Elementary general topology as diagram chasing calculations with finite categories

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a draft of a research proposal

> Die Mathematiker sind eine Art Franzosen: Redet man zu ihnen, so übersetzen sie es in ihre Sprache, und dann ist es alsobald ganz etwas anderes
>
> Johann Wolfgang von Goethe,
> Maximen und Reflexionen.
> Nr. 1005.

Abstract

We observe that some elementary general topology arguments can be seen as diagram chasing computations with finite categories, via a straightforward, naive translation which often involves the Quillen lifting property playing the role of negation.

This paper is a draft of a research proposal: we argue it is worthwhile to try to see whether these observations lead to another axiomatisation of topology and a proof system amenable to automatic theorem proving.

Help in proofreading much appreciated.

1 A synopsis of the results. Structure of the paper

Most of the paper is devoted to the *process* of translation rather than the result: we quote (Bourbaki) extensively and give diagrams corresponding to parts of what we quote. Theorem 1 defines subcategories of the category of topological spaces in terms of a category with an appropriately embedded full subcategory of finite preorders (i.e. transitive reflexive relations) of size at most 4. In §2.9 we observe that a uniform space gives rise to a simplicial object in the category of topological spaces.

In §3 we discuss the Quillen lifting property, which we call Quillen negation. We observe that several elementary definitions are Quillen negations of (a class consisting of) one or more counterexamples, in particular the definitions of surjective, injective, connected, separation axioms T0-T6, closed, limit. We conjecture that quasi-compactness, or rather proper maps, can be defined as the double Quillen negation of a class consisting of four closed morphisms of finite spaces. In §3.1 we discuss the notion of a limit in view of Quillen negation, and observe that the proof that a closed subset of a complete metric space is complete amounts to two application of the Quillen negation.

Last but not least, in §4 we speculate how our diagram chasing calculus should express quasi-compactness, as a rule to remove a dependency, somewhat similar to dead code optimisation by a compiler.

In somewhat more detail (perhaps overstating a bit for clarity):

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- combinatorics of partial orders underlies some basic arguments and definitions in general topology
- the basics of general topology are, in disguise, diagram chasing with the simplest finite categories, namely preorders or, equivalently, finite categories such that there is at most one morphism between any two objects
- the Quillen lifting property is a pattern underlying many of these definitions

Accordingly, we suggest to:

- develop a convenient notation to express the combinatorics of partial orders underlying basic arguments and definitions in general topology, e.g. the notion of surjectivity, injectivity or separation axioms as Quillen lifting properties
- develop an axiomatisation of topology in terms of arrows between categories

1.1 Open Questions and Speculations

These observations suggest the following directions for research. See also speculations in [DMG].

- define a calculus (proof system) capturing proof by diagram chasing; use it in an automatic theorem prover
- define a decidable fragment of topology talking about connected, Hausdorff, normal, and compact spaces. Perhaps these notions could be expressed as Quillen lifting properties wrt finite maps not involving endomorphisms (and there is a decidable fragment of category theory without endomorphisms [GLS]).
- develop notation to conveniently write diagram chasing arguments, in particular those implicitly appearing in elementary topology
- consider the diagram chasing interpretations of topological arguments in the context of (GG)'s "detailed attention to human thought processes". Compare a diagram chasing proof in §3.1 and in (GG,§2.2, Problem 2) of the fact that "a closed subspaces of a complete metric is complete", cf. slides (GM). Does human reasoning follow diagram chasing? For example, do errors in diagram chasing calculation correspond to errors of intuition in elementary topology?

2 Axioms of a topological space as a diagram chasing formalism of $\{0\} \longrightarrow T$ and $T \longrightarrow \{0<1\}$

What follows is based on the following observation. A finite topological space is the same as a preorder (i.e. a finite transitive reflexive binary relation) is the same as a finite category such that there is at most one morphism from one object to another.

We identify a preorder with the topological space where a subset is open iff it is downward closed; conversely, relation $y \in cl\, x$, $x, y \in X$, defines the *specialisation* preorder on the points of a topological space X. If the space has finitely many points, this preorder determines the topology: a subset is closed iff it is the (necessarily finite) union of the closures of its points. We write $x \leq y$, $x \to y$ iff $y \in cl\, x$.

Our notation for preorders and monotone maps is best explained on the examples (used below). Consider the arrow $\{u \to U \leftrightarrow \bar{U}\} \longrightarrow \{u = U \leftrightarrow \bar{U}\}$ above. The first preorder has three elements $u, U, \bar{U}$ where both $U \to \bar{U}$ and $\bar{U} \to U$, and $u \to U$ and $u \to \bar{U}$. The second preorder has two equivalent elements $u = U$ and $\bar{U}$. The map sends each element to "itself", i.e. u and U goes to $u = U$, and $\bar{U}$ to $\bar{U}$. $\{\bullet \to \star\}$, $\bullet < \star$ denotes the two point space with $\bullet$ the open point and $\star$ the closed point. $\{U \leftrightarrow \bar{U}\}$ and $\{\bullet \gtrless \star\}$ denote the two point space with antidiscrete topology, i.e. the only open subsets are $\emptyset$ and $\{U, \bar{U}\}$. The discrete topological space is denoted by either $\{\bullet, \star\}$ or $\{\bullet \not\leftrightarrow \star\}$ if we want to emphasise that there is no relation between $\bullet$ and $\star$. Elements of preorders are denoted by $U, \bar{U}, u, \bullet, \star, ...$; the choice of symbol depends on the intended meaning:

U usually stands for a subset (thus $X \longrightarrow \{U{\leftrightarrow}\bar{U}\}$), u for an element (thus $\{x\} \longrightarrow X$), $\bullet, \star$ have no interpretation.

Note there is no convenient notation for endomorphisms.

This is a feature: there is a decidable fragment of diagram chasing without endomorphisms [GLS] and we hope this notation may help to define a decidable fragment of topology. We shall see that our examples do not use endomorphisms much.

2.1 Hausdorff axioms of a topological space

Hausdorff formulated axioms of topology as follows. A *topology* on a set X is a collection of subsets, called open subsets, satisfying the following properties.

(1) the empty set $\emptyset$ and the whole set X is an open subset of X

(2) if U and V are open subsets of X, then so in $U \cap V$.

(3) a set U is open iff for every point $u \in U$ there is an open subset U' such that $u \in U' \subset U$.

(4) a map is continuous iff the preimage of an open subset is open

A category theorist would denote these notions by:

(1) *a point of* $x \in X$ by an arrow $\{x\} \longrightarrow X$

(2) *a subset* $U \subset X$ by an arrow $X \longrightarrow \{U{\leftrightarrow}\bar{U}\}$ representing the characteristic function of U where $\{U{\leftrightarrow}\bar{U}\}$ is the antidiscrete topological space where points U and $\bar{U}$ are neither open nor closed

(3) *an open subset* $U \subset X$ by its characteristic function $X \longrightarrow \{U{\rightarrow}\bar{U}\}$ where $\{U{\rightarrow}\bar{U}\}$ denotes the topological space with one open point U and one closed point $\bar{U}$

(4) *a point* $u \in U$ *of a subset* U *of* X is an arrow $\{u\} \longrightarrow X$ such that the composition $\{u\} \longrightarrow X \longrightarrow \{U{\leftrightarrow}\bar{U}\}$ coincides with the arrow $\{u\} \longrightarrow \{U{\leftrightarrow}\bar{U}\}$ sending u to U.

¿(4) *a point* $u \in U$ *of a subset* U *of* X is an arrow we denote by either $\{u\} \longrightarrow X \longrightarrow \{u{\rightarrow}\bar{u}\}$ or $\{U\} \longrightarrow X \longrightarrow \{U{\rightarrow}\bar{U}\}$; it is implicit that the composition sends u to u, resp. U to U.

(5) *the preimage* $f^{-1}(U)$ *of a subset* U *of* Y *under a map* $X \xrightarrow{f} Y$ is the composition arrow $X \longrightarrow Y \longrightarrow \{U{\leftrightarrow}\bar{U}\}$

This notation reminds us of category theory and lets us view the axioms of topology as rules for manipulating arrows between finite preorders and formal variables. The description of these rules below is vague and incomplete. One way to make it precise is given by a theorem below.

($1_\rightarrow$) composed arrows $X \longrightarrow \{\bullet\} \longrightarrow \{\bullet{\rightarrow}\star\}$ and $X \longrightarrow \{\star\} \longrightarrow \{\bullet{\rightarrow}\star\}$ are well-defined: the whole of X, resp. the empty (sub)set, is the preimage of the open subset $\{\bullet\}$ under the first, resp. the second, map

($2_\rightarrow$) the product $\{\bullet \rightarrow \star\} \times \{\bullet' \rightarrow \star'\}$ is well-defined and behaves as expected: the intersection of two open subsets corresponds the preimage of the open subset $\{(\bullet, \bullet')\}$ of the product $\{\bullet{\rightarrow}\star\} \times \{\bullet'{\rightarrow}\star'\}$

($3_\rightarrow$) for each arrow $X \underset{\xi_U}{\longrightarrow} \{U \leftrightarrow \bar{U}\}$ it holds

$$\begin{array}{ccc} & & \{U{\rightarrow}\bar{U}\} \\ & \nearrow & \downarrow \\ X & \xrightarrow[\xi_U]{} & \{U \leftrightarrow \bar{U}\} \end{array} \qquad \text{iff for each } \{u\} \longrightarrow X, \qquad \begin{array}{ccc} \{u\} & \longrightarrow & \{u{\rightarrow}U \leftrightarrow \bar{U}\} \\ \downarrow & \nearrow & \downarrow \\ X & \xrightarrow[\xi_U]{} & \{u = U \leftrightarrow \bar{U}\} \end{array}$$

($4_\rightarrow$) the composition arrow $X \longrightarrow Y \longrightarrow \{U{\rightarrow}\bar{U}\}$ behaves as expected

2.2 Axiom ($3_{\to}$) and pointwise equality

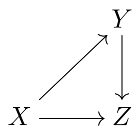
Axiom ($0_{\to}$) says we can check equality of maps, equivalently, commutativity of diagrams. pointwise, i.e. $f = g$ iff for any point x in their domain, $f(x) = g(x)$. We shall see now how ($3_{\to}$) is related to ($0_{\to}$).

($0_{\to}$) for each arrows $X \longrightarrow Y$, $Y \longrightarrow Z$, $X \longrightarrow Z$ the following are equivalent

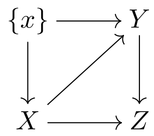
(i) the triangle (i) commutes

(ii) in (ii) below, for each $\{x\} \longrightarrow X$, if the upper triangle commutes, so does the square

(i)



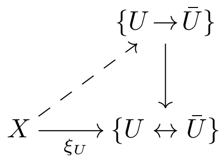
(ii)



Axiom ($3_{\to}$) can be reformulated as follows. Note the similarity of (ii) above and (b) below.

($3'_{\to}$) for each arrow $X \underset{\xi_U}{\longrightarrow} \{U \leftrightarrow \bar{U}\}$ the following are equivalent:

(i)

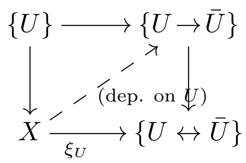


(ii) for each $\{U\} \longrightarrow X$ there is $X \longrightarrow \{U \to \bar{U}\}$ (depending on $\{U\} \longrightarrow X$) such that

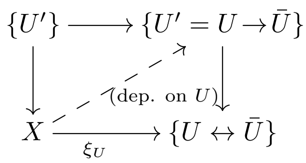
(a) in (a) below, if the square commutes, so the upper triangle

(b) in (b) below, if the upper triangle commutes, so does the square commutes

(a)



(b)



Note also that (b) may be informally described as: in diagram (a), to prove the implication that if the upper triangle commutes, so does the square, we need not use any information about arrow $\{U\} \longrightarrow X$ except that the upper triangle commutes

2.3 $h_{\{x\}} = Hom(\{x\}, -)$ is a functor to topological spaces.

The observations above lead to the following theorem.

Theorem 1 *Let $\mathcal{T}$ be a category containing the category of finite preorders as a full subcategory. Assume that ($3_{\to}$) above holds and that*

($1_{\to}$) the one-point order $\{\bullet\}$ is a terminal object of $\mathcal{T}$

($2_{\to}$) the product $\{\bullet \to \star\} \times \{\bullet' \to \star'\}$ of preorders is also the product in the category $\mathcal{T}$

Then the functor $h_{\{\bullet\}} = Hom(\{\bullet\}, -)$ defines a functor from $\mathcal{T}$ to the category of topological spaces. Moreover, this functor is faithful iff $\{\bullet\}$ co-generates $\mathcal{T}$, i.e. satisfies Axiom ($0_{\to}$).

($0_{\to}$) For $f, g : X \longrightarrow Y$, it holds $f = g$ provided $f \circ x = g \circ x$ for any arrow $x : \{\bullet\} \longrightarrow X$.

It is enough to assume that $\mathcal{T}$ contains the category of preorders on at most 4 elements.

Proof. The proof is trivial: the set $h_{\{\bullet\}} = Hom(\{\bullet\}, -)$ is equipped with topology in the following way: open subsets are of the form $\{\, g \in Hom(\{\bullet\}, X) \,:\, u(g(\bullet)) = U \,\}$ where $u : X \longrightarrow \{U < \bar{U}\}$; equivalently but in somewhat different notation, the open subsets are $U_u := \{\, g \in Hom(\{\bullet\}, X) \,:\, g \circ u = u_\bullet \,\}$ where $u_\bullet : \{\bullet\} \longrightarrow \{\bullet \to \star\}$, $\bullet \mapsto \bullet$ and u varies through all the arrows $X \longrightarrow \{\bullet \to \star\}$.

To check Axiom (1), consider composed arrows $X \longrightarrow \{\bullet\} \longrightarrow \{\bullet \to \star\}$ and $X \longrightarrow \{\star\} \longrightarrow \{\bullet \to \star\}$. They correspond to $Hom(\{\bullet\}, X)$ and the empty subset.

To check Axiom (2), consider the product the product $\{\bullet \to \star\} \times \{\bullet \to \star\}$. To check that $U_u \cap U_v$ is open, note it corresponds to the arrow

$$X \xrightarrow{u \times v} \{\bullet \to \star\} \times \{\bullet \to \star\} \longrightarrow \{(\bullet, \bullet) \to (\bullet, \star) = (\star, \bullet) = (\star, \star)\} \longrightarrow \{\bullet \to \star\}.$$

To check Axiom (3), note that the preimage of u under the diagonal arrow is an open subset of (the preimage of) U containing an arbitrary point u in U.

To see that the induced maps $f_* : Hom(\{\bullet\}, X) \longrightarrow Hom(\{\bullet\}, Y)$ are continuous, i.e. the preimage of an open subset is open, consider the composition $\{\bullet\} \longrightarrow X \longrightarrow Y \longrightarrow \{\bullet \to \star\}$: an arrow $Y \longrightarrow \{\bullet \to \star\}$ give rise to an arrow $X \longrightarrow \{\bullet \to \star\}$.

Axioms $(1_\rightarrow) - (3_\rightarrow)$ do not provide ways to construct neither new maps nor new topological spaces. Accordingly, the functor $h_{\{\bullet\}}$ is neither necessarily full nor surjective. It would be interesting to formulate diagram chasing axioms which would allow to prove, say, existance of the Alexandroff one-point compactification.

2.4 Examples: Connected, Hausdorff spaces, and Urysohn separation lemma

2.5 Bourbaki: connected, Hausdorff, Compact spaces and proper maps

In the next three sections we quote Bourbaki and see how parts of their text describing connected, Hausdorff, or quasi-compact spaces and proper maps may be rewritten in the language of arrows and finite preorders as instances of the Quillen lifting property.

Then we look at the Uryhson metrisation lemma and observe it is also a Quillen lifting property and its standard proof is an infinitary diagram chasing argument. Finally, we look at the Bourbaki definition of a uniform structure, an axiomatic approach to metric space, and see it describes a simplicial object.

A reader may prefer to skip all the text, look at the diagrams and decipher their meaning.

2.5.1 Connected Spaces

We quote from (Bourbaki, I§11.2) as an example of an argument we cannot quite translate to the language of arrows.

> 2. IMAGE OF A CONNECTED SET UNDER A CONTINUOUS MAPPING
>
> PROPOSITION 4. *Let A be a connected subset of a topological space X, and let f be a continuous mapping of X into a topological space X' . Then $f(A)$ is connected.*
>
> Suppose $f(A)$ is not connected. Then there exist two sets M', N' which are open in $f(A)$ and which form a partition of $f(A)$; hence $A \cap f^{-1}(M')$ and $A \cap f^{-1}(N')$ are open in A and form a partition of A; this contradicts the hypothesis that A is connected.
>
> The inverse image of a connected set under a continuous mapping need not be connected; consider for example a mapping of a discrete space into a space consisting of one point.
>
> From Proposition 4 we derive another characterization of non-connected spaces :
>
> PROPOSITION 5. *For a topological space X to be not connected it is necessary and sufficient that there exists a surjective continuous mapping of X onto a discrete space*

containing more than one point.

The condition is sufficient by Proposition 4. Conversely, if X is not connected, there exist two non-empty disjoint open subsets A, B whose union is X, and the mapping f of X onto a discrete space of two elements $\{a, b\}$, defined by $f(A) = \{a\}$ and $f(B) = \{b\}$, is continuous.

Proposition 5 becomes a definition in our approach: X is connected iff

$$\begin{array}{ccc} X & \longrightarrow & \{a,b\} \\ \downarrow & \nearrow & \downarrow \\ \{\bullet\} & \longrightarrow & \{a=b\} \end{array}$$

here $\{a, b\} \longrightarrow \{a = b\}$ is the map sending the discrete space on two points to a single point.

We cannot nicely translate Proposition 4 and so we make do with the following weaker statement that a surjective map from a connected space necessarily goes to a connected space: (i) $X \longrightarrow X'$ is surjective and (ii) X is connected implies (iii) X' is connected.

(i) $\begin{array}{ccc} \emptyset & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ \{\bullet\} & \longrightarrow & X' \end{array}$ and (ii) $\begin{array}{ccc} X & \longrightarrow & \{a,b\} \\ \downarrow & \nearrow & \downarrow \\ \{\bullet\} & \longrightarrow & \{a=b\} \end{array}$ imply (iii) $\begin{array}{ccc} X' & \longrightarrow & \{a,b\} \\ \downarrow & \nearrow & \downarrow \\ \{\bullet\} & \longrightarrow & \{a=b\} \end{array}$

The proof of our weaker version of Proposition 4 is a diagram chasing argument: draw the square of (iii), add arrow $X \longrightarrow X'$, by (ii) add arrow $\{\bullet\} \longrightarrow \{a, b\}$ which fits as a diagonal arrow in (iii). To prove (iii) commutes, use (i) and $(0_{\rightarrow})$.

2.5.2 Hausdorff spaces

We translate a definition and a proposition from a section on Hausdorff spaces in (Bourbaki, General Topology, I§8.1.2).

A space X is *Hausdorff* iff any of the following equivalent conditions holds:

(H) (Bourbaki) Any two distinct points of X have disjoint neighbourhoods.

Two distinct points of X is an injective arrow $\{x, y\} \longrightarrow X$, i.e. an arrow such that either of the following equivalent diagrams holds:

$$\begin{array}{ccc} \{x,y\} & \longrightarrow & \{x,y\} \\ \downarrow & \nearrow & \downarrow \\ \{x=y\} & \longrightarrow & X \end{array} \qquad \begin{array}{ccc} \{x,y\} & \longrightarrow & \{x \leftrightarrow y\} \\ \downarrow & \nearrow & \downarrow \\ X & \longrightarrow & \{x=y\} \end{array}$$

Points x and y of X *have disjoint neighbourhoods* is the following diagram:

$$\begin{array}{ccc} \{x,y\} & \longrightarrow & \{x < U > y\} \\ \downarrow & \nearrow & \\ X & & \end{array}$$

The whole definition may be written as follows: X is Hausdorff iff

$$\begin{array}{ccc} \{x,y\} & \longrightarrow & \{x < U > y\} \\ {\scriptstyle (injective)}\downarrow & \nearrow & \\ X & & \end{array}$$

Here we use that the preimage of the two open points define two non-intersecting open subsets of X containing x and y, resp.

Proposition 2 (Bourbaki, I§8.1.2). *Let f, g be two continuous mappings of a topological space X into a Hausdorff space Y; then the set of all $x \in X$ such that $f(x) = g(x)$ is closed in X.*

In the language of arrows, this can be stated as follows:

For Y Hausdorff, there exists an arrow $X \xrightarrow{\Delta} \{x \leftarrow \bar{x}\}$ such that

$$\begin{array}{ccc} \{x\} & \xrightarrow{x} & X \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{g} & Y \end{array} \qquad \text{iff} \qquad \begin{array}{ccc} \{x\} & \xrightarrow{x} & X \\ & \searrow & \downarrow{\scriptstyle \Delta} \\ & & \{x \leftarrow \bar{x}\} \end{array}$$

The proof is a diagram chasing calculation. We need a new "extensionality" second-order axiom which says that each property of arrows $\{x\} \longrightarrow X$ defines a subset $X \longrightarrow \{x \leftrightarrow \bar{x}\}$.

($5_\to$) For any property A of arrows $\{x\} \longrightarrow X$, there is an arrow $X \xrightarrow{A} \{x \leftrightarrow \bar{x}\}$

property $A(\{x\} \longrightarrow X)$ holds iff

$$\begin{array}{ccc} \{x\} & \xrightarrow{x} & X \\ & \searrow & \downarrow{\scriptstyle \Delta} \\ & & \{x \leftrightarrow \bar{x}\} \end{array}$$

commutes

By Extensionality Axiom ($5_\to$) there is an arrow $X \xrightarrow{\Delta} \{x \leftrightarrow \bar{x}\}$ satisfying the condition above; that is, the condition $f(x) = g(x)$ defines a subset. To show it is closed, we need to show that $X \xrightarrow{\Delta} \{x \leftrightarrow \bar{x}\}$ factors as $X \longrightarrow \{x \leftarrow \bar{x}\} \longrightarrow \{x \leftrightarrow \bar{x}\}$. By ($3_\to$), it is enough, for each $\{u\} \longrightarrow X$,

$$\begin{array}{ccc} \{u\} & \longrightarrow & \{u \to \bar{x} \leftrightarrow x\} \\ \downarrow & \nearrow & \downarrow \\ X & \xrightarrow[\xi_U]{} & \{u = \bar{x} \leftrightarrow x\} \end{array}$$

An arrow $\{u\} \longrightarrow X$ and arrows $X \xrightarrow{f} Y$, $X \xrightarrow{g} Y$ gives rise to an arrow $\{f(u), g(u)\} \longrightarrow Y$, which is injective by (i). Y is Hausdorff and hence construct an arrow $Y \xrightarrow{h} \{f(u) > y < g(u)\}$. Now consider arrows $X \xrightarrow{f} Y \xrightarrow{h} \{f(u) > y < g(u)\}$, and $X \xrightarrow{g} Y \xrightarrow{h} \{f(u) > y < g(u)\}$ and their product

$$X \xrightarrow{f \circ h \times g \circ h} \{f(u) > y < g(u)\} \times \{f(u) > y < g(u)\} \longrightarrow \{u = (f(u), g(u)) \to \bar{u} = (\text{ the others })\}$$

Finally, the required arrow is the pushout of this arrow and $X \xrightarrow{\Delta} \{x \leftrightarrow \bar{x}\}$.

We may use ($3'_\to$) instead of ($3_\to$). We then need not introduce a new variable u and we skip the last step as well. By ($3'_\to$), we need to construct an arrow depending on $\{x\} \longrightarrow X$, with certain properties. It is easy to check that the displayed arrow above does have those properties.

$$\begin{array}{ccc} \{\bar{x}\} & \longrightarrow & \{x \to \bar{x}\} \\ \downarrow & \nearrow & \downarrow \\ X & \xrightarrow[\xi_U]{} & \{\bar{x} \leftrightarrow x\} \end{array}$$

2.6 Compact spaces and proper maps

We quote (Bourbaki, I§10.2):

DEFINITION I. *Let f be a mapping of a topological space X into a topological space Y. f is said to be proper if f is continuous and if the mapping $f \times i_Z : X \times Z \longrightarrow Y \times Z$ is closed, for every topological space Z.*

THEOREM I. *Let $f^{-1} : X \longrightarrow Y$ be a continuous mapping. Then the following four statements are equivalent:*

a) f is proper.
b) f is closed and $f^{-1}(y)$ is quasi-compact for each $y \in Y$.
c) If $\mathfrak{F}$ is a filter on X and if $y \in Y$ is a cluster point of $f(\mathfrak{F})$ then there is a cluster point x of such that $f(x) = y$.
d) If $\mathfrak{U}$ is an ultrafilter on X and if $y \in Y$ is a limit point of the ultrafilter base $f(\mathfrak{U})$, then there is a limit point x of $\mathfrak{U}$ such that $f(x) = y$.

We quote (Bourbaki, I§6.5, Definition 5, Example):

> *Example.* Let X be a topological space, A a subset of X, x a point of X. In order that the trace on A of *the neighbourhood filter* $\mathfrak{F}$ of x should be a filter on A, it is necessary and sufficient that every neighbourhood of x meets A, i.e. that x lies in the *closure* of A (§1, no. 6, Definition 10). This example of an induced filter is of interest for two reasons: first because it plays an important role in the theory of limits (§7, no. 5) and secondly because *every filter can be defined in this way.* Indeed, let $\mathfrak{F}$ be a filter on a set X and let X' be the set obtained by *adjoining* a new element to X, X being identified with the complement of $\{w\}$ in X' (*Set Theory*, **R**, §4, no. 5) ; let $\mathfrak{F}'$ be the filter on X' consisting of the sets $M \cup \{\omega\}$ where M runs through $\mathfrak{F}$. For each point $x \neq \omega$ of X', let $\mathfrak{B}(x)$ be the set of all subsets of X' which contain x, and let $\mathfrak{B}(\omega)$ be $\mathfrak{F}'$; then the $\mathfrak{B}(x)$ for $x \in X'$ obviously satisfy axioms (V_I), (V_{II}), (V_{III}) and (V_{IV}) and therefore define a topology on X' for which they are the neighbourhood filters of points. Finally ω lies in the closure of X in this topology, and $\mathfrak{F}$ is induced by $\mathfrak{F}' = \mathfrak{B}(\omega)$ on X. The topology thus defined on X' (resp. the set X' with this topology) is called the topology (resp. the topological space) associated with $\mathfrak{F}$.

Item *d*) can be readily rewritten in the language of arrows. Following (Bourbaki, I§6.5, Definition 5, Example), for an ultrafilter U on X, define the topology on the set $X \cup \{\infty\}$ by: a subset Z of X is open iff Z is open in X; $Z \cup \{\infty\}$ is open iff Z is open X and is big according to the ultrafilter U. Denote this space $X \cup_U \{\infty\}$. A point x is a *limit point of* U iff each neighbourhood of x in X is U-big, i.e. iff sending ∞ to x defines a continuous map from $X \cup_U \{\infty\}$.

These considerations let us rewrite *d*) as $d_\rightarrow$) and $d'_\rightarrow$) below. Note $d'_\rightarrow$) immediately implies Tychonoff theorem that a product of quasi-compact spaces is quasi-compact, as a class defined by a lifting property is closed under taking colimt/limit. This is how it is proved in (Bourbaki, I§9.5).

$d_\rightarrow$) A map $X \longrightarrow Y$ is proper iff for each ultrafilter U on X it holds

$$\begin{array}{ccc} X & \xrightarrow{id} & X \\ \downarrow & \nearrow & \downarrow \\ X \cup_U \{\infty\} & \longrightarrow & Y \end{array}$$

$d'_\rightarrow$) A map $X \longrightarrow Y$ is proper iff for each ultrafilter U on a space A it holds

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ A \cup_U \{\infty\} & \longrightarrow & Y \end{array}$$

To see $d'_\rightarrow) \implies d)$, take $A = X$ and $A \longrightarrow X$ to be identity. To see $d) \implies d'_\rightarrow)$, take the ultrafilter on X such that a subset of X is big iff its preimage in A is big.

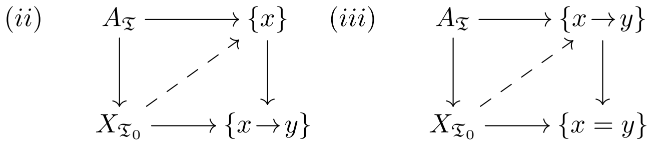
> (p.129 of file) (Bourbaki, I, Exercises §3, Exercise 11, p.129) I I) Let X be a topological space, $\mathfrak{T}$ the topology of X, A a dense subspace of X. Show that the set of topologies on X which are finer than $\mathfrak{T}$, in which A is dense in X, and which induce the same topology as $\mathfrak{T}$ on A, has at least one maximal element; such a maximal element is called an A-maximal topology. Show that a topology $\mathfrak{T}_0$ on X is A-maximal if and

only if every subset M of X, such that $A \cap M$ is dense in M (with respect to $\mathfrak{T}_0$) and open in A, is open in the topology $\mathfrak{T}_0$, and that the subspace $X \setminus A$ is then discrete in the topology induced by $\mathfrak{T}_0$, and is closed in X. Show also that if $\mathfrak{T}_0$ is A-maximal and if the topology induced by $\mathfrak{T}_0$ on A is quasi-maximal (9 2, Exercise 6) then $\mathfrak{T}_0$ is quasi-maximal.

(p.134 of file) d) $\mathfrak{F}$ is an ultrafilter on X if and only if the associated topology on X' is X-maximal (§3, Exercise I I).

topologies on X which are (i) finer than $\mathfrak{T}$, in which (ii) A is dense in X, and which (iii) induce the same topology as $\mathfrak{T}$ on A is described by

(i) $X_{\mathfrak{T}_0} \longrightarrow X_{\mathfrak{T}}$ is continuous



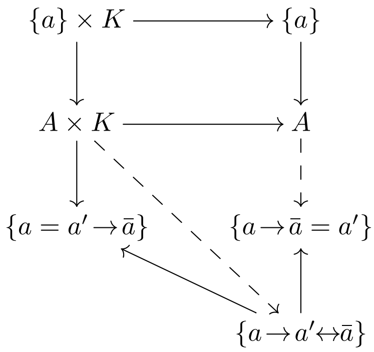
The 'ultrafilter' arrows are the 'shortest' arrows in the following sense:

Claim 1 *A non-identity injective arrow $A \hookrightarrow X$ has form $A \longrightarrow A \cup_U \{\infty\}$ iff it is injective, satisfies the lifting properties (ii-iii) above and there is no non-trivial decomposition $A \hookrightarrow X' \hookrightarrow X$ such that $A \hookrightarrow X'$ satisfies (ii-iii).*

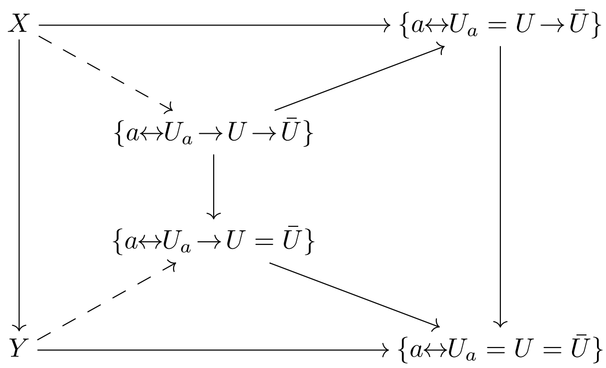
2.7 Quasi-compactness as a diagram.

K is quasi-compact iff the map $A \times K \longrightarrow A$ is closed for each A. By definition that means that the projection of a closed set is closed; by Axiom (3) of topological spaces this can be reformulated as follows: for an open subset $U \subset A \times K$ and a point $a \in A$, if $\{a\} \times K \subset U$, then there is an open neighbourhood $U_a \ni a$ such that $U_a \times K \subset U$.

This can be expressed as the following diagram:



A map $X \longrightarrow Y$ is closed iff the following diagram holds:



2.8 Metric spaces. Uryhson separation lemma.

We give two more examples related to the real numbers.

2.8.1 Uryhson separation lemma

[Hausdorff, Set theory, §25,n.3] state items (i) and (ii); [Hausdorff, Set theory, §26] implies $(ii)'$.

> `Uryhson Separation Lemma`. *Assume X is normal, i.e. (i) for each pair of closed subsets $A, B \subset X$ there exist disjoint open neighbourhoods $U \supset A$ and $V \subset B$. Then each pair of closed subsets $A, B \subset X$ can be separated by a function to the real line, i.e. (ii) there is a function $f : X \longrightarrow [0,1]$ such that $f(a) = 0$ for each $a \in A$ and $f(b) = 1$ for each $b \in B$. Moreover, if X is first countable, then $(ii)'$ there is a function $f : X \longrightarrow [0,1]$ such that $f(a) = 0$ iff $a \in A$ and $f(b) = 1$ iff $b \in B$.*

Let us translate this to the language of arrows.

> `Uryhson Separation Lemma`. For X first countable, it holds

$$(i)_{\rightarrow} \quad \begin{array}{ccc} & & \{A > U < x > V < B\} \\ & \overset{f}{\nearrow} & \downarrow \\ X & \longrightarrow & \{A > U = x = V < B\} \end{array} \qquad \text{implies} \qquad (ii)'_{\rightarrow} \quad \begin{array}{ccc} & & [0,1] \\ & \overset{f}{\nearrow} & \downarrow \\ X & \longrightarrow & \{0 > x < 1\} \end{array}$$

To see the equivalence of (i) and $(i)_{\rightarrow}$, take the neighbourhoods $U \subset A$ and $V \supset B$ to be the preimages of the open two-point subsets $\{A, U\}$ and $\{B, V\}$.

The reformulation of (ii) is somewhat more cumbersome:

> $(ii)_{\rightarrow}$ there is an arrow $X \longrightarrow [0,1]$ such that the following implication holds: if the square commutes, then the upper triangle commutes.
>
> $$\begin{array}{ccc} \{0,1\} & \longrightarrow & [0,1] \\ \downarrow & \overset{f}{\nearrow} & \downarrow \\ X & \longrightarrow & \{0 > x < 1\} \end{array}$$

The standard proof of Uryhson Separation Lemma is then viewed as follows. Points $0 = t_{0..0} > t_{0..1} > .... > t_{1..1} = 1$ split the interval $[0,1]$ into open sets $I_i = (t_i, t_{i+1})$ and closed sets $\{t_i\}$, $1 \leq i \leq n$, and hence give rise to the arrow

$$[0,1] \longrightarrow \{t_{0..0} > I_{0..0} < ... < t_i > I_i < t_{i+1} > ... > I_{1..1} < t_{1..1}\}.$$

Taking strings i longer and points t_i more and more densely, we see that

$$[0,1] = \lim_{n \longrightarrow \infty} \{t_{0..0} > I_{0..0} < ... < t_i > I_i < t_{i+1} > ... > I_{1..1} < t_{1..1}\}.$$

Then use item (i) and the pullback square below to construct arrows to these partial orders

$$\begin{array}{ccc} \{t_1 > I_1 < ... < t_i = t_{i0} > I_{i0} < t_{i1} > I_{i1} < t_{i+1} > ... > I_{n-1} < t_n\} & \longrightarrow & \{t_1 > I_1 < ... < t_i > I_i < t_{i+1} > ... > I_{n-1} < t_n\} \\ \downarrow & & \downarrow \\ \{t_i = t_{i0} > I_{i0} < t_{i1} > I_{i1} < t_{i+1}\} & \longrightarrow & \{t_1 = I_1 = ... = t_i > I_i < t_{i+1} = ... = I_{n-1} = t_n\} \end{array}$$

Finally, pass to the inverse limit to construct an arrow $X \longrightarrow [0,1]$; then prove $(ii)_{\rightarrow}$.

2.9 Uniform spaces

(Bourbaki, II§1.1.1) treats metric spaces as uniform spaces; we observe that the uniform space is a simplicial object.

We quote (Bourbaki, I§6.1.1) and (Bourbaki, II§1.1.1):

DEFINITION I. A filter on a set X is a set F of subsets of X which has the following properties:
(F I) Every subset of X which contains a set of F belongs to F.
(FII) Every finite intersection of sets of F belongs to F.
(FuI) The empty set is not in F.

1. DEFINITION OF A UNIFORM STRUCTURE DEFINITION I. A uniform structure (or uniformity) on a set X is a structure given by a set $\mathcal{U}$ of subsets of $X \times X$ which satisfies axioms (F_I) and (F_{II}) of Chapter I, 6, no. I and also satisfies the following axioms:
(V_I) Every set belonging to $\mathcal{U}$ contains the diagonal d.
(V_{II}) If $V \in U$ then $V^{-1} \in \mathcal{U}$.
(V_{III}) For each $V \in \mathcal{U}$ there exists $W \in \mathcal{U}$ such that $W \circ W \subset V$.

The sets of $\mathcal{U}$ are called entourages of the uniformity defined on X by $\mathcal{U}$. A set endowed with a uniformity is called *a uniform space*. If V is an entourage of a uniformity on X, we may express the relation $(x, x') \in V$ by saying that "x and x' are V-close".

A metric space X is a uniform space: $V \in \mathcal{U}$ iff $\{(x, x') : dist(x, x') < \varepsilon\} \subset V$ for some $\varepsilon > 0$.

Let us translate the above to the language of arrows: we shall see that a uniform space may be viewed as a simplical object in the subcategory of topological spaces associated with filters.

First notice that a filter $\mathcal{F}$ on a set X may be viewed as a topology on X where a subset is open iff it is either $\mathcal{F}$-big or empty. Indeed, Axioms (F_I) and (F_{II}) of a filter imply that the family of subsets $\mathfrak{U} \cup \{\emptyset\}$ is a topology on a set X. Let $f\mathcal{T}op$ be the full subcategory of topological spaces associated with filters.

A uniform structure defines a filter topology on $X \times X$, i.e. an object of $f\mathcal{T}op$.

Axiom (V_{II}) says that permuting the coordinates $X \times X \longrightarrow X \times X, (x_1, x_2) \mapsto (x_2, x_1)$ is continuous in this topology.

Define topology on the set $X \times X \times X$ via the pullback square in the subcategory $f\mathcal{T}op$:

$$\begin{array}{ccc} X \times X \times X & \xrightarrow{(p_1 \times p_2)} & X \times X \\ {\scriptstyle (p_2 \times p_3)}\downarrow & & \downarrow{\scriptstyle p_2} \\ X \times X & \xrightarrow[p_2]{} & X \end{array}$$

Axiom (V_{III}) says that the map $X \times X \times X \xrightarrow{(p_1, p_3)} X \times X, (x_1, x_2, x_3) \mapsto (x_1, x_3)$ is continuous in this topology. Indeed, by definition $W_1 \circ W_2 = \{(x_1, x_3) : (x_1, x_2) \in W_1,\ (x_2, x_3) \in W_2\}$ and the sets

$$\{(x_1, x_2, x_3) : (x_1, x_2) \in W_1,\ (x_2, x_3) \in W_2\} = W_1 \times X \cap X \times W_2,$$

$W_1, W_2 \in \mathfrak{U}$, form a base of the pullback topology on $X \times X \times X$. Hence, (V_{III}) says that the preimage of an open subset of $X \times X$ under (p_1, p_3) contains an open subset of $X \times X \times X$, i.e. is open (as pullback is taken among filter topologies).

Axiom (V_I) implies that the diagonal map $X \xrightarrow{(x,x)} X \times X$ is continuous as a map from the set X equipped with antidiscrete topology to the set $X \times X$ equipped with the topology above.

Note that $W \circ W$ intersects the diagonal and the continuity of the diagonal map $X \xrightarrow{(x,x)} X \times X$ implies $W \circ W$ contains the diagonal. Thus, in presence of (V_{III}), (V_I) is equivalent to the continuity of the diagonal map $X \xrightarrow{(x,x)} X \times X$ in the topologies indicated.

Let X_1 denote the set X equipped with the antidiscrete topology. Let X_2 and X_3 denote the sets $X \times X$ and $X \times X \times X$ equipped with the topologies above. For $n > 3$, let X_n be the pullback in $f\mathcal{T}op$

$$\begin{array}{ccc} X_n & \xrightarrow{(p_2 \times \ldots \times p_n)} & X_{n-1} \\ {\scriptstyle (p_1 \times p_2)}\downarrow & & \downarrow{\scriptstyle p_2} \\ X_2 & \xrightarrow[p_2]{} & X \end{array}$$

The axioms above ensure that the "set-theoretic" face and degeneration maps

$$(p_{i_1},...,p_{i_k}) : X \times ... \times X \longrightarrow X \times ... \times X$$

are continuous. Thus we see that a uniform structure on a set X defines a simplicial complex X_n in $f\mathcal{T}op$,

$$(p_{i_1},...,p_{i_k}) : X_n \longrightarrow X_m$$

Claim 2 *A uniform structure on a set X is a simplical object $X.$ in the subcategory $f\mathcal{T}op$ of filter topological spaces equipped with an involution $i : X. \longrightarrow X.$ such that*

- *X_1 is the set X equipped with antidiscrete topology*
- *the underlying set of X_2 is $X \times X$*
- *$i : X. \longrightarrow X.$ is the involution permuting the coordinates on $X \times X$*
- *for $n > 2$, X_n is the pullback as described above*

Question 1 *Find a categorical description of the simplicial objects obtained from uniform spaces.*

3 Lifting property wrt counterexample as a common pattern

Given a morphism f, perhaps the simplest way to define a property of morphisms that f *not* satisfy, short of $g \neq f$, is by *left or right Quillen lifting property*: we say $f : A \longrightarrow B$ lifts wrt $g : X \longrightarrow Y$, write $f \rightthreetimes g$, iff for each $i : A \longrightarrow X$ and $j : B \longrightarrow Y$ making the square commutative, i.e. $f \circ j = i \circ g$, there is a diagonal arrow $\tilde{j} : B \longrightarrow X$ making the total diagram $A \xrightarrow{f} B \xrightarrow{\tilde{j}} X \xrightarrow{g} Y, A \xrightarrow{i} X, B \xrightarrow{j} Y$ commutative, i.e. $f \circ \tilde{j} = i$ and $\tilde{j} \circ g = j$ (see Figure 1(a)).

Perhaps the simplest way to define a property a given morphism, short of being equal, satisfies is to apply Quillen lifting property twice: we say that g' is *(left) exemplified by* g iff for each f it holds that $f \rightthreetimes g$ implies $f \rightthreetimes g'$. More generally, we say that g' is *(left) exemplified by a class P of morphisms* or that g' *is P-like* iff it holds that a morphism lifts wrt g' if it lifts wrt each $p \in P$.

Left (right) Quillen negation of a property (class) P of morphisms is

$$P^{\rightthreetimes l} := \{\, p^l : \ p^l \rightthreetimes p \ \text{ for each } \ p \in P \,\}$$

$$P^{\rightthreetimes r} := \{\, p^r : \ p \rightthreetimes p^r \ \text{ for each } \ p \in P \,\}.$$

Double Quillen negation we call *right (left) Quillen generalisation*

$$P^{\rightthreetimes lr} := \{\, p^r : \ \text{ for each } p^l \text{ it holds } p^l \rightthreetimes p \ \text{ for each } \ p \in P \text{ implies } p^l \rightthreetimes p^r \,\}$$

$$P^{\rightthreetimes rl} := \{\, p^l : \ \text{ for each } p^r \text{ it holds } p \rightthreetimes p^r \ \text{ for each } \ p \in P \text{ implies } p^l \rightthreetimes p^r \,\}$$

Thus g' is left exemplified by P iff $g' \in P^{\rightthreetimes rl}$ and g' is right exemplified by P iff $g' \in P^{\rightthreetimes lr}$.

The goal of this section is to convince the reader that Quillen lifting property is a surprisingly useful "trick" which often defines a notion given a (counter)example, and that this is worthy of further investigation. We do so by listing examples.

Surjective is both left Quillen negation of $\emptyset \longrightarrow \{x\}$ and right Quillen negation of $\{a\} \longrightarrow \{a \leftrightarrow b\}$. *Injective* is both left Quillen negation of $\{x, y\} \longrightarrow \{x = y\}$ and right Quillen negation of $\{x \leftrightarrow y\} \longrightarrow \{x = y\}$. Right Quillen negation of $\{1\} \longrightarrow \{0 < 1\}$ is *dense image*. X is *connected* iff $X \longrightarrow \{\bullet\} \rightthreetimes \{x, y\} \longrightarrow \{x = y\}$. X is *linearly connected* iff $\{0, 1\} \longrightarrow [0, 1] \rightthreetimes X \longrightarrow \{\bullet\}$. *Every real-valued function on X is bounded* is, for X connected, $\emptyset \longrightarrow X \rightthreetimes \coprod_{n \in \mathbb{N}} [-n, n] \longrightarrow \mathbb{R}$. A subset A of X is *sequentially closed* iff $\{1, \frac{1}{2}, \frac{1}{3}, \ldots\} \longrightarrow \{1, \frac{1}{2}, \frac{1}{3}, \ldots, 0\} \rightthreetimes A \longrightarrow X$ where, as expected, topology on $\{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ and $\{1, \frac{1}{2}, \frac{1}{3}, \ldots, 0\}$ is induced from the real numbers.

All the separation axioms $T_0 - T_4$ and T_6 are Quillen negations:

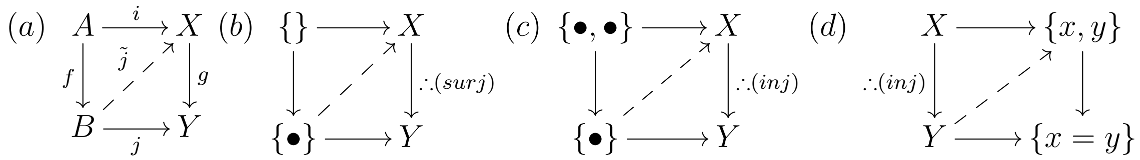


Figure 1: Lifting properties. Dots $\therefore$ indicate free variables and what property of these variables is being defined; in a diagram chasing calculation, "$\therefore$ $(surj)$" reads as: given a (valid) diagram, add label $(surj)$ to the corresponding arrow.
(a) The definition of a lifting property $f \rightthreetimes g$: for each $i : A \longrightarrow X$ and $j : B \longrightarrow Y$ making the square commutative, i.e. $f \circ j = i \circ g$, there is a diagonal arrow $\tilde{j} : B \longrightarrow X$ making the total diagram $A \xrightarrow{f} B \xrightarrow{\tilde{j}} X \xrightarrow{g} Y, A \xrightarrow{i} X, B \xrightarrow{j} Y$ commutative, i.e. $f \circ \tilde{j} = i$ and $\tilde{j} \circ g = j$.
(b) $X \longrightarrow Y$ is surjective.
(c) $X \longrightarrow Y$ is injective; $X \longrightarrow Y$ is an epicmorphism if we forget that $\{\bullet\}$ denotes a singleton (rather than an arbitrary object and thus $\{\bullet, \bullet\} \longrightarrow \{\bullet\}$ denotes an arbitrary morphism $Z \sqcup Z \xrightarrow{(id,id)} Z$)
(d) $X \longrightarrow Y$ is injective, in the category of Sets; $\pi_0(X) \longrightarrow \pi_0(Y)$ is injective, when the diagram is interpreted in the category of topological spaces.

(T_0) $\{\bullet \gtreqless \star\} \longrightarrow \{\bullet\} \rightthreetimes X \longrightarrow \{\bullet\}$

(T_1) $\{\bullet < \star\} \longrightarrow \{\bullet\} \rightthreetimes X \longrightarrow \{\bullet\}$

(T_2) $\{\bullet, \bullet'\} \hookrightarrow X \rightthreetimes \{\bullet > \star < \bullet'\} \longrightarrow \{\bullet\}$ for each injection $\{\bullet, \bullet'\} \hookrightarrow X$

(T_3) $\{a\} \longrightarrow X \rightthreetimes \{a \approx V < x > U_A < A\} \longrightarrow \{a = V = x = U_A \to A\}$ for each arrow $\{a\} \longrightarrow X$

(T_4) $\{\} \longrightarrow X \rightthreetimes \{A > U < x > V < B\} \longrightarrow \{A > U = x = V < B\}$

(T_6) $\{\} \longrightarrow X \rightthreetimes [0,1] \longrightarrow \{0 > x < 1\}$

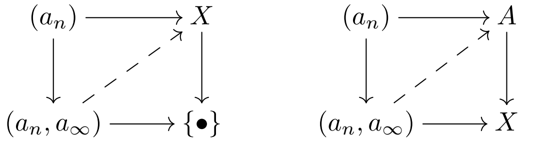
Question 2 *Is compactness (rather, being a proper map) the right Quillen generalisation of the class of maps between finite spaces, or perhaps the right Quillen generalisation of the class consisting of the following four morphisms:*

$$(\{U \to x \leftarrow V\} \longrightarrow \{U = x = V\},\ \{1\} \longrightarrow \{0 \to 1\},\ \{0 \to 1\} \longrightarrow \{0 = 1\},\ \{0 \leftrightarrow 1\} \longrightarrow \{0 = 1\})^{\rightthreetimes lr}$$

It is easy to check that the maps obtained from ultrafilters do lift wrt closed morphisms of finite spaces, and thus the right Quillen generalisation is a subclass of proper maps. If a map $f : A \longrightarrow X$ lifts wrt the latter three morphism, then it has dense image, is injective and the topology on $A \subset X$ is induced.

3.1 Limit as a lifting property

Limit is not a lifting property but is related to the same diagram. Recall that a sequence $\{a_n\}_{n \in \mathcal{N}}$ in X converges to a_∞ in X iff each open neighbourhood of a_∞ in X contains all but finitely many points in the sequence. The latter is a condition on subsets of the sequence which we take as the definition of topology on the set $\{a_n\}_{n \in \mathcal{N}} \cup \{a_\infty\}$. This leads to the following diagrams:



The latter defines the notion of a *sequentially closed* subset A of X.

In the category of metric spaces with Lipschitz or uniformly continuous maps, a space X is *complete* iff $\{1, \frac{1}{2}, \frac{1}{3}, \ldots\} \longrightarrow \{1, \frac{1}{2}, \frac{1}{3}, \ldots, 0\} \rightthreetimes \emptyset \longrightarrow X$. A subset A of X is *closed* iff $\{1, \frac{1}{2}, \frac{1}{3}, \ldots\} \longrightarrow \{1, \frac{1}{2}, \frac{1}{3}, \ldots, 0\} \rightthreetimes A \longrightarrow X$; as expected, the metric on $\{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ and $\{1, \frac{1}{2}, \frac{1}{3}, \ldots, 0\}$ is induced from the real numbers. Note that in this notation the proof that *a closed subspace of a complete metric space is complete* reduces to two applications of the Quillen negation (lifting property), cf. a formal proof of this in Ganesalingam and Gowers (GG, §2.2, Problem 2).

4 Calculus: Compact and Finite

We want to define a calculus (derivation calculus, a proof system) based on the following observations:

- a metric space is compact iff the modulus of continuity of a continuous function does not depend on the point: in physics notation,

$$\delta(x, \varepsilon) = \delta(\varepsilon)$$

  where $\delta(x, \varepsilon)$ is the modulus of continuity of a continuous function on X
- in the topology generated by a family of closed sets, a set is contained in a proper closed subset iff it is contained in a finite union of sets in the family.
- there is a decidable fragment of category theory diagram chasing without endomorphisms

Accordingly, we want a calculus which

- quasi-compactness is expressed as a rule to remove dependency of a term on a free variable in certain situations, not dissimilar to what does a compiler optimizing away dead code
- "finite" is expressed as being closed in topology generated by a family of sets
- there is no (convenient) notation to talk about endomorphisms

Moreover,

- properties of arrows are represented as *labels* on arrows; hence inference rules may add or *remove* labels from arrows
- it is specified *explicitly* what an arrow may depend on, and there are rules to add or remove dependencies. In first-order language, an analogue would be a rule to replace term $t(x, y, z)$ by $t(x, y)$ (that is, removing dependency on z) (in certain situations)
- a proof/derivation is a collection of labelled arrows, commutativity relations

Let us now give several examples of how we want our calculus to function.

4.1 K is quasi-compact iff $X \times K \longrightarrow X$ is closed for any X.

(Bourbaki, Introduction)

> We are thus led at last to the general concept of a topological space, which does not depend on any preliminary theory of the real numbers. We shall say that a set E carries a topological structure whenever we have associated with each element of E, by some means or other, a family of subsets of E which are called neighbourhoods of this element - provided of course that these neighbourhoods satisfy certain conditions (the axioms of topological structures). Evidently the choice of axioms to be imposed is to some extent arbitrary, and historically has been the subject of a great deal of experiment (see the Historical Note to Chapter I). The system of axioms finally arrived at is broad enough for the present needs of mathematics, without falling into excessive and pointless generality. A set carrying a topological structure is called a topological space and its elements are called points. The branch of mathematics which studies topological structures bears the name of *Topology* (etymologically, "science of place", not a particularly expressive name), which is preferred nowadays to the earlier (and synonymous) name of *Analysis situs*.

> To formulate the idea of neighbourhood we started from the vague concept of an element "sufficiently near" another element. Conversely, a topological structure now enables us to give precise meaning to the phrase "such and such a property holds for all points *sufficiently near a*": by definition this means that the set of points which have this property is a neighbourhood of a for the topological structure in question. From the notion of neighbourhood there flows a series of other notions whose study is proper to topology: the interior of a set, the closure of a set, the frontier of a set, open sets, closed sets, and so on (see Chapter I, g I). For example, a subset A is an *open* set if, whenever a point a belongs to A, all the points sufficiently near a belong to A; in other words, if A is a neighbourhood of each of its points.

A vague, if old-fashioned, way to express compactness may be: a parameter¿variable? k varies over a compact domain if the following holds: if a property holds for all points sufficiently near a in some sense (continuously) depending on parameter k, then then the property holds for all points sufficiently near a in some sense independent of k.

Let us translate this vague expression first in the set theoretic language of Bourbaki, and then speculate what that might mean for the calculus we want to construct.

(Bourbaki, Introduction) writes: "[A] topological structure now enables us to give precise meaning to the phrase "such and such a property holds for all points *sufficiently near a*": by definition this means that the set of points which have this property is a neighbourhood of a for the topological structure in question." Recall that for Bourbaki a neighbourhood of a is a superset of an open subset containing a.

It is implicit that a is a point of a space A and variable k varies over a space K. A "property" is a subset of A, call it P. "the property holds for all points sufficiently near a" iff P is a (not necessarily open) neighbourhood of a. "points sufficiently near a in some sense (continuously) depending on parameter k" defines a family of neighbourhoods $P_k \ni a$, $k \in K$ of the point a. "Continuously" may be taken to mean the subset $\tilde{P} := \cup_k P_k \times \{k\} \subset A \times K$ is open, or at least it is a neighbourhood of $\{a\} \times K$ in $A \times K$.

Hence the sentence above says: if $\tilde{P} \subset A \times K$ is open, $\{a\} \times K \subset \tilde{P}$, then there is an open subset $a \in A_a \subset A$ such that $A_a \times K \subset \tilde{P}$. As this holds for each $a \in A$, it is equivalent to say the set $\{a : \{a\} \times K \subset \tilde{P}\}$ is open, i.e. the projection of $\bar{\tilde{P}} := A \setminus \tilde{P}$ is closed.

In our calculus we would like to express this principle as a derivation rule of the following kind: k varies a compact domain iff in certain situations¿contexts? the following derivation rule is valid:

$$\frac{\{a\} \longrightarrow A \xrightarrow[\text{(depends on } k)]{} \{a \to \bar{a}\}}{\{a\} \longrightarrow A \longrightarrow \{a \to \bar{a}\}}$$

or something like

$$\frac{\{a\} \times K \longrightarrow A \times K \longrightarrow \{a \to \bar{a}\}}{\{a\} \longrightarrow A \longrightarrow \{a \to \bar{a}\}}$$

It is implicit in our notation that the composition $\{a\} \longrightarrow \{a \to \bar{a}\}$ sends a to a, hence the conclusion, resp. the hypothesis, define an open neighbourhood of a, resp. a family of such depending on k.

The context should imply two things: (i) that the arrow $A \xrightarrow[\text{(depends on } k)]{} \{a \to \bar{a}\}$ corresponds a neighbourhood $\{a\} \times K$, and, more interestingly, (ii) we only know (care) about $A \longrightarrow \{a \to \bar{a}\}$ that it is a property of points which holds sufficiently near a.

Let us stress: as (iii) indicates, in our calculus we want that an inference rule may tell us to *forget* some information, such as details of the construction of the family $A \xrightarrow[\text{(depends on } k)]{} \{a \to \bar{a}\}$.

Probably, (i) is related to the arrow being definable.

4.2 Axioms $T_2 - T_4$

Let us give a proof of *a Hausdorff quasi-compact space is necessary normal* using terminology of "sufficiently near". It is this proof that we want to rewrite in the calculus we want to construct.

Recall that we want to prove that for two closed non-intersecting subsets $A, B \subset X$ there are neighbourhoods $V \supset A$ and $W \subset B$ such that $V \cap W = \emptyset$. Assume we know that A and B are quasi-compact.

A space is Hausdorff iff any pair of points sufficiently near to a given pair of distinct points are also distinct. That is, if $a \neq b \in X$ and a', b' are sufficiently near to a, b, then $a' \neq b'$; the meaning of "sufficiently near" depends on both a and b. b varies over a compact domain B. Hence for any a' "sufficiently near" to a in a sense independent of b in B, $a' \neq b'$ for b' sufficiently near b varying over B. Variable a ranges over a compact domain as well, hence for a' sufficiently near a ranging over domain A, in a sense independent of both a and b, $a' \neq b'$ for b' sufficiently near b varying over B. Now, "a' sufficiently near a ranging over domain A, in a sense independent of both a and b" means a' varying over a neighbourhood of A, and similarly b' varies over a neighbourhood of B. Hence, the statement is proven.

In terms of our calculus this argument should look as follows. We have the following diagrams:

$$\text{(i)}\quad \begin{array}{ccc} & & \{A\to A'\leftarrow x\to B'\leftarrow B\} \\ & \overset{f}{\nearrow} & \downarrow \\ X & \longrightarrow & \{A\to A' = x = B'\leftarrow B\} \end{array} \qquad \text{(ii)}\quad \begin{array}{ccc} \{A, A', x, B', B\} & \longrightarrow & \{A\to A'\leftarrow x\to B'\leftarrow B\} \\ \downarrow & \overset{f}{\nearrow} & \downarrow \\ X & \longrightarrow & \{A\to A' = x = B'\leftarrow B\} \end{array}$$

We have an arrow $X \longrightarrow \{A \to A' = x = B' \leftarrow B\}$, that is, two closed subsets A and B of X. We want to construct an arrow $X \longrightarrow \{A \leftarrow A' \to x \leftarrow B' \to B\}$ such that the diagram (i) commutes. That is, $A \cup A'$ and $B \cup B'$, are non-intersecting neighbourhoods of A and B.

To check commutativity, it is enough to check it pointwise.

This means that for arbitrary arrow $\{A, A', x, B', B\} \longrightarrow X$, if the square of (ii) commutes (where the upper horizontal arrow $\{A, A', x, B', B\} \longrightarrow \{A \leftarrow A' \to x \leftarrow B' \to B\}$ is the obvious one), so does the upper triangle.

X is Hausdorff and that means you can construct an arrow $X \longrightarrow \{A \to A' \leftarrow x \to B' \leftarrow B\}$ making the upper triangle commute *given* an arrow $\{A, B\} \longrightarrow \{A\to A'\leftarrow x\to B'\leftarrow B\}$ such that the square commutes.

$$(iii)\qquad \begin{array}{ccc} \{A, B\} & \longrightarrow & \{A\to A'\leftarrow x\to B'\leftarrow B\} \\ \downarrow & \overset{f}{\nearrow}\ {\scriptstyle \text{dep.}(A,B)} & \downarrow \\ X & \longrightarrow & \{A\to A' = x = B'\leftarrow B\} \end{array}$$

Now, use quasi-compactness of X to remove the dependency of the diagonal arrow X. Hence you get the diagonal arrow such that for each an arrow $\{A, B\} \longrightarrow \{A\to A'\leftarrow x\to B'\leftarrow B\}$ making the square commute, the upper triangle commute. Now check this implies (i), which is what we want to prove.

4.3 K is quasi-compact iff each open covering has a finite subcovering.

An open covering $\mathcal{U}$ of X is a collection of open subsets such that for each $u \in X$ there is $U = U(u) \in \mathcal{U}$ such that $u \in U(u)$. In our calculus this should be denoted by

$$\{u\} \longrightarrow X \xrightarrow{(U(u))} \{u\to w\} \qquad (*)$$

Here it is implicit that the composition sends u into u. The label $U(u)$ says that the arrow $X \longrightarrow \{u\to v\}$ may depend on u.

To express the notion of a finite subcovering, we use the following observation. Adjoin a point w to X and consider topology on $X \cup \{w\}$ generated by sets in U as basic closed subsets. Then X is closed in this topology iff X is a finite union of basic closed sets, i.e. there is a finite subcovering of $\mathcal{U}$.

A finite subcovering is to be expressed as follows.

Replace $\{u\to w\}$ by $\{u\leftarrow w\}$ and consider the following as a definition of a topological space X_U:

$$\{u\} \longrightarrow X_U \xrightarrow{(U(u))} \{u\leftarrow w\}$$

$$\{w\} \longrightarrow X_U \longrightarrow \{u \leftarrow w\} \text{ –or– } \{w\} \longrightarrow X_U \xrightarrow{(U(u))} \{u \leftarrow w\}$$

..The commutativity identities are inherited from (*) above. ..Arrow $\{w\} \longrightarrow X$ did not appear in (*) and hence w is a new point adjoined to X..

To say that "there is a finite subcovering" is the same as saying we may remove the dependency of $U(u)$ on u. ...

There is an arrow $X_U \longrightarrow \{u \leftarrow w\}$ such that

$$\{u\} \longrightarrow X_U \longrightarrow \{u \leftarrow w\}$$

$$\{w\} \longrightarrow X_U \longrightarrow \{u \leftarrow w\}$$

Same as above, the commutativity identities are inherited from (*) above.

4.4 ¿Proper maps and quasi-compact fibres?

Consider $X \times X_U$ and an open covering $\{u\} \longrightarrow X_U \xrightarrow{U(u)} \{u \rightarrow w\}$. The open subset $U(u) \times \bar{U}(u)$ of $X \times X_U$ can be written as the arrow

$$W := X \times X_U \xrightarrow{(U(u) \times U(u))} \{u \rightarrow w\} \times \{u \leftarrow w\} \longrightarrow \{uw \rightarrow uu = wu = ww\} \qquad (+)$$

Consider the open subset $\bigcup_u U(u) \times \bar{U}(u)$ of $X \times X_U$, which does not depend on u. In terms of our calculus, this should correspond to removing the dependency on u in the arrow $(+)$. Now consider closed set $\bar{W} := X \setminus W$ and its projection $\bar{W} \longrightarrow X_U$. By construction $w \notin Im\bar{W}$ and the diagonal is a subset of $\bar{W}$. Hence, $pr\,\bar{W}$ contains X (as a subset of X_U) and does not contain w.

By assumption(?) $pr\,\bar{W}$ is closed; by the definition of X_U this implies that X is a finite union of basic closed subsets in X_U, i.e. is a finite union of the subsets in the covering family $\mathcal{U}$.

5 Speculations and Remarks

5.1 Algebraic topology

Finite preorders may be thought as homotopy types.

There is a diagram chasing characterisation of spheres by Kline theorem, as follows.

The circle $\mathbb{S}^1$ is the unique 2nd countable space that splits into two connected spaces after removing any two points. Kline theorem provides an analogous characterisation exists for spheres of any dimension. An interval I with endpoints $a, b \in I$ is the unique 2nd countable space that splits into two connected components after removing any point except a and b, and remains connected after removing either a, b or both.

Acknowledgments and historical remarks

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Much of this paper was done in St.Petersburg; it wouldn't have been possible without support of family and friends who created an excellent social environment and occasionally accepted an invitation for a walk or a coffee; alas, I made such a poor use of it.

This note is elementary, and it was embarrassing and boring to think or talk about matters so trivial, but luckily I had no obligations for a time.

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