNaamen}_\kappa^+(L)$ could be equivalent to $\text{QtNaamen}_\lambda^+$ for some λ (with the latter constructed within L).

6. SUGGESTIONS FOR FUTURE RESEARCH

Among the possible objections to the work presented in the present paper there are two which we view as most intriguing. These are the coherence and usefulness of work.

The problem which we call *coherence* is that, as we already hinted above, if $f : \mathfrak{C} \rightarrow \text{On}^\top$ is any function on the posetal model category \mathfrak{C} , then the value of the (cofibrantly replaced) left-derived functor of f is not necessarily invariant under the equivalence of model categories. Namely, if $\mathfrak{C}' \equiv \mathfrak{C}$ (as model categories) and $F : \mathfrak{C}' \rightarrow \mathfrak{C}$ is a witness (of one direction of) this equivalence then $\mathbb{L}_c(F \circ f)(x)$ is not necessarily the same as $\mathbb{L}_c f(F(x))$.

This is most obvious in our calculation of Shelah's revised power function. In deriving the cardinality function on QtNaamen_κ^+ we obtain the desired result, but if we tried doing the same on the equivalent model category QtNaamen_κ , we would have obtained a different answer. The same situation would have occurred if trying to derive cardinality in QtNaamen we worked with the (equivalent) full subcategory of "downward closed" objects.

Because the derivation of a function $f : \mathfrak{C} \rightarrow \text{On}^\top$ on a posetal model category can be viewed as a minimization operation (of $f(x)$ over all $x' \in \text{Ob}\mathfrak{C}$ homotopy equivalent to x) our (informal) approach to this problem was that the "correct" derivation is the one giving the minimal results, i.e., if $\mathfrak{C}' \equiv \mathfrak{C}$ witnessed by the functor $F : \mathfrak{C} \rightarrow \mathfrak{C}'$ which is injective on $\text{Ob}\mathfrak{C}$ then the "correct" function to derive is $(f \circ F)$, rather than F . The first problem for future research is, therefore

Problem 46. *Let \mathfrak{C} be a posetal model category, $f : \mathfrak{C} \rightarrow \text{On}^\top$ any function. Find a functor $\tilde{\mathbb{L}}_f f : \mathfrak{C} \rightarrow \text{On}^\top$ such that*

- (1) $\tilde{\mathbb{L}}_f f(x) \leq \mathbb{L}_f f(x)$ for all x .
- (2) $\mathbb{L}_f f$ is not trivial (unless, say, $L_{(f \circ F)}$ is trivial for every functor $F : \mathfrak{C}' \rightarrow \mathfrak{C}$ with $\mathfrak{C}' \equiv \mathfrak{C}$).
- (3) $\tilde{\mathbb{L}}_f f$ is invariant under the equivalence of model categories (in the sense explained above).

In other words, extend the notion of the left derived functor of a functor $f : \mathfrak{C} \rightarrow \text{On}^\top$ to a larger class of function with the result as invariant as possible under equivalence of model categories.

The second objection to the present work relates to its usefulness. Here is a list of problems, a positive answer to some of which could indicate of the usefulness of the new tools developed in the present work:

Problem 47. *Are there more combinatorial concepts that can be captured by our suggested formalism, e.g., closed unbounded sets, stationary sets, Fodor's lemma, diamond, square etc..*

As a somewhat speculative special case of the previous problem consider the fact that there are no measurable cardinals in L . As we have seen in Remark 45 the statement "there are no measurable cardinals" can be restated in our geometric language. Thus it is natural to ask:

Problem 48. *Can it be proved using (mainly) the language of model categories that there are no measurable cardinals in L . In other words, can an analogue of Scott's theorem [18] stating that if there are measurable cardinals then $V \neq L$ be proved using our geometric language?*

As we do not have any “geometric” characterisation of L (unlike, e.g., the set theoretic characterisation of L being the smallest inner model, i.e., the smallest submodel of V containing all ordinals, or the smallest transitive universal class closed under Gödel operations), the above question is somewhat speculative. As in our treatment of ordinals in Section 4.1, it seems reasonable to use some auxiliary notions such as naming $On \in \mathcal{ObStNaamen}$ to address this problem.

Problem 49. *Apparently, given a model structure on a category \mathfrak{C} , the computation of homotopy limits (i.e. the computation of limits in the associated homotopy category) gives in many cases important information on the category \mathfrak{C} . In the case of $\mathcal{QtNaamen}$, one can easily give an explicit combinatorial interpretation of the limit (at least for set-sized diagrams). Are these objects of set theoretic significance? More generally, is there a set theoretic significance to the class of cute objects? To the homotopy category itself? Are there other derived functors defining invariants of models of ZFC that, say, can be bounded in ZFC?*

In classical homotopy theory, homotopy groups (by themselves) and the associated structures (such as long exact sequences) are powerful tools allowing many calculations. In (pointed) model categories analogues of such constructions exist, such as the groupoid of homotopy classes between any two objects A, B (where A is a cofibrant object and B is a fibrant object) as well as other constructs, analogues of other classical homotopical tools such as the suspension and loop functors, fibration sequences and more. In posetal model categories these constructions degenerate, and much of the computational power of the associated homotopy structure is lost. This may be one of the reasons that while we were able to recover homotopical interpretations of important and non-trivial set theoretic objects we were unable to prove any of their properties using the model category structure on $\mathcal{QtNaamen}$.

In view of the above it is interesting to look for other constructions in $\mathcal{QtNaamen}$ (or $\mathcal{HoQtNaamen}$), which may serve as analogues of the above mentioned model categorical constructions. One possible such construction is the sequence of model categories $\mathcal{QtNaamen}_\kappa$ when κ ranges over all cardinals.

First, recall that we were able to give the category $\mathcal{QtNaamen}_\kappa$ a model structure only under the assumption that κ is a regular cardinal. A first problem is, therefore, to construct a similar model category for singular κ . It seems that such a model category can be constructed inductively (assuming $\mathcal{QtNaamen}_\lambda$ was constructed for all $\lambda < \kappa$) by taking an appropriate “limiting” process. For example, one could define

$$\mathcal{ObStNaamen}_\kappa := \left\{ \bigcup_{\lambda < \kappa} X_\lambda : X_\lambda \in \mathcal{ObStNaamen}_\kappa, X_\lambda \subseteq X_{\lambda'} \text{ if } \lambda < \lambda' \right\}$$

with the additional requirement that if $X = \bigcup_{\lambda < \kappa} X_\lambda$ as above then the X_λ are uniformly definable (this is required in order to assure that X is, indeed, a class). And the labelling

$$X \xrightarrow{(*)}_\kappa Y \iff (\forall^* \lambda)(y \in Y \Rightarrow X \times \{y\} \xrightarrow{(*)}_\lambda Y)$$

where $\forall^* \lambda$ means “for all large enough $\mu < \kappa$ ”. Passing to the full sub-class of *cute* objects (with respect to this labelling) we apparently get a model category $\mathcal{QtNaamen}_\kappa$. It is unclear to us, however, whether this construction is the “correct” one.

There is also an obvious functor $F_\kappa : \text{StNaamen}_\kappa \rightarrow \text{StNaamen}_{\kappa^+}$ given by $X \mapsto \bigcup_{<\kappa^+} X$ for $X \in \mathcal{Ob}\text{StNaamen}_\kappa$. Indeed, this is a functor of model categories: this is obvious for (c) and (f) arrows, and not much harder for (w) -arrows, with the conclusion following from Remark 43. On the level of the associated homotopy categories, it is clear that, $\gamma(F_\kappa(X)) = \perp_{\text{QtNaamen}_\kappa^+}$ for any cofibrant $X \in \mathcal{Ob}\text{QtNaamen}_\kappa$ (where, as above, $\gamma : \text{QtNaamen}_\kappa \rightarrow \text{HoQtNaamen}_\kappa$ is the localization functor). Since the cofibrant objects of any model category suffice to determine the associated homotopy category, it follows that the homotopy category associated with the image of QtNaamen_κ under F_κ is trivial. This gives the sequence of categories QtNaamen_κ a certain flavour of “exactness”, which seem to require some further research.

6.1. Looking back. We conclude these notes looking back to the original motivation leading to the development of the model category QtNaamen , i.e., the goal of developing a homotopy structure for the class of models of an uncountably categorical theory and, more generally, to (quasi-minimal) excellent abstract elementary classes (see, e.g., [3] for the details). The need for homotopy theoretic tools in this contexts arose through the study, by Zilber and his school, of categoricity problems of model theoretic structures such as pseudo-exponentiation [22] and covers of semi-Abelian varieties [4], [5]. The model theoretic analysis needed to show the (uncountable) categoricity of the natural examples studied in the above mentioned references uses known number theoretic and algebro-geometric results and conjectures nowadays understood as being of essentially cohomological character, and formulated in functorial language. Such statements are, e.g., particular cases of André’s generalized Grothendieck conjectures on periods of motives ([6], [2, 7.5.2.1 Conjecture],[15, §4.2 Conjecture,§1.2 Conjecture]), the Mumford-Tate conjecture on the image of Galois action on the first étale cohomology ([19]), Kummer theory ([17]), and more. This does not seem to be entirely coincidental, as the definition of Shelah’s, so called, Excellent classes — the model theoretic machinery employed in this study — and in particular the requirement that there exists a unique prime model over maximally independent tuples of countable (sub) models (and that this requirement makes sense) reminds, at least superficially, some of the axioms of a model category.

However, the common model theoretic language does not seem to have the means to incorporate this functorial language in its full power and generality. Thus, in order to be applied in addressing the above mentioned categoricity problems “old-fashioned” reformulations of these conjectures, deprived of their functorial language and homological character had to be used — Schanuel’s conjecture and its cognates explicated by Bertolin [6] derived from the generalized Grothendieck conjecture on periods, Bashmakov’s original formulations of Kummer theory for elliptic curves ([4], [10]), and Serre’s explicit description of the image of the Galois action on the Tate module as a subgroup of the profinite group $\text{GL}_2(\hat{\mathbb{Z}})$.

It is the first author’s belief that the inability of common model theoretic language to digest these statements in their full power and generality is a major obstacle in further exploring these intriguing connections between Shelah’s excellent classes and deep algebro-geometric conjectures. The homotopy theoretic approach to set theory discussed in the present paper is a toy example exploring the ways in which homotopy theoretic language could be introduced into the realm of model theory.

Unfortunately, we were unable to use the model category QtNaamen to associate such a homotopy structure to those classes of models. In fact, it is not even clear to us when this could be done:

Problem 50. *Let \mathfrak{K} be a (quasiminimal) excellent abstract elementary class (e.g. algebraically closed fields of characteristic p , models of pseudo-exponentiation). Let $\text{QtNaamen}(\mathfrak{K})$ be the sub-category of QtNaamen whose objects are elements of \mathfrak{K} and such that for $\mathcal{M}, \mathcal{N} \in \text{Ob}\mathfrak{K}$ there is an arrow $\mathcal{M} \rightarrow \mathcal{N}$ if $\mathcal{M} \prec \mathcal{N}$. Are there natural model theoretic conditions under which $\text{QtNaamen}(\mathfrak{K})$ is a model category? What about $\text{QtNaamen}_\kappa(\mathfrak{K})$? Is there a similar construction associating a model category to the class \mathfrak{K} ?*

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