The unreasonable power of the lifting property in elementary mathematics

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in memoriam: evgenii shurygin

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> instances of human and animal behavior [...] miraculously complicated, [...] they have little, if any, pragmatic (survival/reproduction) value. [...] they are due to internal constraints on possible architectures of unknown to us functional "mental structures".
>
> Gromov, Ergobrain

Abstract

We illustrate the generative power of the lifting property (orthogonality of morphisms in a category) as a means of defining natural elementary mathematical concepts by giving a number of examples in various categories, in particular showing that many standard elementary notions of abstract topology can be defined by applying the lifting property to simple morphisms of finite topological spaces. Examples in topology include the notions of: compact, discrete, connected, and totally disconnected spaces, dense image, induced topology, and separation axioms. Examples in algebra include: finite groups being nilpotent, solvable, torsion-free, p-groups, and prime-to-p groups; injective and projective modules; injective, surjective, and split homomorphisms.

We include some speculations on the wider significance of this.

1 Introduction.

The purpose of this short note is to draw attention to the following observation which we find rather curious:

> a number of elementary properties from a first-year course can be defined category-theoretically by repeated application of a standard category theory trick, the Quillen lifting property, starting from a class of explicitly given morphisms, often consisting of a single (counter)example

*A draft; comments welcome. `mishap@sdf.org`. current version at `http://mishap.sdf.org/expressive-power-of-the-lifting-property.pdf`. Minor updates 7.17; 8.8.17.

In particular, several elementary notions of topology have Kolmogorov complexity of several bytes in a natural category-theoretic formalism (explained below), e.g.

compactness is "$((\{a\} \longrightarrow \{a \searrow b\})^r_{<5})^{lr}$",
connectedness is "$(\{a, b\} \longrightarrow \{a = b\})^l$",
dense image is "$(\{b\} \longrightarrow \{a \searrow b\})^l$".

We suggest it appears worthwhile to try to develop a formalism, or rather a very short program (kilobytes) based on such a formalism, which supports reasoning in elementary topology.

These observations arose in an attempt to understand ideas of Misha Gromov [Memorandum Ergo] about ergologic/ergostructure/ergosystems. Oversimplifying, ergologic is a kind of reasoning which helps to understand how to generate proper concepts, ask interesting questions, and, more generally, produce interesting rather than useful or correct behaviour. He conjectures there is a related class of mathematical, essentially combinatorial, structures, called *ergostructures* or *ergosystems*, and that this concept might eventually help to understand complex biological behaviour including learning and create mathematically interesting models of these processes.

We hope our observations may eventually help to uncover an essentially combinatorial reasoning behind elementary topology, and thereby suggest an example of an ergostructure.

Related works. This paper continues work started in [DMG], a rather leisurely introduction to some of the ideas presented here. Draft [Gavrilovich, Elementary Topology] shows how to view several topology notions and arguments in [Bourbaki, General Topology] as diagram chasing calculations with finite categories. Draft [Gavrilovich, Tame Topology] is more speculative but less verbose; it has several more examples dealing with compactness, in particular it shows that a number of consequences of compactness can be expressed as a change of order of quantifiers in a formula. Notably, these drafts show how to "read off" a simplicial topological space from the definition of a uniform space, see also Remarks 2 and 7.

Structure of the paper. A mathematically inclined reader might want to read only the first two sections with miscellaneous examples of lifting properties and a combinatorial notation for elementary properties of topological spaces. A logician might want to read in the third section our suggestions towards a theorem prover/proof system for elementary topology based on diagram chasing. Appendix A states separation axioms in terms of lifting properties and finite topological spaces. Appendix B reproduces some references we use.

Finally, the last section attempts to explain our motivation and says a few words about the concept of ergostructure by Misha Gromov.

We would also like to draw attention to Conjecture 1 (a charaterisation of the class of proper maps) and Question 2 asking for a characterisation of the circle and the interval.

2 The lifting property: the key observation

For a property C of arrows (morphisms) in a category, define

$$C^l := \{f : \text{ for each } g \in C \ f \rightthreetimes g\}$$
$$C^r := \{g : \text{ for each } f \in C \ f \rightthreetimes g\}$$
$$C^{lr} := (C^l)^r, ...$$

here $f \rightthreetimes g$ reads " f has the left lifting property wrt g ", " f is (left) orthogonal to g ", i.e. for $f : A \longrightarrow B$, $g : X \longrightarrow Y$, $f \rightthreetimes g$ iff for each $i : A \longrightarrow X$, $j : B \longrightarrow Y$ such that $ig = fj$ ("the square commutes"), there is $j' : B \longrightarrow X$ such that $fj' = i$ and $j'g = j$ ("there is a diagonal making the diagram commute").

The following observation is enough to reconstruct all the examples in this paper, with a bit of search and computation.

> **Observation.**
> A number of elementary properties can be obtained by repeatedly passing to the left or right orthogonal $C^l, C^r, C^{lr}, C^{ll}, C^{rl}, C^{rr}, ...$ starting from a simple class of morphisms, often a single (counter)example to the property you define.

A useful intuition is to think that the property of left-lifting against a class C is a kind of negation of the property of being in C, and that right-lifting is another kind of negation. Hence the classes obtained from C by taking orthogonals an odd number of times, such as $C^l, C^r, C^{lrl}, C^{lll}$ etc., represent various kinds of negation of C, so $C^l, C^r, C^{lrl}, C^{lll}$ each consists of morphisms which are far from having property C.

Taking the orthogonal of a class C is a simple way to define a class of morphisms excluding non-isomorphisms from C, in a way which is useful in a diagram chasing computation.

The class C^l is always closed under retracts, pullbacks, (small) products (whenever they exist in the category) and composition of morphisms, and contains all isomorphisms of C. Meanwhile, C^r is closed under retracts, pushouts, (small) coproducts and transfinite composition (filtered colimits) of morphisms (whenever they exist in the category), and also contains all isomorphisms.

For example, the notion of isomorphism can be obtained starting from the class of all morphisms, or any single example of an isomorphism:

$$(Isomorphisms) = (all\ morphisms)^l = (all\ morphisms)^r = (h)^{lr} = (h)^{rl}$$

where h is an arbitrary isomorphism.

Example.

> Take $C = \{\varnothing \longrightarrow \{*\}\}$ in Sets and Top. Let us show that C^l is the class of surjections, C^{rr} is the class of subsets, C^l consists of maps $f : A \longrightarrow B$ such that either $A = B = \varnothing$ or $A \neq \varnothing$, B arbitrary. Further, in Sets, C^{rl} is the class of injections, and in Top, C^{rl} is the class of maps of form $A \longrightarrow A \cup D$, D is discrete.

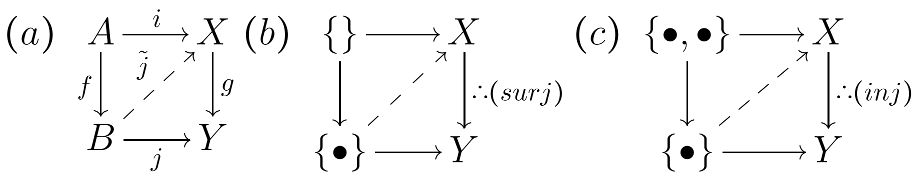

Figure 1: Lifting properties. (a) The definition of a lifting property $f \rightthreetimes g$. (b) $X \longrightarrow Y$ is surjective (c) $X \longrightarrow Y$ is injective

> $A \longrightarrow B \rightthreetimes \varnothing \longrightarrow \{*\}$ iff $A = B = \varnothing$ or $A \neq \varnothing$, B arbitrary. Indeed, if $A \neq \varnothing$, there is no map $i : A \longrightarrow X = \varnothing$ and the lifting property holds vacuously; if $A = \varnothing \neq B$, there exist unique maps $i : A = \varnothing \longrightarrow X = \varnothing$, $j : B \longrightarrow Y = \{*\}$, but no map $j' : B \longrightarrow X = \varnothing$ as $B \neq \varnothing$ by assumption.
>
> $\varnothing \longrightarrow \{*\} \rightthreetimes g$ iff $g : X \longrightarrow Y$ is surjective; indeed, the map $j : B = \{*\} \longrightarrow Y$ picks a point in Y and $j' : B = \{*\} \longrightarrow X$ picks its preimage as $j'g = j$; the other condition $fj' = i : \varnothing \longrightarrow X$ holds trivially. Thus $(\varnothing \longrightarrow \{*\})^r$ is the class of surjections.
>
> In Sets, $(\varnothing \longrightarrow \{*\})^{rl}$ is the class of injections, i.e. $f \rightthreetimes g$ for each surjection g iff f is injective; indeed, for such f and g the following is well-defined: set $j'(b) = i(f^{-1}(b))$ for b in Imf, and $j'(b) = g^{-1}(j(b))$ otherwise; for injective f $j'(b)$ does not depend on the choice of a preimage of b, and for g surjective a preimage always exists.
>
> In Top, $(\varnothing \longrightarrow \{0\})^{rl}$ is the class of maps of form $A \longrightarrow A \cup D$, D is discrete; given a map $A \longrightarrow B$, consider $A \longrightarrow B \rightthreetimes ImA \cup (B \setminus A) \longrightarrow B$ where $ImA \cup D \longrightarrow B$ denotes the disjoint union of the image of A in B with induced topology, and $B \setminus A$ equipped with the discrete topology.
>
> In both Sets and Top, $(\varnothing \longrightarrow \{*\})^{rr}$ is the class of subsets, i.e. injective maps $A \hookrightarrow B$ where the topology on A is induced from B.

Toying with the observation leads to the examples in the claim below which is trivial to verify, an exercise in deciphering the notation in all cases but (vii) proper.

Claim 1. *(i)* $(\varnothing \longrightarrow \{*\})^r$, $(0 \longrightarrow R)^r$, *and* $\{0 \longrightarrow \mathbb{Z}\}^r$ *are the classes of surjections in in tha categories of Sets, R-modules, and Groups, resp., (where* $\{*\}$ *is the one-element set, and in the category of (not necessarily abelian) groups,* 0 *denotes the trivial group)*

(ii) $(\{\star, \bullet\} \longrightarrow \{*\})^l = (\{\star, \bullet\} \longrightarrow \{*\})^r$, $(R \longrightarrow 0)^r$, $\{\mathbb{Z} \longrightarrow 0\}^r$ *are the classes of injections in the categories of Sets, R-modules, and Groups, resp*

(iii) in the category of R-modules,
a module P *is projective iff* $0 \longrightarrow P$ *is in* $(0 \longrightarrow R)^{rl}$
a module I *is injective iff* $I \longrightarrow 0$ *is in* $(R \longrightarrow 0)^{rr}$

(iv) in the category of Groups,

$$\text{a finite group } H \text{ is nilpotent iff } H \longrightarrow H \times H \text{ is in } \{0 \longrightarrow G : G \text{ arbitrary}\}^{lr}$$

a finite group H is solvable iff $0 \longrightarrow H$ *is in* $\{\, 0 \longrightarrow A : A$ *abelian* $\}^{lr} = \{\, [G,G] \longrightarrow G : G$ *arbitrary* $\}^{lr}$

a finite group H is of order prime to p iff $H \longrightarrow 0$ *is in* $\{\mathbb{Z}/p\mathbb{Z} \longrightarrow 0\}^r$

a finite group H is a p-group iff $H \longrightarrow 0$ *is in* $\{\mathbb{Z}/p\mathbb{Z} \longrightarrow 0\}^{rr}$

a group H is torsion-free iff $0 \longrightarrow H$ *is in* $\{n\mathbb{Z} \longrightarrow \mathbb{Z} : n > 0\}^r$

a group F is free iff $0 \longrightarrow F$ *is in* $\{0 \longrightarrow \mathbb{Z}\}^{rl}$

a homomorphism f is split iff $f \in \{\, 0 \longrightarrow G : G$ *arbitrary*$\}^r$

(v) in the category of metric spaces and uniformly continuous maps,
a metric space X is complete iff $\{1/n\}_n \longrightarrow \{1/n\}_n \cup \{0\} \rightthreetimes X \longrightarrow \{0\}$ *where the metric on* $\{1/n\}_n$ *and* $\{1/n\}_n \cup \{0\}$ *is induced from the real line*
a subset $A \subset X$ is closed iff $\{1/n\}_n \longrightarrow \{1/n\}_n \cup \{0\} \rightthreetimes A \longrightarrow X$

(vi) in the category of topological spaces,
for a connected topological space X, each function on X is bounded iff

$$\varnothing \longrightarrow X \rightthreetimes \cup_n(-n,n) \longrightarrow \mathbb{R}$$

(vii) in the category of topological spaces (see notation defined below),

a Hausdorff space K is compact iff $K \longrightarrow \{*\}$ *is in* $((\{a\} \longrightarrow \{a\searrow b\})^r_{<5})^{lr}$

a Hausdorff space K is compact iff $K \longrightarrow \{*\}$ *is in*

$\{\, \{a \leftrightarrow b\} \longrightarrow \{a = b\},\ \{a\searrow b\} \longrightarrow \{a = b\},\ \{b\} \longrightarrow \{a\searrow b\},\ \{a\swarrow o\searrow b\} \longrightarrow \{a = o = b\}\, \}^{lr}$

a space D is discrete iff $\varnothing \longrightarrow D$ *is in* $(\varnothing \longrightarrow \{*\})^{rl}$

a space D is antidiscrete iff $D \longrightarrow \{*\}$ *is in* $(\{a,b\} \longrightarrow \{a = b\})^{rr} = (\{a \leftrightarrow b\} \longrightarrow \{a = b\})^{lr}$

a space K is connected or empty iff $K \longrightarrow \{*\}$ *is in* $(\{a,b\} \longrightarrow \{a = b\})^l$

a space K is totally disconnected and non-empty iff $K \longrightarrow \{*\}$ *is in* $(\{a,b\} \longrightarrow \{a = b\})^{lr}$

a space K is connected and non-empty iff for some arrow $\{*\} \longrightarrow K$
$\{*\} \longrightarrow K$ *is in* $(\varnothing \longrightarrow \{*\})^{rll} = (\{a\} \longrightarrow \{a,b\})^l$

a space K is non-empty iff $K \longrightarrow \{*\}$ *is in* $(\varnothing \longrightarrow \{*\})^l$

a space K is empty iff $K \longrightarrow \{*\}$ *is in* $(\varnothing \longrightarrow \{*\})^{ll}$

a space K is T_0 iff $K \longrightarrow \{*\}$ *is in* $(\{a \leftrightarrow b\} \longrightarrow \{a = b\})^r$

a space K is T_1 iff $K \longrightarrow \{*\}$ *is in* $(\{a\searrow b\} \longrightarrow \{a = b\})^r$

a space X is Hausdorff iff for each injective map $\{x,y\} \hookrightarrow X$ *it holds* $\{x,y\} \hookrightarrow X \rightthreetimes \{x\searrow o\swarrow y\} \longrightarrow \{x = o = y\}$

a non-empty space X is regular (T3) iff for each arrow $\{x\} \longrightarrow X$ *it holds* $\{x\} \longrightarrow X \rightthreetimes \{x\searrow X\swarrow U\searrow F\} \longrightarrow \{x = X = U\searrow F\}$

a space X is normal (T4) iff $\varnothing \longrightarrow X \rightthreetimes \{a\swarrow U\searrow x\swarrow V\searrow b\} \longrightarrow \{a\swarrow U = x = V\searrow b\}$

a space X is completely normal iff $\varnothing \longrightarrow X \rightthreetimes [0,1] \longrightarrow \{0 \swarrow x \searrow 1\}$ *where the map* $[0,1] \longrightarrow \{0 \swarrow x \searrow 1\}$ *sends* 0 *to* 0, 1 *to* 1, *and the rest* $(0,1)$ *to* x

a space X is path-connected iff $\{0,1\} \longrightarrow [0,1] \rightthreetimes X \longrightarrow \{*\}$

a space X is path-connected iff for each Hausdorff compact space K and each injective map $\{x,y\} \hookrightarrow K$ *it holds* $\{x,y\} \hookrightarrow K \rightthreetimes X \longrightarrow \{*\}$

$(\varnothing \longrightarrow \{*\})^r$ *is the class of surjections*

$(\varnothing \longrightarrow \{*\})^{rr}$ *is the class of subsets, i.e. injective maps* $A \hookrightarrow B$ *where the topology on A is induced from B*

$(\varnothing \longrightarrow \{*\})^{lll}$ *is the class of maps* $A \longrightarrow B$ *which split*

$(\{b\} \longrightarrow \{a \searrow b\})^l$ *is the class of maps with dense image*

$(\{b\} \longrightarrow \{a \searrow b\})^{lr}$ *is the class of closed subsets* $A \subset X$, *A a closed subset of X*

$((\{a\} \longrightarrow \{a \searrow b\})^r_{<5})^{lr}$ *is roughly the class of proper maps (see below).*

Proof. In (iv), we use that a finite group H is nilpotent iff the diagonal $\{(h,h) : h \in H\}$ is subnormal in $H \times H$. (vii) is discussed below. QED.

Appendix A shows that the usual formulations of the separation axioms are in fact lifting properties.

Question 1. Find more examples of meaningful lifting properties in various categories. Play with natural classes of morphisms to see whether their iterated orthogonals are meaningful.

Remark 1. Most of the definitions above are in a form useful in a diagram chasing computation. Let us comment on this.

In category theory it is usual to view an object X as either the arrow $\bot \longrightarrow X$ or $X \longrightarrow \top$ from the initial object $\bot$ or to the terminal object $\top$. However, in Claim 1(vii) we sometimes view (the properties of) a space as (properties of a non-unique!) arrow $\{*\} \longrightarrow X$ or $\{a,b\} \hookrightarrow X$. Our purpose is to give definitions useful in a diagram chasing computation, and these definitions can be used in this way.

A group G has cohomological dimension 1 iff each surjections $G' \longrightarrow G$ splits. Item (iv) views this as the following diagram chasing rule: in (the valid diagram corresponding to) $0 \longrightarrow \mathbb{Z} \rightthreetimes G' \longrightarrow G$, it is permissible to replace $\mathbb{Z}$ by an arbitrary group A (thereby obtaining a valid diagram corresponding to $0 \longrightarrow A \rightthreetimes G' \longrightarrow G$).

Remark 2. Claim (v) shows that a computer-generated proof in (Ganesalingam, Gowers; Problem 2) of the claim that completeness is inherited by closed subsets of metric spaces, i.e. a closed subspace of a complete metric space is necessarily complete, translates to two applications of a diagram chasing rule corresponding to the lifting property.

In fact, in any category for an arbitrary class C of morphisms it holds $X \longrightarrow \{*\} \in C^l$ and $f : A \hookrightarrow X \in C^l$ implies $A \longrightarrow \{*\} \in C^l$ whenever f is a monomorphism and where $\{*\}$ denotes the terminal object.

Note that, the definition involved infinite objects which are infinite sequences $\{1/n\}_n \longrightarrow \{1/n\}_n \cup \{0\}$. This can be probably be avoided if instead we consider *uniform spaces*, equivalently *type-definable equivalence relations* as *simplicial objects in the category of topological spaces* (see Remark 7 and Conjecture 2 for details) and interpret the lifting property in the category of simplicial objects in the category of topological spaces; this uses that a Cauchy filter is (almost) a map from a "constant" simplicial object.

Conjecture 2 suggests a formal analogy corresponding to the informal analogy between *compact* topological spaces and *complete* metric (uniform) spaces.

Remark 3. Claim demonstrates that a number of elementary notions can be concisely expressed in terms of a simple diagram chasing rule. However, it appears there is no (well-known) logic or proof system based on diagram chasing in a category. We make some suggestions towards such a proof system in the last two sections.

3 Elementary topological properties via finite topological spaces

First we must introduce notation for maps of finite topological spaces we use. Two features are important for us:

1. it reminds one that a finite topological space is a category (degenerate if you like)
2. it does not allow one to talk conveniently about non-identity endomorphisms of finite topological spaces. We hope this may help define a decidable fragment of elementary topology because there is a decidable fragment of diagram chasing without endomorphisms, see [GLZ].

A topological space comes with a *specialisation preorder* on its points: for points $x, y \in X$, $x \leq y$ iff $y \in clx$, or equivalently, a category whose objects are points of X and there is a unique morphism $x \searrow y$ iff $y \in clx$.

For a finite topological space X, the specialisation preorder or equivalently the category uniquely determines the space: a subset of X is closed iff it is downward closed, or equivalently, there are no morphisms going outside the subset.

The monotone maps (i.e. functors) are the continuous maps for this topology.

We denote a finite topological space by a list of the arrows (morphisms) in the corresponding category; '$\leftrightarrow$' denotes an isomorphism and '=' denotes the identity morphism. An arrow between two such lists denotes a continuous map (a functor) which sends each point to the correspondingly labelled point, but possibly turning some morphisms into identity morphisms, thus gluing some points.

Thus, each point goes to "itself" and

$$\{a, b\} \longrightarrow \{a \searrow b\} \longrightarrow \{a \leftrightarrow b\} \longrightarrow \{a = b\}$$

denotes

$$(\textit{discrete space on two points}) \longrightarrow (\textit{Sierpinski space}) \longrightarrow (\textit{antidiscrete space}) \longrightarrow (\textit{single point})$$

In $A \longrightarrow B$, each object and each morphism in A necessarily appears in B as well; we avoid listing the same object or morphism twice. Thus both

$$\{a\} \longrightarrow \{a,b\} \text{ and } \{a\} \longrightarrow \{b\}$$

denote the same map from a single point to the discrete space with two points. Both

$$\{a \swarrow U \searrow x \swarrow V \searrow b\} \longrightarrow \{a \swarrow U = x = V \searrow b\} \text{ and } \{a \swarrow U \searrow x \swarrow V \searrow b\} \longrightarrow \{U = x = V\}$$

denote the morphism gluing points U, x, V.

In $\{a \searrow b\}$, the point a is open and point b is closed.

Let

$$C_{<n} := \{f : f \in C, \text{ both the domain and range of } f \text{ are finite of size less than } n\}.$$

Claim 2. *The following is a list of properties defined using the lifting property starting from a single morphism between spaces of at most two points.*

In the category of topological spaces, it holds:

$((\{a\} \longrightarrow \{a \searrow b\})^r_{<5})^{lr}$ *is almost the class of proper maps, namely a map of T4 spaces is in the class iff it is proper*

$(\{b\} \longrightarrow \{a \searrow b\})^l$ *is the class of maps with dense image*

$(\{b\} \longrightarrow \{a \searrow b\})^{lr}$ *is the class of maps of closed inclusions $A \subset X$, A is closed*

$(\varnothing \longrightarrow \{*\})^r = (\{0\} \longrightarrow \{0 \leftrightarrow 1\})^l$ *is the class of surjections*

$(\varnothing \longrightarrow \{*\})^{rl}$ *is the class of maps of form $A \longrightarrow A \cup D$, D is discrete*

$(\varnothing \longrightarrow \{*\})^{rll} = (\{a\} \longrightarrow \{a,b\})^l$ *is the class of maps $A \longrightarrow B$ such that each open closed non-empty subset of B intersects ImA.*

$(\varnothing \longrightarrow \{*\})^l$ *is the class of maps $A \longrightarrow B$ such that $A = B = \varnothing$ or $A \neq \varnothing$, B arbitrary*

$(\varnothing \longrightarrow \{*\})^{ll}$ *is the class of maps $A \longrightarrow B$ such that either $A = \varnothing$ or the map is an isomorphism*

$(\varnothing \longrightarrow \{*\})^{lll}$ *is the class of maps $A \longrightarrow B$ which split*

$(\varnothing \longrightarrow \{*\})^{rr}$ *is the class of subsets, i.e. injective maps $A \hookrightarrow B$ where the topology on A is induced from B.*

$(\{a \leftrightarrow b\} \longrightarrow \{a = b\})^l$ *is the class of injections*

$(\{a \searrow b\} \longrightarrow \{a = b\})^l$ *is the class of maps $f : X \longrightarrow Y$ such that the topology on X is induced from Y*

$(\{a,b\} \longrightarrow \{a = b\})^l$ *describes being connected, and is the class of maps $f : X \longrightarrow Y$ such that $f(U) \cap f(V) = \varnothing$ for each two open closed subsets $U \neq V$ of X; if both X and Y are unions of open closed connected subsets, this means that the map $\pi_0(X) \hookrightarrow \pi_0(Y)$ is injective*

$(\{a \leftrightarrow b\} \longrightarrow \{a = b\})^r$ *fibres are T0 spaces*

$(\{a \searrow b\} \longrightarrow \{a = b\})^r$ *fibres are T1 spaces*

$(\{a, b\} \longrightarrow \{a = b\})^r$ *is the class of injections*

$(\{a\} \longrightarrow \{a \leftrightarrow b\})^l$ *is the class of surjections*

$(\{a\} \longrightarrow \{a \leftrightarrow b\})^r$ *is the class of surjections*

$(\{b\} \longrightarrow \{a \searrow b\})^l$ *something T1-related but not particularly nice*

$(\{a\} \longrightarrow \{a \searrow b\})^l$ *something T0-related*

$(\{a\} \longrightarrow \{a, b\})^l$ *is the class of maps* $f : X \longrightarrow Y$ *such that either* X *is empty or* f *is surjective*

Proof. All items are trivial to verify, with the possible exception of the first item. [Bourbaki, General Topology, I§10.2, Thm.1(d), p.101], quoted in Appendix B, gives a characterisation of proper maps by a lifting property with respect to maps associated to ultrafilters. Using this it is easy to check that each map in $(\{a\} \longrightarrow \{a \searrow b\})^r_{<5}$ being closed, hence proper, implies that each map in $((\{a\} \longrightarrow \{a \searrow b\})^r_{<5})^{lr}$ is proper. A theorem of [Taimanov], cf. [Engelking, 3.2.1,p.136], quoted in Appendix B, states that for a compact Hausdorff space K, a Hausdorff space K is compact iff the map $K \longrightarrow \{*\}$ is in C_T^{lr} where

$$C_T := \{\ \{a \leftrightarrow b\} \longrightarrow \{a = b\},\ \{a \searrow b\} \longrightarrow \{a = b\},\ \{b\} \longrightarrow \{a \searrow b\},\ \{a \swarrow o \searrow b\} \longrightarrow \{a = o = b\}\ \}$$

It is easy to check that all the maps listed in the formula above are closed, hence proper, and therefore

$$C_T^{lr} \subseteq ((\{a\} \longrightarrow \{a \searrow b\})^r_{<5})^{lr}$$

Finally, note that the proof of Taimanov theorem generalises to give that a proper map between normal Hausdorff (T4) spaces is in the larger class. QED.

Conjecture 1. *In the category of topological spaces,*

$$((\{a\} \longrightarrow \{a \searrow b\})^r_{<5})^{lr}$$

is the class of proper maps.

The formalism suggests[1] the following generalisation describing both compactness and completeness of metric spaces based on the following observation: the category Top of topological spaces and the category Mtr of metric spaces and uniformly continuous maps embed as full subcategories into the category sTop of simplicial topological spaces.

Let E : Top $\longrightarrow$ sTop be the natural embedding functor sending a space X into the simplicial objects (X^n) of its Cartesian powers and coordinate maps. Let s : Mtr $\longrightarrow$ sTop be the functor described below in Remark 7 which embeds the category of metric spaces and uniformly continuous maps into sTop. For a class C of morphism in $sTop$, let C_{finite} denote the subclass consisting of those objects whose components are finite topoloical spaces.

[1] I thank Sergei V. Ivanov for the question and Sergei Sinchuk for extensive discussions.

Conjecture 2. *In the category* sTop *of simplicial topological spaces, consider the class* C

$$C = ((E(\{a\} \longrightarrow \{a \searrow b\}))^r_{finite})^{lr}.$$

- *A topological space* K *is compact iff* $E(K \longrightarrow pt) \in C$.
- *A metric space* M *is complete iff* $s(M \longrightarrow pt) \in C$.

It is easy to verify that a Cauchy sequence in a metric space M corresponds to a simplicial map $E(\mathbb{N}_{\text{cofinite}}) \longrightarrow s(M)$, and that a metric space M is complete iff the following lifting property holds in sTop:

$$E(\mathbb{N}_{\text{cofinite}} \longrightarrow \mathbb{N}_{\text{cofinite}} \cup \{\infty\}) \rightthreetimes s(M \longrightarrow pt)$$

Here $\mathbb{N}_{\text{cofinite}}$ denotes the natural numbers with cofinite topology, and the topology on $\mathbb{N}_{\text{cofinite}} \cup \{\infty\}$ is such that a subset is open iff it is cofinite and contains ∞.

Remark 4. It is easy to see that $((\{a\} \longrightarrow \{a \searrow b\})^r_{<m})^{lr} \subset ((\{a\} \longrightarrow \{a \searrow b\})^r_{<n})^{lr}$ for any $m < n$. However, I do not know whether there is $n > m > 3$ such that the inclusion is strict. An example using cofinite topology (suggested by Sergei Kryzhevich) shows that C_T^{lr} does not define the class of compact spaces: indeed, consider infinite sets $A \subset B$, $\omega \le \operatorname{card} A < \operatorname{card} B$, equipped with cofinite topology (i.e. a subset is closed iff it is finite). Then $A \subseteq B \in C_T^l$ yet $A \subseteq B \rightthreetimes A \longrightarrow \{*\}$ fails: for a map $f : B \longrightarrow A$ the preimage of some (necessarily closed) point is infinite as $\operatorname{card} B > \operatorname{card} A$, hence not closed, and the map is not continious. Hence, $A \longrightarrow \{*\} \notin C_T$ yet A is compact (non-Hausdorff). This example could probably be generalised to show that that $((\{a\} \longrightarrow \{a \searrow b\})^r_{<4})^{lr} \subsetneq ((\{a\} \longrightarrow \{a \searrow b\})^r_{<5})^{lr}$.

Question 2. (a) Calculate

$$((\{b\} \longrightarrow \{a \searrow b\})^r_{<5})^{lr},\ ((\{b\} \longrightarrow \{a \searrow b\})^{lrr}, \text{ and } (\{a \swarrow U \searrow x \swarrow V \searrow b\} \longrightarrow \{a \swarrow U = x = V \searrow b\})^{lr}$$

Could either be viewed as a "definition" of the real line?

(b) Characterise the interval $[0, 1]$, a circle $\mathbb{S}^1$ and, more generally, spheres $\mathbb{S}^n$ using their topological characterisations provided by the Kline sphere charterisation theorem and its analogues. An example of such a characterisation is that a topological space X is homomorphic to the circle $\mathbb{S}^1$ iff X is a connected Hausdorff metrizable space such that $X \setminus \{x, y\}$ is not connected for any two points $x \ne y \in X$ ([Hocking,Young, Topology, Thm.2-28,p.55]); another example is that a topological space X is homomorphic to the closed interval $[0, 1]$ iff X is a connected Hausdorff metrizable space such that $X \setminus \{x\}$ is not connected for exactly two points $x \ne y \in X$ ([Hocking,Young, Topology, Thm.2-27,p.54]).

Remark 5. Urysohn lemma and Tietze extension theorem relate lifting properties involving $\mathbb{R}$ and those involving opens maps of finite topological spaces, and this is why we hope the question above might have something to do with the real line. Let us give some more details.

Note a map f of finite spaces is open iff f is in $(\{b\} \longrightarrow \{a \searrow b\})^r$, and that $\{a \swarrow U \searrow x \swarrow V \searrow b\} \longrightarrow \{a \swarrow U = x = V \searrow b\}$ is an open map.

A space X is normal (T4) iff $\varnothing \longrightarrow X \rightthreetimes \{a \swarrow U \searrow x \swarrow V \searrow b\} \longrightarrow \{a \swarrow U = x = V \searrow b\}$, hence the Uryhson lemma can be stated as follows:

$$\varnothing \longrightarrow X \rightthreetimes \{a \swarrow U \searrow x \swarrow V \searrow b\} \longrightarrow \{a \swarrow U = x = V \searrow b\}$$

iff

$$\varnothing \longrightarrow X \rightthreetimes \{0'\} \cup [0,1] \cup \{1'\} \longrightarrow \{0 = 0' \searrow x \swarrow 1 = 1'\}$$

where points $0', 0$ and $1, 1'$ are topologically indistinguishable in $\{0'\} \cup [0,1] \cup \{1'\}$, the closed interval $[0,1]$ goes to x, and $0, 0'$ map to point $0 = 0'$, and $1, 1'$ map to point $1 = 1'$.

Tietze extension theorem states that for a normal space X and a closed subset A of X, $A \longrightarrow X \rightthreetimes \mathbb{R} \longrightarrow \{*\}$, i.e. in notation $\mathbb{R} \longrightarrow \{*\}$ is in

$$\left((\{b\} \longrightarrow \{a \searrow b\})^{lr} \cap \{A \longrightarrow X : \varnothing \longrightarrow X \rightthreetimes \{a \swarrow U \searrow x \swarrow V \searrow b\} \longrightarrow \{a \swarrow U = x = V \searrow b\} \right)^r$$

Note that $\varnothing \longrightarrow X \rightthreetimes [0,1] \longrightarrow \{0 \searrow x \swarrow 1\}$ is the stronger property X is perfectly normal. A normal space X is perfectly normal provided each closed subset of X is the intersection of a countably many open subsets.

For example, is $\{0'\} \cup [0,1] \cup \{1'\} \longrightarrow \{0 = 0' \searrow x \swarrow 1 = 1'\}$ in $(\{a \swarrow U \searrow x \swarrow V \searrow b\} \longrightarrow \{a \swarrow U = x = V \searrow b\}^{lr} \subset ((\{b\} \longrightarrow \{a \searrow b\})^r_{<5})^{lr}$?

Remark 6. Is there a model category or a factorisation system of interest associated with any of these lifting properties, for example compactness/properness?

Many of the separation axioms can be expressed as lifting properties with respect to maps involving up to 4 points and the real line, see [Appendix A].

4 Elementary topology as diagram chasing computations with finite categories

Early works talk of topology in terms of *neighbourhood* systems U_x where U_x varies though *open neighbourhoods of points* of a topological space; this is how the notion of topology was defined by Hausdorff. In the notation of arrows, *a neighbourhood system U_x, $x \in X$* would correspond to a system of arrows

$$\{x\} \longrightarrow X \xrightarrow{U} \{x \searrow x'\}$$

and Hausdorff's axioms (A),(B),(C) (see Appendix B) would correspond to diagram chasing rules.

Here we show the axioms of topology stated in the more modern language of open subsets can be seen as diagram chasing rules for manipulating diagrams involving notation such as

$$\{x\} \longrightarrow X,\ X \longrightarrow \{x \searrow y\},\ X \longrightarrow \{x \leftrightarrow y\}$$

in the following straightforward way; cf. [Gavrilovich, Elementary Topology,§.2.1] for more details.

As is standard in category theory, identify a point x of a topological space X with the arrow $\{x\} \longrightarrow X$, a subset Z of X with the arrow $X \longrightarrow \{z \leftrightarrow z'\}$, and an open subset U of X with the arrow $X \longrightarrow \{u \searrow u'\}$. With these identifications, the Hausdorff axioms of a topological space become rules for manipulating such arrows, as follows.

Both the empty set and the whole of X are open says that the compositions

$$X \longrightarrow \{c\} \longrightarrow \{o \searrow c\} \text{ and } X \longrightarrow \{o\} \longrightarrow \{o \searrow c\}$$

behave as expected (the preimage of {o} is empty under the first map, and is the whole of X under the second map).

The intersection of two open subsets is open means the arrow

$$X \longrightarrow \{o \searrow c\} \times \{o' \searrow c'\}$$

behaves as expected (the "two open subsets" are the preimages of points $o \in \{o \searrow c\}$ and $o' \in \{o' \searrow c'\}$; "the intersection" is the preimage of (o, o') in $\{o \searrow c\} \times \{o' \searrow c'\}$).

Finally, *a subset U of X is open iff each point u of U has an open neighbourhood inside of U* corresponds to the following diagram chasing rule:

for each arrow $X \underset{\xi_U}{\longrightarrow} \{U \longleftrightarrow \bar{U}\}$ it holds

$$\begin{array}{ccc} & & \{U \to \bar{U}\} \\ & \nearrow & \downarrow \\ X & \xrightarrow[\xi_U]{} & \{U \longleftrightarrow \bar{U}\} \end{array} \quad \text{iff for each } \{u\} \longrightarrow X, \quad \begin{array}{ccc} \{u\} & \longrightarrow & \{u \to U \longleftrightarrow \bar{U}\} \\ \downarrow & \nearrow & \downarrow \\ X & \xrightarrow[\xi_U]{} & \{u = U \longleftrightarrow \bar{U}\} \end{array}$$

The preimage of an open set is open corresponds to the composition

$$X \longrightarrow Y \longrightarrow \{u \searrow u'\} \longrightarrow \{u \leftrightarrow u'\}.$$

This observation suggests that some arguments in elementary topology may be understood entirely in terms of diagram chasing, see [Gavrilovich, Elementary Topology] for some examples. This reinterpretation may help clarify the nature of the axioms of a topological space, in particular it offers a constructive approach, may clarify to what extent set-theoretic language is necessary, and perhaps help to suggest an approach to "tame topology" of Grothendieck, Does this lead to tame topology of Grothendieck, i.e. a foundation of topology "without false problems" and "wild phenomena" "at the very beginning" ?

Let us state two problems; we hope they help to clarify the notion of an ergosystem and that of a topological space.

Question 3. Write a short program which extracts diagram chasing derivations from texts on elementary topology, in the spirit of the ideology of ergosystems/ergostructures.

Question 4. Develop elementary topology in terms of finite categories (viewed as finite topological spaces) and labelled commutative diagrams, with an emphasis on labels (properties) of morphisms defined by the Quillen lifting property. Does this lead to tame topology of Grothendieck, i.e. a foundation of topology "without false problems" and "wild phenomena" "at the very beginning" ?

Question 5. (Ganesalingam, Gowers) wrote an automatic theorem prover trying to make it "thinking in a human way". In a couple of their examples their generated proofs amount to diagram chasing, e.g. Claim (v) shows the generated proof of the claim that a closed subspace of a complete metric space is necessarily complete translates to two applications of a diagram chasing rule corresponding to the lifting property, see [Gavrilovich, Slides] for details. Arguably, both examples on the first page of (ibid.) also correspond to diagram chasing. In their approach, can this be seen as evidence that humans are really thinking by diagram chasing and the lifting property in particular? Note that Claim (v) involves examples typically shown to students to clarify the concepts of a metric space being complete or closed. Can one base a similar theorem prover on our observations, particularly in elementary topology?

Let us comment on an approach to Problem 1.

We observed that there is a simple rule which leads to several notions in topology interesting enough to be introduced in an elementary course. Can this rule be extended to a very short program which learns elementary topology?

We suggest the following naive approach is worth thinking about.

The program maintains a collection of directed labelled graphs and certain distinguished subgraphs. Directed graphs represent parts of a category; distinguished subgraphs represent commutative diagrams. Labels represent properties of morphisms. Further, the program maintains a collection of rules to manipulate these data, e.g. to add or remove arrows and labels.

The program interacts with a flow of signals, say the text of [Bourbaki, General Topology, Ch.1], and seeks correlations between the diagram chasing rules and the flow of signals. It finds a "correlation" iff certain strings occur nearby in the signal flow iff they occur nearby in a diagram chasing rule. To find "what's interesting", by brute force it searches for a valid derivation which exhibits such correlations. To guide the search and exhibit missing correlations in a derivation under consideration, it may ask questions: are these two strings related? Once it finds such a derivation, the program "uses it for building its own structure". Labels correspond to properties of morphisms. Labels defined by the lifting property play an important role, often used to exclude counterexamples making a diagram chasing argument fail. In [DeMorgan] we analysed the text of the definitions of surjective and injective maps showing what such a correlation may look like in a "baby" case.

A related but easier task is to write a theorem prover doing diagram chasing in a model category. The axioms of a (closed or not) model category as stated in [Quillen,I.1.1] can be interpreted as rules to manipulate labelled commutative diagrams in a labelled category. It appears straightforward how to formulate a logic (proof system) based on these rules which would allow to express statements like: Given a labelled commutative diagram, (it is permissible to) add this or that arrow or label. Moreover, it appears not hard to write a theorem prover for this logic doing brute force guided search. What is not clear whether this logic is complete in any sense or whether there are non-trivial inferences of this form to prove.

Writing such a theorem prover is particularly easy when the underlying category of the model category is a partial order [GavrilovichHasson] and [BaysQuilder] wrote some code for doing diagram chasing in such a category. However, the latter problem is particularly severe as well.

The two problems are related; we hope they help to clarify the notion of an ergosystem and that of a topological space.

The following is a more concrete question towards Problem 4.

Question 6. (a) Prove that a compact Hausdorff space is normal by diagram chasing; does it require additional axioms? Note that we know how to express the statement entirely in terms of the lifting property and finite topological spaces of small size.

(b) Formalise the argument in [Fox, 1945] which implies the category of topological spaces is not Cartesian closed. Namely, Theorem 3 [ibid.] proves that if X is separable metrizable space, R is the real line, then X is locally compact iff there is a topology on $X^{\mathbb{R}}$ such that for any space T, a function $h : X \times T \longrightarrow \mathbb{R}$ is continuous iff the corresponding function $h^* : T \longrightarrow X^{\mathbb{R}}$ is continuous (where $h(x,t) = h^*(t)(x)$) Note that here we do not know how to express the statement.

Remark 7. In [Gavrilovich, Tame Topology,§6] and [Gavrilovich, Elementary Topology,§2.9] we note that uniform spaces may be viewed as simplicial objects in the category of topological spaces in the following way. We wish to emphasise that this observation can be easily "read off" from [Bourbaki, General Topology, II§1.1.1] if one is inclined to translate everything into diagram chasing.

Let X be a set. Let

$$X \Longleftrightarrow X \times X \Longleftrightarrow X \times X \times X ...$$

be the "trivial" simplicial set where degeneracy and face maps are coordinate projections and diagonal embeddings. A uniform structure on a set X is a filter, hence a topology, on the set $X \times X$ satisfying certain properties; equip $X \times X$ with this topology. Put the antidiscrete topology on X; put on $X \times X \times X$ the topology which is the pullback of the topologies on $X \times X$ and X along the *two* projection maps on two of the three pairs of coordinates, and similarly for $X \times X \times X \times X$ etc. A straightforward verification shows this is well-defined whenever the topology on $X \times X$ corresponds to a uniform structure on X.

We[2] note that the uniform structure can be viewed model-theoretically as being the same thing as a a binary relation defined as the conjunction of infinitely many formulas such that in a saturated model it defines an equivalence relation.

Remark 8. In [Gavrilovich, Tame Topology, §5.4] we observe that a number of consequences of compactness can be expressed as a change of order of quantifiers in a formula, i.e. are of form $\forall\exists\phi(...) \implies \exists\forall\phi(...)$ namely that a real-valued function on a compact is necessarily bounded, that a Hausdorff compact is necessarily normal, that the image in X of a closed subset in $X \times K$ is necessarily closed, Lebesgue's number Lemma, and paracompactness.

Such formulae correspond to inference rules of a special form, and we feel a special syntax should be introduced to state these rules.

[2]I thank Martin Bays for this observation

For example, consider the statement that "a real-valued function on a compact domain is necessarily bounded". As a first order formula, it is expressed as

$$\forall x \in K \exists M (f(x) \leq M) \Longrightarrow \exists M \forall x \in K (f(x) \leq M)$$

Another way to express it is:

$$\exists M : K \longrightarrow \mathbb{R} \forall x \in K (f(x) \leq M(x)) \Longrightarrow \exists M \in \mathbb{R} \forall x \in K (f(x) \leq M)$$

Note that all that happened here is that a function $M : K \longrightarrow \mathbb{R}$, become a constant $M \in \mathbb{R}$, or rather expression "M(x)" of type $K \longrightarrow M$ which used (depended upon) variable "x" become expression "M" which does not use (depend upon) variable "x".

We feel there should be a special syntax which would allow to express above as an inference rule *removing dependency of "M(x)" on "x"*, and this syntax should be used to express consequences of compactness in a diagram chasing derivation system for elementary topology.

To summarise, we think that compactness should be formulated with help of inference rules for expressly manipulating which variables are 'new', in what order they 'were' introduced, and what variables terms depend on, e.g. rules replacing a term t(x,y) by term t(x).

Something like the following:

```
... f(x) =< M(x) ...
--------------------
...  f(x) =< M   ...
```

5 Ergo-Structures/Ergo-Systems Conjecture of Misha Gromov.

We conclude with a section which aims to explain our motivation and hence is speculative and perhaps somewhat inappropriate in what is mostly a mathematical text.

Misha Gromov [Memorandum Ergo] conjectures there are particular mathematical, essentially combinatorial, structures, called *ergostructures* or *ergosystems*, which help to understand complex biological behaviour including learning and create mathematically interesting models of these processes.

We hope our observations may eventually help to uncover an essentially combinatorial reasoning behind elementary topology, and thereby suggest an example of an ergostructure.

An ergosystem/ergostructure may be thought of as an "engine" producing (structurally or mathematically) *interesting* behaviour which is then misappropriated into a *useful* behaviour by a biological system.

"Behaviour" is thought of as an interaction with a flow of signals. By itself, such an "engine" produces *interesting* behaviour, with little or no concern for any later use; it "interacts with an incoming flow of signals; it recognizes and selects what is *interesting* for itself in such a flow and uses it for building its own structure" [Gromov, Memorandum Ergo].

An analogy might help. Consider a complex mechanical contraption powered by an engine, such as a loom. By itself, there is nothing directly useful done by its engine; indeed, the very same engine may be used by different mechanisms for all sorts of useful and useless tasks. To understand the workings of a mechanism, sometimes you had better forget its purpose and ask what is the engine, how is it powered, and what keeps the engine in good condition. Understanding the loom's engine (only) in terms of how it helps to weave is misleading.

Thus, the concept of an ergostructure/ergosystems suggests a different kind of questions we should ask about biological systems and learning.

A further suggestion is that these "engines" might be rather universal, i.e. able to behave interestingly interacting with a diverse range of signal flows. At the very beginning an ergosystem/structure is a "crisp" mathematically interesting structure of size small enough to be stumbled upon by evolution; as it grows, it becomes "fuzzy" and specialised to a particular kind of flow of signals.

However, we want to draw attention to the following specific suggestion:

> ""The category/functor modulated structures can not be directly used by ergosystems, e.g. because the morphisms sets between even moderate objects are usually unlistable. But the ideas of the category theory show that there are certain (often non-obviuos) rules for generating proper concepts. (You ergobrain would not function if it had followed the motto: "in my theory I use whichever definitions I like".) The category theory provides a (rough at this stage) hint on a possible nature of such rules. [Gromov, Ergobrain]

Our observations give an example of a simple rule which can be used "to generate proper concepts", particularly in elementary topology. We hope that our observations can make the hint less rough, particularly if one properly develops elementary topology in terms of diagram chasing, with an emphasis on the lifting property.

Problem 1. Write a short program which extracts diagram chasing derivations from texts on elementary topology, in the spirit of the ideology of ergosystems/ergostructures. That is, it considers a flow of signals interesting if it correlates with diagram chasing rules.

In the previous section we give some suggestions, albeit naive, what such a program might look like.

Problem 2. Develop elementary topology in terms of labelled commutative diagrams involving finite categories (viewed as finite topological spaces), with an emphasis on labels (properties) of morphisms defined by the Quillen lifting property.

Does this lead to the tame topology of Grothendieck, i.e. a foundation of topology "without false problems" and "wild phenomena" "at the very beginning" ?

In the previous section we give some suggestions, albeit naive, what such a program might look like and how to express elementary topology in terms of labelled diagram chasing.

Acknowledgements. To be written.

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Much of this paper was done in St.Petersburg; it wouldn't have been possible without support of family and friends who created an excellent social environment and who occasionally accepted an invitation for a walk or a coffee or extended an invitation; alas, I made such a poor use of it all.

This note is elementary, and it was embarrassing and boring, and embarrassingly boring, to think or talk about matters so trivial, but luckily I had no obligations for a time.

6 Appendix A. Separation axioms as lifting properties (from Wikipedia)

The separation axioms are lifting properties with respect to maps involving up to 4 points and the real line. What follows below is the text of the Wikipedia page on the separation axioms where we added lifting properties formulae expressing what is said there in words.

Let X be a topological space. Then two points x and y in X are *topologically distinguishable* iff the map $\{x \leftrightarrow y\} \longrightarrow X$ is not continuous, i.e. iff at least one of them has an open neighbourhood which is not a neighbourhood of the other.

Two points x and y are *separated* iff neither $\{x \searrow y\} \longrightarrow X$ nor $\{x \searrow y\} \longrightarrow X$ is continuous, i.e each of them has a neighbourhood that is not a neighbourhood of the other; in other words, neither belongs to the other's closure, $x \notin cl\, x$ and $y \notin cl\, x$. More generally, two subsets A and B of X are *separated* iff each is disjoint from the other's closure, i.e. $A \cap clB = B \cap clA = \varnothing$. (The closures themselves do not have to be disjoint.) In other words, the map $i_{AB} : X \longrightarrow \{A \leftrightarrow x \leftrightarrow B\}$ sending the subset A to the point A, the subset B to the point B, and the rest to the point x, factors both as

$$X \longrightarrow \{A \leftrightarrow U_A \searrow x \leftrightarrow B\} \longrightarrow \{A = U_A \leftrightarrow x \leftrightarrow B\}$$

and

$$X \longrightarrow \{A \leftrightarrow x \swarrow U_B \leftrightarrow B\} \longrightarrow \{A \leftrightarrow x \leftrightarrow U_B = B\}$$

here the preimage of x, B, resp. x, A is a closed subset containing B, resp. A, and disjoint from A, resp. B. All of the remaining conditions for separation of sets may also be applied to points (or to a point and a set) by using singleton sets. Points x and y will be considered separated, by neighbourhoods, by closed neighbourhoods, by a continuous function, precisely by a function, iff their singleton sets $\{x\}$ and $\{y\}$ are separated according to the corresponding criterion.

Subsets A and B are *separated by neighbourhoods* iff A and B have disjoint neighbourhoods, i.e. iff $i_{AB} : X \longrightarrow \{A \leftrightarrow x \leftrightarrow B\}$ factors as

$$X \longrightarrow \{A \leftrightarrow U_A \searrow x \swarrow U_B \leftrightarrow B\} \longrightarrow \{A = U_A \leftrightarrow x \leftrightarrow U_B = B\}$$

here the disjoint neighbourhoods of A and B are the preimages of open subsets A, U_A and U_B, B of $\{A \leftrightarrow U_A \searrow x \swarrow U_B \leftrightarrow B\}$, resp. They are *separated by closed neighbourhoods* iff they have disjoint closed neighbourhoods, i.e. i_{AB} factors as

$$X \longrightarrow \{A \leftrightarrow U_A \searrow U'_A \swarrow x \searrow U'_B \swarrow U_B \leftrightarrow B\} \longrightarrow \{A \leftrightarrow U_A = U'_A = x = U'_B = U_B \leftrightarrow B\}.$$

They are *separated by a continuous function* iff there exists a continuous function f from the space X to the real line $\mathbb{R}$ such that $f(A) = 0$ and $f(B) = 1$, i.e. the map i_{AB} factors as

$$X \longrightarrow \{0'\} \cup [0,1] \cup \{1'\} \longrightarrow \{A \leftrightarrow x \leftrightarrow B\}$$

where points $0', 0$ and $1, 1'$ are topologically indistinguishable, and $0'$ maps to A, and $1'$ maps to B, and $[0,1]$ maps to x. Finally, they are *precisely separated by a* *continuous function* iff there exists a continuous function f from X to $\mathbb{R}$ such that the preimage $f^{-1}(\{0\}) = A$ and $f^{-1}(\{1\}) = B$. i.e. iff i_{AB} factors as

$$X \longrightarrow [0,1] \longrightarrow \{A \leftrightarrow x \leftrightarrow B\}$$

where 0 goes to point A and 1 goes to point B.

These conditions are given in order of increasing strength: Any two topologically distinguishable points must be distinct, and any two separated points must be topologically distinguishable. Any two separated sets must be disjoint, any two sets separated by neighbourhoods must be separated, and so on.

The definitions below all use essentially the preliminary definitions above.

In all of the following definitions, X is again a topological space.

X is T0, or Kolmogorov, if any two distinct points in X are topologically distinguishable. (It will be a common theme among the separation axioms to have one version of an axiom that requires T0 and one version that doesn't.) As a formula, this is expressed as

$$\{x \leftrightarrow y\} \longrightarrow \{x = y\} \ \rightthreetimes\ X \longrightarrow \{*\}$$

X is R0, or symmetric, if any two topologically distinguishable points in X are separated, i.e.

$$\{x \searrow y\} \longrightarrow \{x \leftrightarrow y\} \ \rightthreetimes\ X \longrightarrow \{*\}$$

X is T1, or accessible or Frechet, if any two distinct points in X are separated, i.e.

$$\{x \searrow y\} \longrightarrow \{x = y\} \ \rightthreetimes\ X \longrightarrow \{*\}$$

Thus, X is T1 if and only if it is both T0 and R0. (Although you may say such things as "T1 space", "Frechet topology", and "Suppose that the topological space X is Frechet", avoid saying "Frechet space" in this context, since there is another entirely different notion of Frechet space in functional analysis.)

X is R1, or preregular, if any two topologically distinguishable points in X are separated by neighbourhoods. Every R1 space is also R0.

X is weak Hausdorff, if the image of every continuous map from a compact Hausdorff space into X is closed. All weak Hausdorff spaces are T1, and all Hausdorff spaces are weak Hausdorff.

X is Hausdorff, or T2 or separated, if any two distinct points in X are separated by neighbourhoods, i.e.

$$\{x, y\} \hookrightarrow X \ \rightthreetimes\ \{x \searrow X \swarrow y\} \longrightarrow \{x = X = y\}$$

Thus, X is Hausdorff if and only if it is both T0 and R1. Every Hausdorff space is also T1.

X is $T2\frac{1}{2}$, or Urysohn, if any two distinct points in X are separated by closed neighbourhoods, i.e.

$$\{x,y\} \hookrightarrow X \rightthreetimes \{x \searrow x' \swarrow X \searrow y' \swarrow y\} \longrightarrow \{x = x' = X = y' = y\}$$

Every T2$\frac{1}{2}$ space is also Hausdorff.

X is completely Hausdorff, or completely T2, if any two distinct points in X are separated by a continuous function, i.e.

$$\{x,y\} \hookrightarrow X \rightthreetimes [0,1] \longrightarrow \{*\}$$

where $\{x,y\} \hookrightarrow X$ runs through all injective maps from the discrete two point space $\{x,y\}$.

Every completely Hausdorff space is also T2$\frac{1}{2}$.

X is regular if, given any point x and closed subset F in X such that x does not belong to F, they are separated by neighbourhoods, i.e.

$$\{x\} \longrightarrow X \rightthreetimes \{x \searrow X \swarrow U \searrow F\} \longrightarrow \{x = X = U \searrow F\}$$

(In fact, in a regular space, any such x and$F$ will also be separated by closed neighbourhoods.) Every regular space is also R1.

X is regular Hausdorff, or T3, if it is both T0 and regular.[1] Every regular Hausdorff space is also $T2\frac{1}{2}$.

X is completely regular if, given any point x and closed set F in X such that x does not belong to F, they are separated by a continuous function, i.e.

$$\{x\} \longrightarrow X \rightthreetimes [0,1] \cup \{F\} \longrightarrow \{x \searrow F\}$$

where points F and 1 are topologically indistinguishable, $[0,1]$ goes to x, and F goes to F.

Every completely regular space is also regular.

X is Tychonoff, or T3$\frac{1}{2}$, completely T3, or completely regular Hausdorff, if it is both T0 and completely regular.[2] Every Tychonoff space is both regular Hausdorff and completely Hausdorff.

X is normal if any two disjoint closed subsets of X are separated by neighbourhoods, i.e.

$$\varnothing \longrightarrow X \rightthreetimes \{x \swarrow x' \searrow X \swarrow y' \searrow y\} \longrightarrow \{x \swarrow x' = X = y' \searrow y\}$$

In fact, by Urysohn lemma a space is normal if and only if any two disjoint closed sets can be separated by a continuous function, i.e.

$$\varnothing \longrightarrow X \rightthreetimes \{0'\} \cup [0,1] \cup \{1'\} \longrightarrow \{0 = 0' \searrow x \swarrow 1 = 1'\}$$

where points $0', 0$ and $1, 1'$ are topologically indistinguishable, $[0,1]$ goes to x, and both $0, 0'$ map to point $0 = 0'$, and both $1, 1'$ map to point $1 = 1'$.

X is normal Hausdorff, or T4, if it is both T1 and normal. Every normal Hausdorff space is both Tychonoff and normal regular.

X is completely normal if any two separated sets A and B are separated by neighbourhoods $U \supset A$ and $V \supset B$ such that U and V do not intersect, i.e.????

$$\varnothing \longrightarrow X \prec \{X \swarrow A \leftrightarrow U \searrow U' \swarrow W \searrow V' \swarrow V \leftrightarrow B \searrow X\} \longrightarrow \{U = U', V' = V\}$$

Every completely normal space is also normal.

X is perfectly normal if any two disjoint closed sets are precisely separated by a continuous function, i.e.

$$\varnothing \longrightarrow X \prec [0,1] \longrightarrow \{0 \swarrow X \searrow 1\}$$

where $(0,1)$ goes to the open point X, and 0 goes to 0, and 1 goes to 1.
Every perfectly normal space is also completely normal.

X is extremally disconnected if the closure of every open subset of X is open, i.e.

$$\varnothing \longrightarrow X \prec \{U \searrow Z', Z \swarrow V\} \longrightarrow \{U \searrow Z' = Z \swarrow V\}$$

or equivalently

$$\varnothing \longrightarrow X \prec \{U \searrow Z', Z \swarrow V\} \longrightarrow \{Z' = Z\}$$

7 Appendix B. Quotations from sources.

For reader's convenience we quote here from the several sources we use.

[Bourbaki, General Topology, I§10.2, Thm.1(d), p.101]:

> THEOREM I. Let $f : X \longrightarrow Y$ be a continuous mapping. Then the following four statements are equivalent:
>
> a) f is proper.
>
> b) f is closed and $f^{-1}(y)$ is quasi-compact for each $y \in Y$.
>
> c) If $\mathfrak{F}$ is a filter on X and if $y \in Y$ is a cluster point of $f(\mathfrak{F})$ then there is a cluster point x of such that $f(x) = y$.
>
> d) If $\mathfrak{U}$ is an ultrafilter on X and if $y \in Y$ is a limit point of the ultrafilter base $f(\mathfrak{U})$, then there is a limit point x of $\mathfrak{U}$ such that $f(x) = y$.

[Engelking, 3.2.1,p.136] ("compact" below stands for "compact Hausdorff"):

> 3.2.1. THEOREM. Let A be a dense subspace of a topological space X and f a continuous mapping of A to a compact space Y. The mapping f has a continuous extension over X if and only if for every pair B_1, B_2 of disjoint closed subsets of Y the inverse images $f^{-1}(B_1)$ and $f^{-1}(B_2)$ have disjoint closures in the space X.

[Hausdorff, Set theory, §40, p.259] ("ε" stands for "$\in$", and "$U_x V_x$" stands for "$U_x \cap V_x$") :

> From the theorems about open sets we derive the following properties of the neighborhoods:
>
> (A) Every point x has at least one neighborhood U_x; and U_x always contains x.
>
> (B) For any two neighborhoods U_x and V_x of the same point, there exists a third, $W_x \leq U_x V_x$.
>
> (C) Every point $y \varepsilon U_z$ has a neighborhood $U_y \leq U_x$.
>
> It is now again possible to treat neighborhoods as unexplained concepts and to use them as our starting point, postulating Theorems (A), (B), and (C) as neighborhood axioms.[1] Open sets G are then defined as sums of neighborhoods or as sets in which every point $x \varepsilon G$ has a neighborhood $U_x \leq G$ (the null set included). Theorems (1), (2), and (3) about open sets are then provable.
>
> ....
>
> [1] Such a program was carried through in the first edition of this book. [Grund- zügeder Mengenlehre. (Leipzig, 1914; repr. New York, 1949),]

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