Ein Sommernachtstraum
a construction of a model category
a very early draft

notes by misha gavrilovich

Woody: Can I pour you a draft, Mr. Peterson?
Norm: A little early, isn't it, Woody?

Unless hours were cups of sack, and minutes capons, and clocks the tongues of bawds, and dials the signs of leaping houses, and the blessed sun himself a fair, hot wench in flame-colored taffeta, I see no reason why thou shouldst be so superfluous to demand the time of the day. I wasted time and now doth time waste me.

He draweth out the thread of his verbosity finer than the staple of his argument.

Abstract: In 1967 Quillen introduced *model categories* "to cover in a uniform way" "a large number of arguments [in the different homotopy theories encountered] that were formally similar to well-known ones in algebraic topology". We show the same formalism "covers in a uniform way" a number of arguments in (naive) set theory. We argue that the formalism is curious as it suggests to look at a *homotopy-invariant* variant of Generalised Continuum Hypothesis which has less independence of ZFC, and first appeared in PCF theory independently but with a similar motivation.

Techically, we show how a naive, diagramme chasing homotopy theory approach to set theory leads to a construction of a model category (in the sense of Quillen) modelling some invariants in set theory. These invariants, the covering numbers of PCF theory, appear, in homotopy theory, as values of (minor variations of) the derived functor of cardinality.

notes by misha gavrilovich. Parts of these notes, especially those connecting Quillen's model categories with Shelah's approach to cardinal arithmetic, arose in the course of a joint work with Assaf Hasson, and will eventually appear in the form of a joint paper. These notes are a draft at a very early stage, needlessly verbose and repetitive, and not properly proofread. Any help in proofreading is appreciated. An alternative shorter exposition on 15 pages is also available, at `http://corrigenda.ru/by:gavrilovich/what:work-in-progress/blat.pdf`, and it is somewhat more up-to-date.

Homotopy theory suggests to look at a homotopy-invariant version of Generalised Continuum Hypothesis (hGCH) replacing cardinality by its homotopy invariant approximation, the derived functor, and we observe that ZFC proves strong bounds towards hGCH for many cardinals, either by PCF theory or trivially.

The little of homotopy theory and model category formalism we use, is rendered in a rather explicit computational, combinatorial manner. We speculate about a possibility of a connection to the ergosystems of Gromov.

An alternative, somewhat updated and rather dense 15-page summary exposition is available at `http://corrigenda.ru/by:gavrilovich/what:work-in-progress/blat.pdf`.

Contents

1. Introduction.

This text is a draft of work-in-progress that would not normally be distributed widely; still, we find it makes sense to put it online and ask for help anyone who cares to read it.

this text hasnt yet been proofread even for english.

Please treat this text as an alpha-version of a software....

read it at your own risk.

In particular, I ask that however bad impression this text makes e.g. by bein full of mistakes, this should not inform your opinion on the texts the authors of these paper publish.

....not been proofread yet;

....

1.0.1. Observations and Constructions.

1. We suggest a way to represent (some) statements in set theory (first-order formulae in ZFC) as pictures of labelled coloured graphs (representing commutative diagrammes in a model category with extra data).

2. We suggest that these pictures possess a graphical calculus and a "functional semantics" motivated by homotopy theory, specifically by Quillen's formalism of model categories; in particular, we observe that it seems possible to interpret axioms of a model category as rules to draw arrows in labelled graphs.

3. We construct a model category whose structure describes notions of set theory. for example, for sets A and B, $A \subseteq B$ iff there is a (necessarily unique) arrow/morphism $\{A\}\longrightarrow\{B\}$, the difference $B \setminus A$ is finite iff the (unique) arrow/morphism $\{A\} \xrightarrow{(wc)} \{B\}$ carries both labels w and c (=is both a weak ("homotopy") equivalence and a cofibration), and for sets A and B infinite $\mathrm{card}\, A = \mathrm{card}\, B$ iff the arrow $\{A\} \xrightarrow{(c)} \{B\}$ carries label c(=is a cofibration), and B is countable iff $\varnothing \xrightarrow{(c)} \{B\}$ is a cofibration. An ordinal λ is regular, $\mathrm{cf}\lambda = \lambda$ iff $\lambda \xrightarrow{(wf)} \{\lambda\}$ carries labels w and f. In fact we define (the quasi-poset structure and arrows and) labelling on arrows between *all* sets, and our model category is a full subcategory inheriting the labelling. For example, a singleton $\{A\}$ lies always in the model category but for $A = \lambda$ an ordinal, the object λ is in the model category iff $\mathrm{cf}\lambda = \omega$ or $\mathrm{cf}\lambda = \lambda$ is regular.

4. We observe that some set-theoretic invariants, the covering numbers of PCF theory, thought of as a "better measure of size", are a standard homotopy theory homotopy-invariant approximation to cardinality, as values of (minor variations of a slightly generalised) derived functor. In suggestive notation,[1]

$$\mathrm{cov}(\aleph_\omega, \aleph_1, \aleph_1, 2) = \mathbb{L}_c\mathrm{card}\,(\{\aleph_\omega\}) = \mathbb{L}_c\mathrm{card}\,(2^{\aleph_\omega})$$

5. Generally, we remark that there seem to be a similarity between Shelah's ideology in PCF and the ideology of homotopy theory. We observe that homotopy theory suggests to look at a homotopy-invariant version of Generalised Continuum Hypothesis (hGCH) replacing cardinality by its homotopy invariant approximation, the derived functor, and we observe that ZFC proves strong bounds towards hGCH for many cardinals, either by PCF theory or trivially. This parallels the ideology of PCF.

6. To sum up, we hope that homotopy theory provides a non-trivial analogy between Poincare's continuous and Cantor's infinite.

[1] For reader's convenience, we explain the notation: $\mathrm{cov}(\lambda, \kappa, \theta, \sigma)$ is the *least size of a family $X \subseteq [\lambda]^{<\kappa}$ of subsets of λ of cardinality less than κ, such that every subset of λ of cardinality less than θ, lies in a union of less than σ subsets in X*. The definition of the derived functor is given in §1.4; the letter $\mathbb{L}$ stands for *left derived functor* and the subscript $_c$ indicates passing to cofibrant replacement.

1.0.2. Moreover,

1. These pictures of labelled coloured graphs appear in Quillen's formalisation of a model category.

2. We employ a lot of verbosity to mention explicitly every idea of category theory and model theory we use, and to represent it as a computation, in our example.

3. It seems not inconceivable that any straightforward, computational attempt to connect the calculus of pictures and (basics of) set theory leads to our interpretation of the pictures as statements in set theory (first-order formulae of ZFC) or something different but similar.

4. The little of homotopy theory and model category formalism and yoga we use, is rendered, in our example, in a rather explicit computational, combinatorial manner, as arrow-drawing rules concerning labelled graphs. Our computation arriving at the definition of the covering number, seem to consist entirely of computations with labelled graphs.

1.0.3. Drawbacks:

1. This text is a draft. Unfortunately, we do not fully develop or explain some(many?) of things mentioned above. Exposition here is quite sketchy and repetitive; typesetting and English are deplorable. Read at your own risk. However, I believe the theorems and definitions are correct as stated, and there are no misleading misprints in the main definitions and theorems. I also believe that the construction of the covering numbers as homotopic invariants is explained in detail.

2. The technical content of the paper may be reduced to a couple of longish sentences (accessible to someone familiar with basics of PCF and Quillen's model categories)

3. What we do, appears to be standard, indeed very basic, in homotopy theory, specifically in Quillen's formalism. In other worlds, we try to put set theory into the Procrustean limit of a model category.

4. We do not bother to develop or describe the formalism of pictures as such; rather we attempt to explain it by an example of a calculation in the formalism (leading to the definition of a covering number).

5. We do define a model category, but most of standard tools in homotopy theory, particularly Quillen's formalism, seem to degenerate in our example, e.g. the notion of a path and cylinder objects.

6. The categories we consider, would be usually thought as rather degenerate as categories. They are quasi-partially ordered sets, and every diagramme (that exists) in such a category, commutes. perhaps this is what enables our analysis to be so straightforward.

7. we say nothing about the proof of bounds of PCF.
8. we take a very naive approach to set theory. Though, it should be possible to perform our constructions in a Grothendieck universum, below an inaccessible cardinal, or in an model of ZFC.

1.0.4. Speculations:

Contrary to custom, we mention some highly suppositional speculations, and expound on them in detail. We are tempted to speculate that our approach is not entirely unrelated to the paper [Gromov, 2009+] and particularly the following quote.

> The category/functor modulated structures can not be directly used by ergosystems, e.g. because the morphisms sets between even moderate objects are usually unlistable.
>
> But the ideas of the category theory show that there are certain (often non-obvious) rules for generating proper concepts. (You ergobrain would not function if it had followed the motto: "In *my* theory *I* use whichever definitions *I* like".) The category theory provides a (rough at this stage) hint on a possible nature of such rules.

In the category we use, the morphism sets between any objects are listable (as the category is a quasi-partially ordered set and every diagramme necessarily commutes).

What we do in this paper, is we try to put set theory (textbook) into the Procrustean limit of a model category, and observe that this is easily done by following in a greedy manner a number of tricks appearing common/motivated by category/homotopy theory/ Quillen's model category formalism. Speculations below are on the basis of this observation.

We are tempted to speculate that everything in this paper, can be done by an "ergosystem" "directly using the category/functor modulated structures" indulged into "self-propelled learning". The category/functor modulated structures are simply commutative diagrammes (whose edges are) labelled by three labels (c), (f), (w). It is tempting to further speculate that the "ergosystem" performs an analysis of *syntactic structure* analysis of introductory first chapters on set theory in textbooks, along the lines of [Gromov,2009+,§6]; see §1.3 for more details and speculations.

Further it seems that it may be said that the reasoning employed to construct the model category seem to never raise above the level exhibited in the first and second solutions of von Neumann bird puzzle suggested by [Gromov, 2009, p.37].

We discuss these speculations, and particularly the von Neumann bird puzzle, in §1.3.

There are many obvious problems with this. We presume (very) basic understanding of English syntax and mathematics of books our ergosystem reads. It is unclear, and we have not investigated, whether our strategy gives interesting results in other cases. And perhaps more.

A technical exposition presuming familiarity with homotopy theory. Imagine a (naive student) homotopy theorist trying to understand a 1st introductory set-theoretic chapter in a basic textbook on, e.g., topology (todo: suggest a book). What is the (model) category which helps to understand the chapter? The "in", "being an element" relation is not transitive, and does not lead to a category; the "being a subset" relation is transitive and does lead to a category of sets reflecting the $\subseteq$-relation. An infinite increasing chain of sets $M_0 \subseteq M_1 \subseteq ...$ reminds of a (semi-)simplicial object $X_0 \longrightarrow X_1 \rightrightarrows X_2 ...$ or a chain complex $X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow ...$ in the $\subseteq$-category. So an infinite increasing chain (a common notion), is to be an object; taking its union (a common operation) is to be an arrow. Whether a subset of the union lies in a single element of a chain (a common question about a chain), should also be (a common question about) an arrow. So set the objects to be *arbitrary sets* (equivalently, sets of sets), and an arrow $A \longrightarrow B$ to mean *every element of the antecedent set A is a subset of some element of the succeedant set B*. Adding a single element to a single set, now an arrow $\{\{a, b, ...\}\} \xrightarrow{(wc)} \{\{a, b, ..., \bullet\}\}$, suggest itself as *weak equivalences*; as these arrows have the *left* lifting property with respect to taking the unions of any increasing chain of sets, declare these also as (generating acyclic) *cofibrations* as well as weak equivalences. Further declare an arrow $\{A\} \xrightarrow{(c)} \{B\}$ to be a (generating) *cofibration* iff $\mathrm{card}\, B \leqslant \mathrm{card}\, A + \aleph_0$, to capture the notion of equicardinality of infinite sets as well as that of countability (as $\emptyset \xrightarrow{(c)} \{B\}$ is a cofibration). Consider the full subcategory $QtNaam$ consisting[2] of sets X such that for all $A \subseteq B$, $\mathrm{card}\, B \leqslant \mathrm{card}\, A + \aleph_0$ it holds $\{A\} \cup \{L : L \subseteq B, \mathrm{card}\, L < \mathrm{card}\, B + \aleph_1\} \xrightarrow{(c)} \{B\} \rightthreetimes X \longrightarrow \{\text{universe}\}$ where $\rightthreetimes$ denotes the lifting property. Define the model category cofibrantly generated by these two explicitly defined classes of arrows; The construction gives homotopy theory meaning to concepts of a finite set ($\emptyset \xrightarrow{(wc)} \{X\}$ is a cofibration and a weak equivalence), a countable set ($\emptyset \xrightarrow{(c)} \{X\}$ is a cofibration) and equicardinality (for $A \subseteq B$ infinite, $\{A\} \xrightarrow{(c)} \{B\}$ is a cofibration). The principle that if you wait long enough, every finitely many, or small enough, steps would have happen becomes a lifting property of Quillen's Axiom M1. For example, for an increasing chain $M_0 \subseteq M_1 \subseteq M_2 \subseteq ...$ and a finite X, the common set-theory argument that for a subset X of the union $\cup_i M_i$ of a chain if X is finite than X is a subset of some element M_i of the chain, becomes an instance of Quillen's axiom M1 $\{\} \xrightarrow{(wc)} X \rightthreetimes \{M_0, M_1, ...,\} \xrightarrow{(f)} \{M_0 \cup M_1 \cup ...\}$.

Somewhat less trivially, a homotopy meaning acquires the notion of a covering family ($X \xrightarrow{(wf)} \{A\}$ is a weak equivalence and a fibration) and the covering numbers ($\mathbb{L}_c\mathrm{card}\,(\{A\}) = \mathrm{cov}(\mathrm{card}\, A, \aleph_1, \aleph_1, 2)$, e.g. $\mathbb{L}_c\mathrm{card}\,(\{\aleph_\omega\}) = \mathrm{pp}(\aleph_\omega)$) as values of the left-derived functor-like construction associated with cofibrantly replaced cardinality. The Homotopy version of Generalised Continuum Hypothesis (hGCH) for regular cardinals is trivially true

$$\mathbb{L}_c\mathrm{card}\,(2^A) = \mathbb{L}_c\mathrm{card}\,(\{A\}) = \mathrm{cov}(\mathrm{card}\, A, \aleph_1, \aleph_1, 2) = \mathrm{card}\, A$$

[2] In suggestive notation, $X \in \mathcal{O}bQtNaam$ iff X is isomorphic to $\cup\{X'' : X \longleftarrow X_0 \xrightarrow{(c)} X'' \xleftarrow{(wf)} X' \longrightarrow X\}$

for $\mathrm{cf}\,\mathrm{card}\, A = \mathrm{card}\, A$. Bounds on the covering numbers provided by PCF theory, e.g.

$$\mathbb{L}_c\mathrm{card}\,(2^{\aleph_\omega}) = \mathbb{L}_c\mathrm{card}\,(\{\aleph_\omega\}) < \aleph_{\omega_4}$$

are strong partial results towards the Homotopy version of Generalised Continuum Hypothesis for limit cardinals. In general, it seems possible to speculate that PCF is the part of ZFC homotopy-invariant with respect to the model category structure just defined or its analogues. There seem to be a close connection between the yoga/ideology of PCF and the homotopy theory.

This summary skips a number of things, e.g. speculations about ergosystems.

Shelah's yoga of cofinalities as right measures of size not leading to independence. Artificially/naturality thesis. Concluding the introduction, we find it difficult to refrain from citing a direct comparison to homotopy invariants made by Shelah [Shelah, Cardinal Arithmetic,p.457]; see §5 of [Shelah, Logical Dreams] for an extended discussion.

> *Artificially / naturality thesis.* Probably you will agree that for a polyhedron v (number of vertices), e (number of edges) and f (number of faces) are natural measures, whereas $e + v + f$ is not, but from deeper point of view [the homotopy-invariant Euler characteristic] $v - e + f$ runs deeper than all. In this vain we claim: for λ regular 2^λ is the right measure of [set of all subsets] $P(\lambda)$, and λ^κ is a good measure of $S_{\leqslant\kappa}(\lambda)$. However, the various cofinalities [such as(?) $\mathrm{cf}(S_{\leqslant\aleph_0}(\lambda),\omega)$, $\mathrm{cf}(S_{\leqslant\lambda}(\lambda))$, also $\mathrm{cov}(\lambda,\mu,\theta,\sigma)$, $\mathrm{pp}_\kappa(\lambda)$...(?)] are better measures. λ^κ is an artificial combination of more basic things of two kinds: the function $\lambda\mapsto 2^\lambda$ (λ regular which is easily manipulated) and the various cofinalities we discuss (which are not). For example $\mathrm{pp}(\aleph_\omega) < \aleph_{\omega_4}$ is the right theorem, not $\aleph_\omega^{\aleph_0} < \aleph_{\omega_4} + (2^{\aleph_0})^+$ (not to say: $2^{\aleph_\omega} < \aleph_{\omega_4}$ when $\aleph_\omega$ is strong limit). Also the equivalence of the different definitions which give apparently weak and strong measures, show naturality[.]

Power of ideas/yoga of homotopy theory and tricks of Quillen's model category formalism. We take great pains and great verbosity to explicate category and homotopy theory yoga we use, and explicitly reduce what's used to a number of simple computational tricks using Quillen's formalism of a model category.

We find it extremely important, for the purposes of the current paper, that we can reconstruct all our definitions (and most proofs) by a sequence of simple steps each employing *most trivial*, or even *syntactically short*, guess possible, and *visibly* moving towards our goal by satisfying yet another axiom (of a category or a model category), capturing yet another important notion of set theory in model category theory framework. To explain this better, we find it convenient to employ a fictional simple-minded homotopy theorist character and describe her feelings explicitly whenever useful. However, we switch back to formal 'we' whenever possible.

1.1. Structure of the paper

First we very briefly give some generalities on category theory and Quillen's homotopy theory. Our exposition of category theory owes to that of [Gromov,2009+]. We take liberty to freely use extended quotes from the books and papers of [Gromov,2009+], [Quillen, 1967], [Shelah, Cardinal Arithmetic], and, later, [Dwyer,Spalinski,2005].

In Appendix we reproduce, in their entirety, a few pages from some of the sources we use. We have no hopes to improve upon the expositions of the authors we cite!

The main body of the paper consists of a fairy tale of a fictional homotopy theorist discovering the definition of the covering number; in the story we attempt to mention explicitly every category theory yoga, intuition or trick we use.

A one-page summary of our definitions and results appears near the end(??) of the introduction.

Definitions and theorems. We define (Def.1) a category $StNaam$ whose objects are arbitrary sets, and label its arrows by (c),(f),(w) (Def.2); the category structure on $StNaam$ is equivalent to that of a quasi partially ordered set (rather, class). We then define (Def.3) a full subcategory $QtNaam$ of $StNaam$ such that the labelling induces on $QtNaam$ the structure of Quillen's model category (Lemma 5) where arrows labelled (c) are cofibrations, those labelled (f) are fibrations, and those labelled (w) are weak equivalences. Lemma 5 is followed by a proof clarifying the structure of $QtNaam$. Claim 3, Example 4 characterise some of homotopy notions explicitly in terms of set theory. In §1.4 we introduce a slight generalisation of *left derived functors* and *cofibrant replacement* applicable to function $\mathrm{card}\ : StNaam \dashrightarrow Ordinals$ which is *not* a functor. Lemma 5 identifies the (generalised) left derived functor of cardinality $\mathrm{card}\ : StNaam \dashrightarrow Ordinals$ or $\mathrm{card}\ : QtNaam \dashrightarrow Ordinals$ (after cofibrant replacement) as the covering number $\mathbb{L}_c\mathrm{card}\,(\{\lambda\}) = \mathrm{cov}(\lambda, \aleph_1, \aleph_1, 2)$. Lemma 12 identifies a "finer" cofinality measure $pp(\lambda) = \mathrm{cov}(\lambda, \lambda, \aleph_1, 2)$ (for $\lambda \neq \aleph_\lambda$) as values of (generalised) derived functors. Theorem 13 uses new terminology to list some known PCF bounds on the covering numbers. A discussion following Theorem 13 hints on a connection between PCF and homotopy theory yogas/ideologies. Particularly, we explain why to view Shelah's bound $\mathrm{cov}(\aleph_\omega, \aleph_1, \aleph_1, 2) < \aleph_{\omega_4}$ as a bound towards a Homotopy version of Generalised Continuum Hypothesis. In §1.3 and §1.0.4 we speculate on possibility of a connection to Gromov's ergosystems.

Our use of set theory is intentionally naive; to avoid any problems, it is sufficient to conduct all our constructions in an inaccessible cardinal or a Grothendieck universum.

How (not) to read this paper. ....

A person not familiar with homotopy theory may wish to start by reading the fairy tale, reading background and small print as necessary. ..... A person familiar with homotopy theory may wish to start by reading the one-page summary of our construction in the introduction, and the statement of the Theorem picking up the definitions as necessary. Even such a person may find our notation not entirely unuseful.

1.2. Generalities on Category Theory and Homotopy Theory

Category theory. "Combinatorially[3], a category is a directed graph equipped [with a notion of *composition* of some arrows, such that for every $X \xrightarrow{f} Y \xrightarrow{g} Z$, there is a unique arrow $h : X \longrightarrow Z$ called the composition of f and g, denoted $h = fg$. We allow multiple edges between two vertices as well as loop edges leaving and entering the same vertex.]

"A basic notion in the category theory is that of a commutative diagram. Firstly, a diagram in a category C is a (usually finite) subgraph D in C, i.e. a set of objects and morphisms between them. A diagram is commutative if for every two object $\bullet_1$ and $\bullet_2$ in D and every two chains of composable morphisms, both starting at $\bullet_1$ and terminating at $\bullet_2$, the compositions of the morphisms in the two chains (which are certain morphisms from $\bullet_1$ to $\bullet_2$) are *equal*.

"Importantly, axioms imply that every object comes along with a distinguished identity morphism into itself, say id., the composition with which does not change other morphisms composable with it, i.e. issuing from or terminating at this object. [4]

"The underlying principle of the category theory/language is that the internal structural properties of a mathematical object are fully reflected in the combinatorics of the graph (or rather [the graph with the collection of commutative diagrammes]) of morphisms-arrows around it.

"*Amazingly*, this language, if properly (often non-obviously) developed, does allow a concise uniform description of mathematical structures in a vast variety of cases. [5]

Contract to a point all arrows which are isomorphisms in the category, to obtain a *skeleton* set/graph whose elements/vertices are isomorphism classes of objects of the original category. It is no longer a category as we cannot compose morphisms uniquely. (As an example, consider the category of $Sets$ (with arbitrary functions as morphisms), and composition $\{\bullet_1, \bullet_2\} \xrightarrow{i} \{\bullet_1, \bullet_2, \bullet_3\} \xrightarrow{p} \{\bullet_1, \bullet_2\}$. For i an injection and p a surjection, any *endo*morphism of $\{\bullet_1, \bullet_2\}$ decomposes as $\xrightarrow{i}\xrightarrow{\sigma}\xrightarrow{p}$ for σ an *auto*morphism of the 3-element set.)

What do we mean when we talk about a property P of a *ring* R, a *field* F, *Lie group* L, ... ? We mean (going down to set-theoretic basics) that for every R', F', L' *isomorphic* to R, F, L, resp., as a *ring*, *field*, *Lie group*, resp., the property P (makes sense and) holds of R', F', L' iff it holds of R, F, L. If the property P involves morphisms, functions, etc, we adjust (implicitly) the notion of isomorphic accordingly.

[3]Next few paragraphs are taken from [Gromov,2009+]almost verbatim; even though, we eviscerated the few paragraphs we cite for our purpose. As customary, any changes are marked like [this]. We apologise for not daring to bother the authors to ask permission: our justification is that citing a in-text is, for all practical purposes, the same as inserting a link to the files which are available online.

[4]"This may strike you as a pure pedantry, couldn't one formally introduce such morphisms? The point is that it is not the name but the position of this identity in the set of all selfmorphisms plays the structural role in the category, you can not change this position at will. ([Gromov,2009+])

[5]"(Some mathematicians believe that no branch of mathematics can claim maturity before it is set in a category theoretic or similar framework and some bitterly resent this idea.) ([Gromov,2009+])

In other words, many(any?) mathematical theorem or definition about rings, fields, Lie groups,... are *isomorphism-invariant*, i.e. do not distinguish between isomorphic objects, although constructions involved most certainly do (the value $f(x)$ does not make sense for f a morphism(?) between isomorphism classes of objects). Put in yet another way, a theorem or definition concerns the *skeleton* set/graph of isomorphism classes "with all extra structure which comes by performing constructions in the original category(graph with the collection of commutative diagrammes)." [6]

Quillen's Model category and homotopy theory. Take a category, label some of its arrows with some or all of three letters c, f, w and, mimicking the construction of a free group on letters of an alphabet[7], force all (w)-arrows into isomorphism by adding formal inverses. This means the following; see p.145,§2.2 of [Gelfand-Manin][8] for details. For every (w)-arrow $f : X\longrightarrow Y$, add a new arrow $f^{-1} : Y\longrightarrow X$ in the opposite direction. Further add as an arrow any directed *path* in the new graph. Call two paths are equivalent iff one can be transformed into the other by finitely many operations of replacing two consecutive edges $X\longrightarrow^f Y\longrightarrow^g Z$ by their composition when it is known, i.e. either f and g are arrows in the original category or $g = f^{-1}$ or $f = g^{-1}$. Concatenation of paths provides their composition as arrows. Finally, define a morphism(arrow) in the new category as an equivalence class of paths. There is a functor from the original category to the new one sending an object into itself, and an arrow into (the equivalence class of) itself (as the corresponding path of length 1).

In this way we get a category which we shall call a *homotopy category* of the labelled category. Further insist that the labelling induces no immediate, obvious further structure on the new category. This is somewhat vague; in particular, require the following. Say that an arrow $X\longrightarrow Y$ in the homotopy category *inherits* label (x) (up to isomorphism) iff it decomposes (in the homotopy category) as $X \xrightarrow{(iso)} X_1 \xrightarrow{(x)} Y_1 \xrightarrow{(iso)} Y$ where, as shown, $X \xrightarrow{(iso)} X_1$ and $Y_1 \xrightarrow{(iso)} Y$ are isomorphisms (in the new category), and $X_1 \xrightarrow{(x)} Y_1$ carries label (x) in the old category. Require that each arrow inherits both labels (f) and (c), and isomorphisms in the new category, and only isomorphisms, inherit the (w)-label.

Albeit somewhat trivially, every category C can be labelled in this way: put (wcf) on every isomorphism, and put (fc) on every arrow. For this labelling, the homotopy category HoC of C is C itself.

Quillen noticed that

Amazingly, this language of a labelled category has "sufficient generality to cover in a uniform way the different homotopy theories encountered" if "properly, often non-obviously, developed" to express "a large number of arguments that [are] for-

[6]Though we cannot justify relevance of this remark here, we wish to point out that it is consistent with ZFC that, in a certain sense, any *well-defined construction* $A \mapsto F(A)$ is close to being functorial, see [Hodges-Shelah, 2000](Naturality and Definability) for an exact statement.

[7]"Except for set-theoretic difficulties, the category $S^{-1}C$ exists and may be constructed by "mimic[k]ing the construction of a free group (see Gabriel-Zisman)" [Quillen, §1.12, p.12]

[8]For reader's convenience, we reproduce this page in the appendix.

mally similar to well-known ones in algebraic topology", and that these "homotopy theories" are the (non-obvious) structure the homotopy category *does* inherit.[9]

In fact, Quillen felt it necessary to give/formulate explicitly an "obviously unsatisfactory" "vague definition" of *a homotopy theory* associated to the labelled category; his "vague definition of the homotopy theory associated to a model category, namely [was that of] the [homotopy] category HoC with all extra structure which comes by performing constructions in [the labelled category] C." By a model category, short for "a category of models for a homotopy theory", he means the labelled category; he gives very concise axioms explicitly and we reproduce his axioms in the appendix and also give our graphical rendering of the axioms.[10] [11]

Note the following parallel of homotopy theory and category theory described before. A mathematician(?) thinks of isomorphic objects as equivalent/the same for all practical purposes. A homotopy theorists thinks of objects joined by an (w)-arrows as equivalent/the same for all her/his purposes; s/he calls them *weakly homotopy equivalent*. By adding formal inverses s/he forces (w)-arrows into isomorphisms, and these *(weak) homotopy types* so obtained are the main object of her interest. As was just pointed out, working with isomorphism types only is impossible; one needs to start by picking a representative. Similarly a homotopy theorist notices with disdain that to work in the homotopy category, she has to refer back to the "higher" level category with the labelling: pick representatives there and do diagramme chasing in the labelled category, not the homotopy category.

Particularly, a homotopy theorist feelings towards a non-homotopy invariant notion are akin to those of a mathematician towards a non-isomorphism type invariant notion.

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[9]The extended quote from the preface of [Quillen, Homotopical Algebra], is:

> but there were a large number of arguments that were formally similar to well-known ones algebraic topology, so it was decided to define the notion of a homotopy theory in sufficient generality to cover in a uniform way the different homotopy theories encountered.

[10]For reader familiar with terminology we note that we read weak homotopy equivalence for (w), cofibration for (c), fibration for (f), a cofibration being a weak equivalence, also called acyclic cofibration, for (wc), etc

[11]For reader's convenience, we provide here the extended quotation from [Quillen, Homotopical Algebra,§0,p.0.4]:

> The term "model category" is short for "a category of models for a homotopy theory", where the homotopy theory associated to a model category C is defined to be the homotopy category Ho C with the extra structure defined in 2-3 on this category when C is pointed. The same homotopy theory may have several different models, e.g. ordinary homotopy theory with basepoint is ([10], [15]) the homotopy theory of each of the following model categories: O-connected pointed topological spaces, reduced simplicial sets, and simplicial groups. In section 4 we present an abstract form of this result which asserts that two model categories have the same homotopy theory provided there are a pair of adjoint functors between the categories satisfying certain conditions.
>
> This definition of the homotopy theory associated to a model category is obviously unsatisfactory. In effect, the loop and suspension functors are a kind of primary structure on Ho C, and the families of fibration and cofibration sequences are a kind of secondary structure since they determine the Toda bracket (see 3) and are equivalent to the Toda bracket when Ho C is additive. (This last remark is a result of Alex Heller.) Presumably there is higher order structure ([8J, [17]) on the homotopy category which forms part of the homotopy theory of a model category, but we have not been able to find an inclusive general definition of this structure with the property that this structure is preserved when there are adjoint functors which establish an equivalence of homotopy theories.

The model category we construct, is not pointed.

1.3. Our fairy tale

> if a man bred to the seafaring life ... and if he should take it into his head to philosophize concerning the faculties of the mind, it cannot be doubted, but he would draw his notions from the fabric of the ship, and would find in the mind, sails, masts, rudder, and compass.
>
> ---
>
> As a scholar, meantime, he was trivial, and incapable of labor.

This section describes how our hero, who (unlike the authors of this paper) is a homotopy theorist and who (like the authors of this paper) is not too bright, and knows very little (if at all) set theory could have discovered PCF theory (had Shelah not already discovered it). Whether our hero could have managed to do anything with his discovery remains unclear.

Though not necessarily stupid, our hero is looking for simple - even simplistic - clues, reminiscent of his home world of homotopy theory, in the alien world of set theory he found herself in. Our hero's lack of any knowledge of set theory (and having to start his exploration at some point), is guided by his love for simplicity, puns and syntactic analysis. Or perhaps our hero is an ergosystem device (see [Gromov,2009+]) performing a syntactic structure analysis of set theory books along the lines of 6 of [Gromov, 2009+] with hard-wired model category diagram chasing.

His algorithm is, essentially, greedy. At every step he invents a simple task (indeed, the simplest he can think of) *visibly* advancing his understanding and/or bringing him to the goal of having a model category, e.g. by satisfying yet another axiom, sometimes looking back, correcting and readjusting his advancement as he goes. Most(all?) of tasks and tricks he employs, are quite standard in category and homotopy theory.

Our hero views axioms of a category, a model category as rules to draw arrows, as recipes for action. For example, he has a rule: given arrows $A{\longrightarrow}B$ and $B{\longrightarrow}C$, draw an arrow $A{\longrightarrow}C$. Axiom M1 of model categories (explained below) is for him the following rule : given arrows $A{\longrightarrow}X$, $B{\longrightarrow}Y$, an $A \xrightarrow{(wc)} B$ carrying label (wc), and an arrow $X \xrightarrow{(f)} Y$ carrying label (f), draw the upward "diagonal" arrow $B{\longrightarrow}X$ (and the downward "diagonal" arrow $A{\longrightarrow}Y$ provided simply by composition). In the same way he views the definitions he makes. (This "functional semantics" of axioms of a model category works well for categories being quasi posets where every diagramme is commutative: in this case, e.g., our hero does not have to keep in (his feeble) mind that $A{\longrightarrow}C$ is the composition of arrows $A{\longrightarrow}B$ and $B{\longrightarrow}C$, and not some other arrow.)

Our hero's thinking is very limited. We say he *understands* a concept, he finds a concept *amazing* if he can represent it as a picture of a labelled commutative diagramme common in homotopy theory; our hero likes to make himself amazed. He is not very good with quantifiers and logic, and perhaps he has had no chance to use much of either during his journey we describe.

Our hero's level of thinking seem to never raise above the level exhibited in the first and second solutions of von Neumann bird puzzle suggested by [Gromov, 2009, p.37], and we do wish to speculate that he seems to think in a way 'a baby/animal ergobrain would do it'. No less tempting is it to speculate that he performs the "ergosystem" analysis described in §6 of [Gromov,2009+]. [12]

We finish the description of our hero by a remark that our hero deals with a rather degenerate category where there is at most one morphism between any two objects. Curiously, this makes the following Gromov's argument not apply to this case: "The category/functor modulated structures can not be directly used by ergosystems, e.g. because the morphisms sets between even moderate objects are usually unlistable."

We switch back to the formal 'we' whenever possible.

1.3.1. Pictures of commutative diagrammes as sentences in ZFC

Define a *coloured labelled sentence* as the following data:

(a) a sequence of *colours* $r_1, ..., r_n$ and quantifiers $Q_1, ..., Q_n$

(b) a sequence of labels, e.g. $(c), (w), (f), ...$, and special labels ►, ◄ for free variables, and labels $\because, \therefore$ for "because" and "therefore".

(c) a directed graph whose edges are coloured in colours $r_1, .., r_n$ equipped with a labelling assigning each edge some (or none) of the labels $(c), (w), (f), ...$, ►, ◄..

An important convention is that dashed arrows are always existentially quantified.

Given a category C with a labelling on morphisms, we interpret a coloured labelled sentence as a first-order formula in the category (TODO: Definition) representing the corresponding commutative diagramme with arrows quantified over according to the list of colours and quantifiers.

In the "functional semantics", we interpret signs $\because, \therefore$ as "given a commutative diagramme as shown, add the label following the special signs $\because, \therefore$". A commutative diagramme representing an $\forall\exists$-formula, reads as a rule "give the commutative diagramme of solid arrows labelled as shown, add dashed arrows and their labels".

For reader's convenience, we quote a couple of paragraphs from an extended discussion of [Gromov,2009+] p.37, §1.8.

> A bird flies back and forth between two trains travelling toward each other at 40 and 60 km/h, respectively. The initial distance between the trains is 100 km and the bird flies 100 km/h. What is the distance covered by the bird before the trains meet?
>
> [Solution] 1. Imagine, your English is poor and you missed all words except for the numbers: 40, 60, 100, 100. Which number would you give in response? Obvi- ously, the best bet is 100, even if you miss the third hundred=40+60. [Solution] 2. There was only one *distance*-number in the question = 100 km; therefore this is likely to be the *distance*-solution. (This remains correct even if the distance was 150 km.) ([Gromov,2009+])
>
> These 1 and 2 are how a baby/animal ergobrain would do it; you need ≈ 0.3 sec. in either case. And it is not as silly as it may seem to a mathematician: if 100 km stands for a whiff of a predator, you have no time for computing the total length of its expected jumps.

12

1.3.2. The journey begins

Our category.

The lifting property. Commutative diagrammes as ∀∃-formulae. Quillen identified *the lifting property* as often used in algebraic topology and "any homotopy theory encountered", to concisely define basic notions and tools, and moreover, proving things often reduces to diagramme chasing using the lifting property of various arrows. Below Fig.0($\rightthreetimes$) we give some examples of properties obtained simply by doing the lifting property wrt a *fixed* morphism (in items (e) and (f) this needs to be done twice).

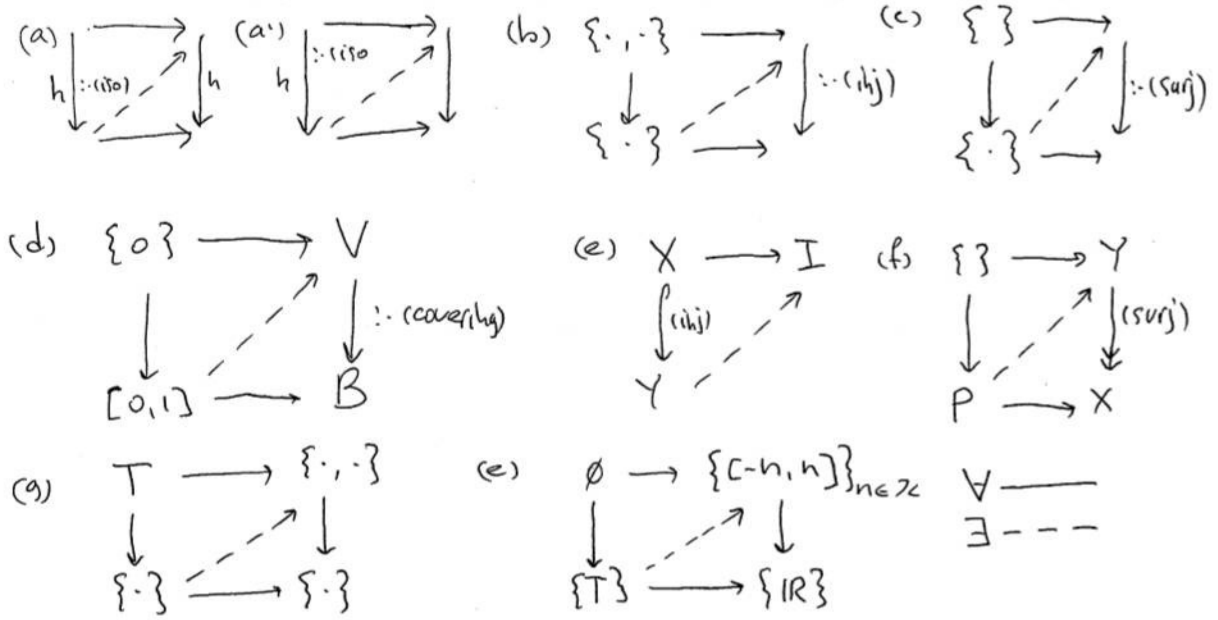


Fig.0($\rightthreetimes$). Read a diagramme as an ∀∃-formula with parameters: for the arrows labelled ► or ◄, the following property holds: "for each commutative diagramme of solid arrows carrying labels as shown, there exists dashed arrows carrying labels as shown, making the diagramme of all the arrows commutative" (a) Isomorphism. In a category an arrow is an isomorphism iff it has (either left or right) lifting property wrt itself (and consequently (a') any other arrow). (b) an arrow is injective iff it has the right lifting property wrt $\{\cdot,\cdot\}\longrightarrow\{\cdot\}$ whenever(=in most categories where) the latter notation/arrow makes sense. (c) an arrow is surjective iff it has left lifting property wrt $\{\}\longrightarrow\{\cdot\}$ whenever(=in most categories where) the latter notation/arrow makes sense. (d) Let $I=[0,1]$ be the unit interval of the real line, and let $0\in[0,1]$ be its end point; the morphism $V\longrightarrow B$ is a *covering* of topological spaces iff there is always exists a unique lifting arrow $I\longrightarrow V$ making the diagramme commute. (e) an object I is injective iff for each injective arrow $X\longrightarrow Y$ and any arrow $X\longrightarrow I$, there exists an arrow $Y\longrightarrow I$. (f) dually, an object P is a projective object, e.g. a free module, iff for each surjective arrow $X\longleftarrow Y$ and an arrow $X\longleftarrow P$, there exists an arrow $Y\longleftarrow P$. (g) a topological space T is connected iff $T\longrightarrow\{\cdot\}$ has the right lifting property wrt to $\{\cdot,\cdot\}\longrightarrow\{\cdot\}$ in the category of topological spaces) (e) this diagram shall become clear later; it says that a topological space T is compact iff every continuous map $T\longrightarrow\mathbb{R}$ factors via an interval $[-n,n]$ for some $n\in\mathbb{Z}$.

In pictures, an arrow $A\longrightarrow B$ has the (left) lifting property wrt $X\longrightarrow Y$, denoted $A\longrightarrow B \rightthreetimes X\longrightarrow Y$, iff

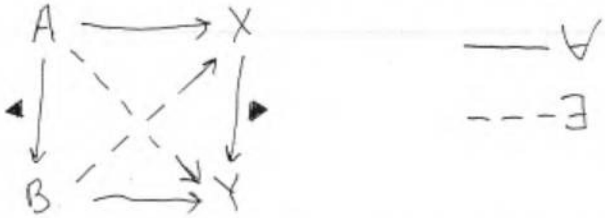


Fig.1(⋌) The diagramme reads: for every pair of horizontal arrows $A \longrightarrow X$ and $B \longrightarrow Y$ making the diagramme commutative, there exists the *lifting* arrow in the diagonal direction $B \longrightarrow X$ and the *composition* arrow $A \longrightarrow Y$ making the diagramme commutative. The *composition* arrow $A \longrightarrow Y$ exists in any category simply by composition. We describe this situation by saying that the arrow $B \longrightarrow Y$ *lifts to* $B \longrightarrow X$.

The journey begins. Our hero suddenly finds himself in the alien world of set theory. Everything is new to him; he is desperate for signs of a familiar homotopy world, for any signs of Quillen's model category formalism.

Bewildered and lost, our hero - who has a strong belief in the all embracing descriptive power of the lifting property - desperately looks for its familiar pattern within the alien world of set theory. At this stage, as a new arrival, his abilities are limited to syntactic analysis: he looks for any syntactic construction occurring (more than once) and resembling the lifting property. He runs several times into arguments similar to the following argument:

If you represent a set as a union of an increasing chain $M_0 \subseteq M_1 \subseteq M_2 \subseteq ...$ (i.e. choose a filtration of a set) and take finitely many elements of the union $\cup_i M_i$, then you infer they all lie in a single element of the chain. (todo: can one get rid of set theory terminology completely, e.g. does this work if you replace 'taking a union' with 'performing infinitely many steps..')

Our chap draws a diagramme:

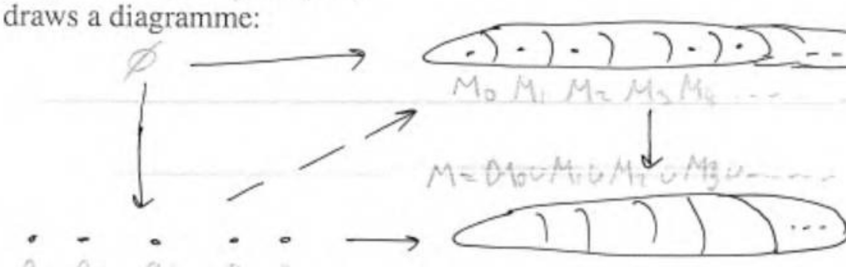


Fig.2(⋌) Our hero considers this diagramme to informally represents the set-theoretic argument described above, and at the same time resembles a lifting property diagramme.

Our hero is happy to observe that this diagramme looks like a lifting property diagramme, a first sign of a familiar word: vertices/arrows on the left and right sides have special properties (being finite or taking the union of a chain) while horizontal arrows seem to be rather arbitrary (lie in). He proceeds to build up on this observation to make the diagramme to represent a 'honest' formal *commutative diagramme* in a *category*. So far he knows that the bottom arrow $a_1, .., a_k \longrightarrow \cup M_i$ is supposed to mean that every a_i lies in the union $\cup M_i$; the right vertical arrow is supposed to mean taking the union, and the diagonal arrow is supposed to mean that all the a_i

lie in a single element of the chain. Standard set-theory notation suggests to think of a chain as a set of its elements. But the right arrow becomes $\{M_i\}_i \longrightarrow \cup M_i$ and it's slightly strange e.g. when the chain consist of a single element, taking union does nothing and the arrow seemingly does something. So think of the union as a singleton set with the union being its unique element. The $\in$-relation is not transitive and seems not to fit into category-theoretic thinking, unlike the relation $\subseteq$. This leaves two options for the bottom arrow $a_1, .., a_n \longrightarrow \cup M_i$: either $\{\{a_1\}, ..., \{a_n\}\} \longrightarrow \{\cup M_i\}$ or $\{\{a_1, ..., a_n\}\} \longrightarrow \{\cup M_i\}$.

The diagramme now becomes

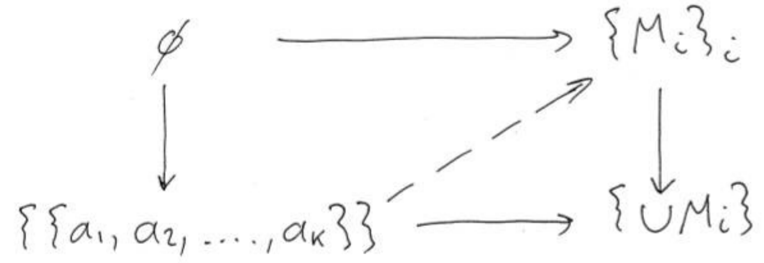


Fig.3(⋌) This is a lifting property diagramme in the category where objects are arbitrary sets, and $X \longrightarrow Y$ iff *every element of the antecedent set X is a subset of some element of the succeedant set Y*.

In the diagramme, an arrow $X \longrightarrow Y$ reads *every element of the antecedent set is a subset of some(or every) element of the succeedant set*. However, in a category there must be an identity arrow $\{M_i\}_i \longrightarrow \{M_i\}_i$, and this excludes the second "every" interpretation (because every M_i is a subset of some M_i but not necessarily a subset of each M_i).

In a category, an essential part of structure is the notion of composition of arrows; how can we compose these arrows ? An easy check establishes transitivity: if $X \longrightarrow Y$ and $Y \longrightarrow Z$, then $X \longrightarrow Z$. With this interpretation of an arrow, there is a unique way to draw an arrow from a set X to a set Z; so define the unique arrow $X \longrightarrow Z$ to be the composition of the unique arrows $X \longrightarrow Y$ and $Y \longrightarrow Z$.

So our homotopy theorists has found himself a category where the above diagram describes an actual lifting property. Moreover, any diagramme in the category that exists, commutes, and this makes the journey of our hero so much easier!

Following his gut feelings, our hero tries to investigate lifting properties of some simple arrows in the new category, and joyfully observes, that combining simple set theoretic properties, with lifting diagrammes quite often results in other set theoretic properties he has already run into.

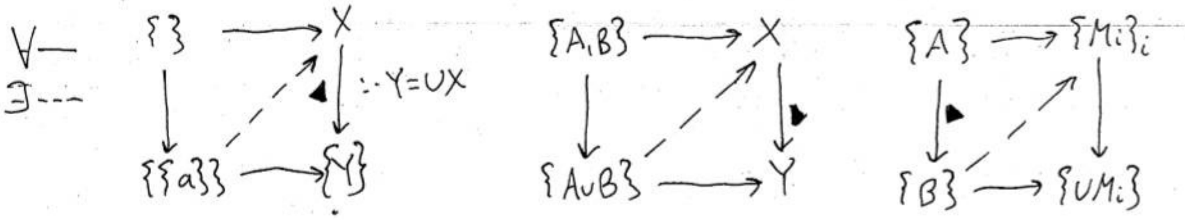


Fig.4($\rightthreetimes$)

($\bigcup X$) $\emptyset \longrightarrow \{\{a\}\} \rightthreetimes X \longrightarrow \{Y\}$ for all a iff $Y = \cup X := \bigcup_{x \in X} x$

(lim) Y is a directed limit/union of X iff $\{A, B\} \longrightarrow \{A \cup B\} \rightthreetimes X \longrightarrow Y$ for all A, B

($fini$) for $A \subseteq B$, $B \setminus A$ is finite iff $\{A\} \longrightarrow \{B\} \rightthreetimes \{M_i\}_{i \in I} \longrightarrow \{\cup_{i \in I} M_i\}$ for any increasing chain M_i (i.e. $M_i \subseteq M_j$ or $M_j \subseteq M_i$ for every i, j)

Of course, our hero has to use set theoretic concepts (such as linearly ordered sets, or - given a set A - form the set $\{A\}$, but other than that he is quite satisfied. The world of set theory is no longer totally alien to him as he realises that in the new world objects still form a category and people apply the lifting property to define new concepts from old ones. So he is happy to define:

Definition 1 (category StNaam) *Objects $ObStNaam$ are arbitrary sets. In $StNaam$, morphisms are unique when they exist, and there is a morphism $X \longrightarrow Y$ iff $\forall x \in X \exists y \in Y (x \subseteq y)$. Composition of morphisms is defined by requiring that that all diagrammes that exist, commute. (In fact this follows from the uniqueness of morphisms)*

1.3.3. Constructing a model category labelling.

With these tools in hand, the labelling of arrows can now proceed almost axiomatically a task on which our homotopy theorist now embarks. What he is trying to do now, is a well-known construction of a *cofibrantly generated* model structure starting by defining the classes of *generating cofibrations* and *generating cofibrations which are also weak equivalences*.

Now we embark to do the labelling. We introduce the Axioms of a model category when (and if) we need them.

Two lifting properties. Axiom M1. Quillen's axiom M1 requires that any (c)-arrow should have the lifting property w.r.t any (wf)-arrow, and dually (interchanging (f) and (c) labels and reversing the direction of every arrow), that any (wc)-arrow should have the lifting property wrt any (f)-arrow.

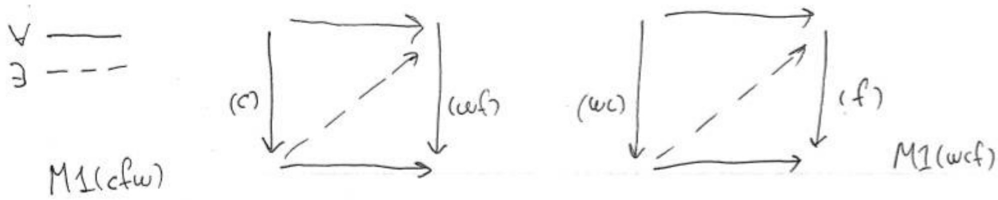


Fig.M1($\measuredangle$). For classes (a) and (b) of arrows, let $(a) \measuredangle (b)$ denote that for every $a \in (a)$ and $b \in (b)$, it holds $a \measuredangle b$. Then these diagrammes read : $(c) \measuredangle (wf)$ and $(wc) \measuredangle (f)$ where $(c), (wf), (wc), (f)$ denote the classes of arrows labelled $(c), (wf), (wc)$ and (f), resp.

Labels (wc) and (f). Dual's dual argument. Our hero first ponders which arrows to label (w). In algebraic topology (and in Quillen's model categories) these correspond to morphisms called *weak homotopy equivalences*, and indeed are thought of as some sort of equivalence, in the sense, as explained above, that our homotopy theorist is only interested in properties preserved under this weak equivalence.

What is the simplest thing to do in set theory that never matters, particularly in a theorem or a definition ?

Adding a single element to a single set seems something basic and common, and seldom, if ever, matters in set theory; this idea is short enough to be stumbled upon by for our hero, and its connection to our hero's inner world of model categories is explicit enough to be recognised as such immediately. In our notation this corresponds to the arrow $\{\{a, b, ...\}\} \longrightarrow \{\{a, b, ..., \bullet\}\}$. So it is reasonable to think of these arrows as some sort of equivalences, and therefore put (w) on these.

But we have already seen somewhat similar arrows in the lifting property Fig. 1-3($\measuredangle$), Fig. 4($\bigcup$). Our hero is aware that the arrows $\{\{a, b, ...\}\} \longrightarrow \{\{a, b, ..., \bullet\}\}$ appear on the left side of the only lifting property diagramme he has seen so far! He knows that in Quillen's axioms, arrows appearing on the left side of a lifting property, are always labelled (c) (for example, in the both diagrammes of the Quillen's axiom M1 the arrows on the left are labelled (c)). That makes him/her add label (c) onto these arrows.

Now that we have put label (wc) on some arrows, we might put (f) on all arrows which have the right lifting property wrt to all arrows labelled (wc).

Now put label (wc) on all arrows which has the left lifting argument wrt to every arrow already labelled (f). We might continue and try to put (f) on some more arrows but our hero would not. The following claim is an innate knowledge to him/her, but we do the proof.

Claim 1 ((wcf)) *The labelling defined satisfies M1(wcf). If we add labels (wc) or (f) to any arrow that does not have them already, M1(wcf) is no longer satisfied. (Axiom M6a) An arrow is labelled (f) iff it has the left lifting property wrt any arrow labelled (wc). An arrow is labelled (wc) iff it has the right lifting property wrt any arrow labelled (f). All isomorphisms are labelled (wcf).*

Proof. The argument is the same as the one showing that, for a vector space V, the duality $V^* = V^{***}$ always holds. Let $(wc)_0$ be the class of arrows labelled (wc) at the first step, i.e. $(wc)_0 := \{\{A\} \longrightarrow \{A \cup \{a\}\} : A \text{ is a set}\}$. Let $(f) := ((wc)_0)_{\measuredangle}$ be the class of all arrows f such that

$(wc)_0 \curlywedge f$, and $(wc) := {}^{\curlywedge}(f) = {}^{\curlywedge}(((wc)_0)_{\curlywedge})$. Continue $(f)_2 := (wc)_{\curlywedge}$. Now do the $V^* = V^{***}$-argument: as $(wc)_0 \curlywedge (f)$, we get $(wc)_0 \subseteq (wc)$ and infer by duality $(f) \supseteq (f)_2$. By construction $(wc) \curlywedge (f)$, i.e. $(f)_2 := (wc)_{\curlywedge} \supseteq (f)$. We are done. By definition, if we add an (wc)- or (f)-arrow while preserving M1(wcf), it must lie in ${}^{\curlywedge}(f)$ and $(wc)_{\curlywedge}$, resp. But we added these arrows already; this proves the second claim. The third and forth claims follow from the second one. The last claim holds as an isomorphism has both left and right lifting property wrt any arrow.

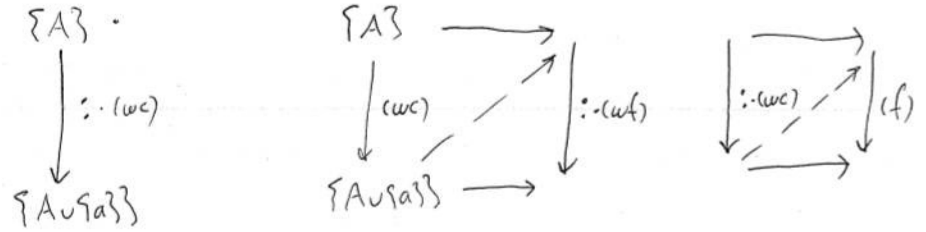


Fig. M1(wc-dual). These three diagrammes represent the three steps taken in the labelling defining (wc)- and (f)-arrows.

Meantime, our hero feels happy : he grokkes [13] the concept of a *finite* set — that's when $\{\emptyset\} \xrightarrow{(wc)} \{A\}$.

Labels (c) and (wf). Now we see whether we can extend the labelling to satisfy M1(cwf). In a model category, we need to have enough enough labels (c) and (f), so that, when factored out by (w)-arrows, in the homotopy category every arrow inherits both (f) and (c) labels. (We said this in the introduction and this is implied by Quillen's Axiom M2). So far this is not true: no arrow in the homotopy category inherits a (c)-label as we have not put any yet; all (wc)-arrows get contracted. So we have to continue and go back to our hero...

Meantime, our hero heard a lot of talk about sets of equal cardinality, and put (c) on all arrows $\{A\} \longrightarrow \{B\}$, card $A =$ card B trying to get to grips with this. Alas, belatedly he noticed that his (wc)-arrows $\{\{a, .., b\}\} \longrightarrow \{\{a, .., b, c\}\}$ between singletons of finite sets, do not fall in this class: indeed, for *finite* sets A,B, card $A =$ card B iff $A = B$. Or perhaps he has not give it a thought (he tends not to think much), being preoccupied trying to understand what *countability* is—apparently a very useful

[13] The word *to grok* is taken from hacker's slang. Here is the definition: from [Jargon File (4.4.4, 14 Aug 2003)]:

grok /grok/, /grohk/, vt.

[common; from the novel Stranger in a Strange Land, by Robert A. Heinlein, where it is a Martian word meaning literally 'to drink' and metaphorically 'to be one with'] The emphatic form is grok in fullness. 1. To understand. Connotes intimate and exhaustive knowledge. When you claim to 'grok' some knowledge or technique, you are asserting that you have not merely learned it in a detached instrumental way but that it has become part of you, part of your identity. For example, to say that you "know" LISP is simply to assert that you can code in it if necessary – but to say you "grok" LISP is to claim that you have deeply entered the world-view and spirit of the language, with the implication that it has transformed your view of programming. Contrast zen, which is similar supernal understanding experienced as a single brief flash. See also glark.

2. Used of programs, may connote merely sufficient understanding. "Almost all C compilers grok the void type these days."

concept! His (only) way of understanding a concept, is, of course, is by putting some labels, and his freedom is severely limited now – he has only about (c)-labels to put left. So he puts (c) label on all arrows $\{\emptyset\}\longrightarrow\{A\}$ for A countable.

So we label these arrows (c) and extend the labelling as we just did above. In symbols,

$$(c)_0 := \{\{A\}\longrightarrow\{B\} :\ A \subseteq B \text{ are sets, } \mathrm{card}\, B = \mathrm{card}\, A\} \cup \{\emptyset\longrightarrow\{B\} :\ B \text{ a set, } \mathrm{card}\, B \leqslant \aleph_0\},$$

and $(wf) := ((c)_0)_{\rightthreetimes}$, and finally $(c) := {}^{\rightthreetimes}(wf) = {}^{\rightthreetimes}(((c)_0)_{\rightthreetimes})$. Evidently $(wc)_0 \subseteq (c)_0$ and therefore by $\rightthreetimes$-duality $(f) \supseteq (wf)$ and $(wc) \subseteq (c)$.

Note that an arrow $\{A\}\longrightarrow\{B\}$ acquires the label (c) iff $\mathrm{card}\, B \leqslant \mathrm{card}\, A + \aleph_0$ (and $A \subseteq B$). This follows from the observation that $\{\}\longrightarrow\{B\} \rightthreetimes X\longrightarrow Y$ implies $\{A\}\longrightarrow\{B\} \rightthreetimes X\longrightarrow Y$ for any $A \subseteq B$ and any $X\longrightarrow Y$.

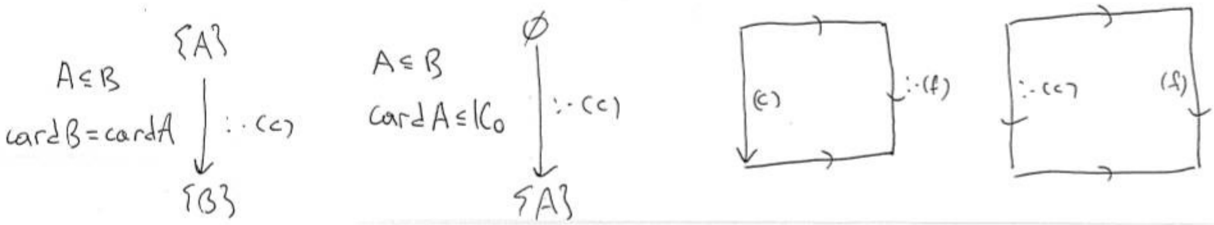


Fig. M1(c-dual). These three diagrammes represent the three steps taken in the labelling defining (wc)- and (f)-arrows.

We state some properties of labelling just defined.

Claim 2 ((cfw)) *The labelling defined satisfies M1(cfw). If we add labels (c) or (wf) to any arrow that does not have them already, M1(cfw) is no longer satisfied. (Axiom M6b) An arrow is labelled (c) iff it has the left lifting property wrt any arrow labelled (wf). An arrow is labelled (wf) iff it has the right lifting property wrt any arrow labelled (c). All isomorphisms are labelled (wcf).*

Labels (w). Axiom M2: (w)=(wc)(wf). The composition of two (w)-arrows is required to be an (w)-arrow; e.g., this is implied by Axiom M2(2-out-of-3). So we put labels (w) on the arrows that are compositions of a (wc)-arrow and a (wf)-arrow.

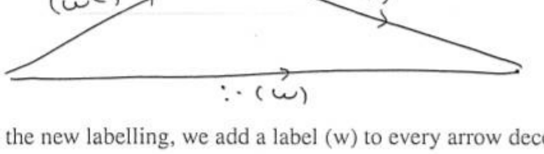


Fig.M2(w) In the new labelling, we add a label (w) to every arrow decomposing (wc)(wf).

We need to check that at this step we introduced no new (wf)- and (wc)-arrows. We know that an arrow carries label (wc) iff it has the left lifting property wrt any (f)-arrow. As $(wc) = {}^{\rightthreetimes}(f)$ and $(wf) = (c)^{\rightthreetimes}$ by construction, it is enough to show that $(wc)(wf) \cap (c) \rightthreetimes (f)$ and $(c) \rightthreetimes (wc)(wf) \cap (f)$. The figure 5(w) depicts the proof.

M3. Fibrations are stable under composition, base change, and any isomorphism is a fibration.

~~Cofibrations are stable under composition, co-base change, and~~ any isomorphism is a cofibration.

Fig. 1: Axiom M3 as it appears in the book of Quillen (1967).

But let us go back to our hero. Can s/he do this proof ; does s/he understand the formulae $(wc) = {}^{\leftthreetimes}(f)$ and $(wf) = (c)^{\leftthreetimes}$? Yes, but only in a rather practical sense: to conclude that an arrow in a commutative diagramme is labelled (wc) or (wf), add a new lifting square including that arrow, and prove the lifting property for the new square.

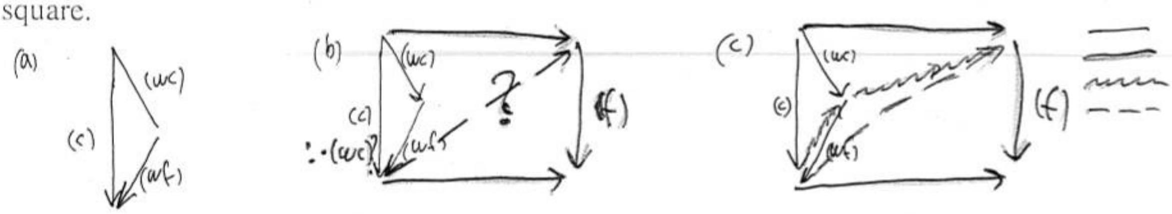


Fig.5(w) (a) first draw the diagramme how s/he added (w)-labels (b) to use $(wc) = {}^{\leftthreetimes}(f)$, draw an (f)-arrow and the lifting square (c) find the lifting arrow in the lifting square by using all available lifting properties.

1.3.4. Axioms M6, M1, M3, and M4 in $StNaam$

Axiom M6. $(f) = {}^{\leftthreetimes}(wc), (c) = (wf)^{\leftthreetimes}, (w) = (f)^{\leftthreetimes\leftthreetimes}(c)$, *and* $(wf) = {}^{\leftthreetimes}(c), (wc) = (f)^{\leftthreetimes}, (w) = (wc)(wf)$. The axiom M6 states that $(f) = {}^{\leftthreetimes}(wc), (c) = (wf)^{\leftthreetimes}, (w) = (f)^{\leftthreetimes\leftthreetimes}(c)$. To our hero it is clear that the axiom M6 is satisfied by construction, and that "it is clear that M6 implies M1, M3, and M4". (Quillen,§5.2, Remark 1 following the introduction of M6). The axiom M6 has been verified explicitly above. Let us now verify the axiom M3; we skip the others.

Axiom M3. Now our hero embarks to check Axiom M3. He repeatedly uses the formulae $(f) = (wc)^{\leftthreetimes}$ and $(c) = (wf)^{\leftthreetimes}$.

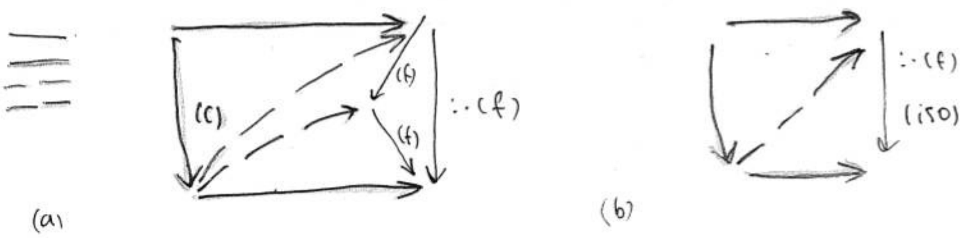


Figure M3f. (a) (f)-arrows are stable under composition (c) Any isomorphism is labelled (f).

Figure M3c. (a) (c)-arrows are stable under composition (c) Any isomorphism is labelled (c).

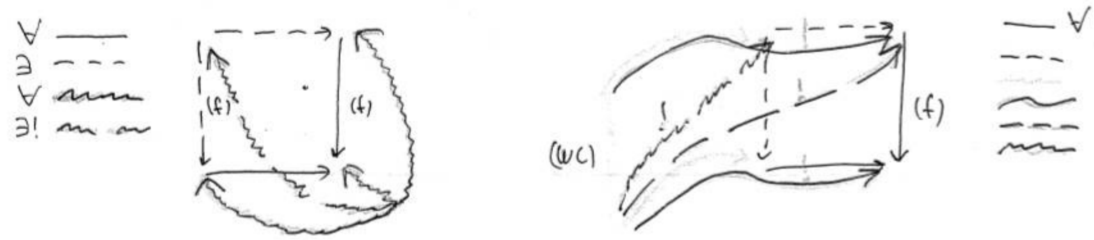


Figure M3f. (b1) the vertical arrow on the left side, is the base change, i.e. pull back, of the arrow on the right side (b2) (f)-arrows are stable under base change

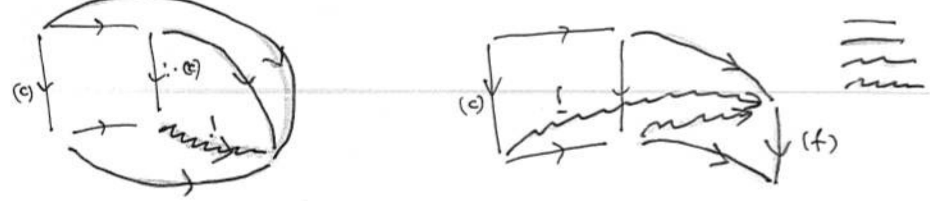


Figure M3c. (b1) the vertical arrow on the right side, is the cobase change,i.e. push out, of the arrow on the left side (b2) (c)-arrows are stable under co-base change

Summary. Self-consistency of the labelling. So far, we put a number of labels (w),(c),(f) on the arrows of the category $StNaam$. We have done in three steps possibly interfering with each other and we need to check that everything's ok. (**sorry, i seem unable to spell things precisely at the moment. what follows is utterly unreadable and useless...**). Let us check that the labelling satisfies Quillen's axioms M1 and M6. This follows from the construction.

First notice that $(wc)_0 \subseteq (c)_0$ and consequently $(wf)_0 \subseteq (f)_0$. Now forget for a moment the last step of adding (w)-labels; then every (w)-label on a (wc)- or (wf)-arrow was put as a part of a (wc)- or (wf)-label on the same step. Consider a (wc)-arrow and show that (wc)-label was put on it at step 1 $((wc) \ \rightthreetimes\ (f))$: otherwise both (w)- and (c)-labels were asquired on the step $2((c) \ \rightthreetimes\ (wf))$, i.e. the arrow acquired (wcf)-label at the 2nd step; in particular, this implies it has the left lifting property wrt itself, and is then an isomorphism. Finally this implies that the arrow was put (wcf)-label also at the step $1((wc) \ \rightthreetimes\ (f))$. Analogously we show that the (wf)-label was put on an (wf)-arrow always at the step $2((c) \ \rightthreetimes\ (wf))$. (**This explanation seems utterly unclear and useless..**)

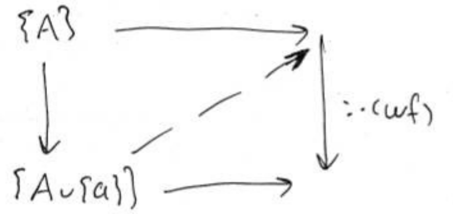
1.3.5. Homotopy/diagrammatic meaning of set-theoretic concepts

Our hero recounts his trophies: with great feeling of fulfilment he observes that he *grokkes finiteness* ($\{\} \xrightarrow{(wc)} \{A\}$), *countability* $\{\} \xrightarrow{(c)} \{B\}$, and *equicardinality* $\{A\} \xrightarrow{(c)} \{B\}$ for infinite sets $A \subseteq B$. He has an uneasy feeling of understanding of what's *regular* for an ordinal ($\lambda \xrightarrow{(wf)} \{\lambda\}$) (but doesn't know what an ordinal is.) He also has *an idea about* unions ($Y = \cup X := \cup_{x\in X} x$ iff $\{\}\longrightarrow\{\{\bullet\}\} \not\perp X\longrightarrow\{Y\}$) and directed systems of sets (X is a directed set iff $\{A, B\}\longrightarrow\{A\cup B\} \not\perp \{X\}\longrightarrow\{universe\}$ for all A, B)

He sees the typical argument about long chains as applications of Quillen's Axiom M1, e.g. $\{\} \xrightarrow{(wc)} \{\{a_1, .., a_k\}\} \not\perp \{M_i\}_i \xrightarrow{(f)} \{\cup_i M_i\}$—every finitely many elements of the union of an increasing chain are contained in one of its elements—, and $\{M_i\} \xrightarrow{(c)} \{B\} \not\perp \{M_i\}_i \xrightarrow{(wf)} \{\cup_i M_i\}$ (if you split a set of sufficiently inaccessible cardinality into a chain, each small set is contained in an element of the chain). [14]

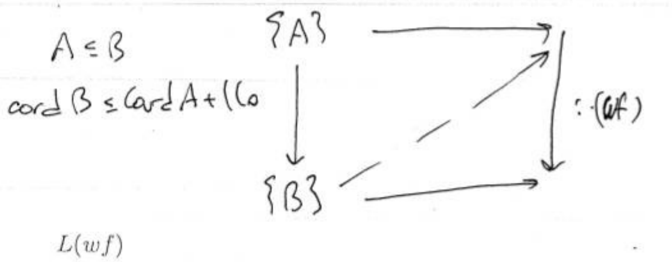
While Alice is still feeling amazed, gratified or fideistic, we summarise the new definitions and spell out their meanings in set theory.

Definition 2 ("model category" "cofibrantly generated" labelling) *(f) an arrow $X\longrightarrow Y$ is labelled (f) iff it has the right lifting property w.r.to any arrow $\{A\}\longrightarrow\{A\cup\{a\}\}$*



$L(f)$

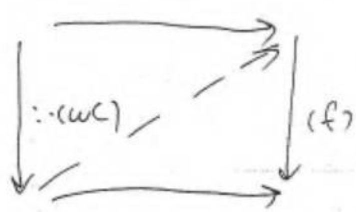
(wf) an arrow $X\longrightarrow Y$ is labelled (wf) iff it has the right lifting property w.r.to $\{A\}\longrightarrow\{B\}$ for $\mathrm{card}\, B \leqslant \mathrm{card}\, A + \aleph_0$



$L(wf)$

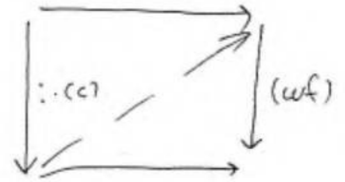
(wc) an arrow $A\longrightarrow B$ is labelled (wc) iff it has the left lifting property w.r.to any arrow labelled (f)

[14]TODO: use interest/amazement to parallel gromov's terminology, e.g." Thus various degrees of interest/amusement and surprise/amazement are rough indicators of a relation of the structure of your egobrain to the in- formational structure presented in the flow of signals. (If you are bored by a sequence of letters, this may be because you are not familiar with the language or, on the contrary, if you were obliged to memorize this sequence as a child at your school lessons.) "



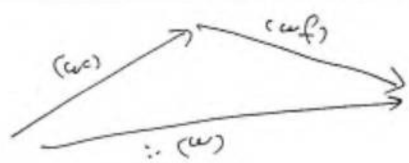
$L(wc)$

(c) an arrow $A \longrightarrow B$ is labelled (c) iff it has the left lifting property w.r.to any arrow labelled (wf)



$L(c)$

(w) an arrow $A \longrightarrow Y$ is labelled (w) iff it can be decomposed as $A \longrightarrow B = X \longrightarrow Y$ where the first arrow is labelled (wc) and the second one is labelled (wf)



$L(w)$

Claim 3 (labels (c),(f),(w),(wc),(wf) in ZFC) *In words these definitions mean:*

- *(f) an arrow $X \longrightarrow Y$ is labelled (f) iff for every $x \in X \cup \{\{\}\}$, $y \in Y$ and a finite subset $y_1, .., y_n \in y$ there exists $x' \in X$ such that $(x \cap y) \cup \{y_1, ..., y_n\} \subseteq x'$.*
- *(wf) an arrow $X \longrightarrow Y$ is labelled (wf) iff for every $x \in X \cup \{\{\}\}$, $y \in Y$ and a subset $y' \subseteq y$ of y such that $\operatorname{card} y' \leqslant \operatorname{card}(x \cap y) + \aleph_0$, there exists $x' \in X$ such that $y' \subseteq x'$.*
- *(wc) an arrow $A \longrightarrow B$ is labelled (wc) iff every $b \in B$ is contained, up to finitely many elements, in an $a \in A \cup \{\{\}\}$, $b \setminus a$ is finite.*
- *(c) an arrow $A \longrightarrow B$ is labelled (c) iff every element of B of cardinality λ is connected to an element of $A \cup \{\{\}\}$ by a finite $\geqslant\lambda$-connected chain of elements of B; a finite $\geqslant\lambda$-connected chain is a finite sequence $b_0, ..., b_n$ such that $\operatorname{card}(b_i \cap b_{i+1}) + \aleph_0 \geqslant \lambda$ for all $0 \leqslant i < n$.*
- *(w) an arrow $A \longrightarrow Y$ is labelled (w) iff for every $a \in A \cup \{\{\}\}$, $y \in Y$ and subset $y' \subseteq y$, $\operatorname{card} y' \leqslant \operatorname{card}(a \cap y) + \aleph_0$, there exists $a' \in A$ such that y' is contained in a' up to finitely many elements.*

Let us prove this claim although we do not really need it. Items (f) and (wf) are straightforward to check. To prove (wc), notice that it is straightforward to check that if if every $b \in B$ is contained, up to finitely many elements, in an $a \in A$, then $A \longrightarrow B \rightthreetimes X \xrightarrow{(f)} Y$ for any arrow $X \xrightarrow{(f)} Y$ labelled (f).

To prove (wc) and (c), first define objects

$$\tilde{A} := \{(a \cap b) \cup b_{fini} \ : \ a \in A, b \in B, b_{fini} \subseteq b \text{ finite}\}$$

$$\tilde{B} := \{b \ : \ b \subseteq b' \in B, b \text{ is connected to an element of } A \text{ by a finite } \geqslant \text{card } b\text{-connected chain of elements of } B\}$$

It is straightforward to check by definitions that arrows $A \longrightarrow \tilde{A} \xrightarrow{(f)} B$ and $A \longrightarrow \tilde{B} \xrightarrow{(wf)} B$ are labelled as shown, and that $A \longrightarrow \tilde{A}$ and $A \longrightarrow \tilde{B}$ satisfy (wc) and (c). Using items (f) and (wf), it is straightforward to check that any arrow as in item (wc) has the left lifting property with respect to any arrow labelled (f); induction on the length of the $\geqslant\lambda$-connected chain shows that any arrow as in item (c) has the left lifting property with respect to any arrow labelled (wf). To finish the proof, we need to show that, conversely, any arrow $A \longrightarrow B$ that has the lifting property with respect to all (f) or (wf), is labelled (wc) or (c), respectively. In particular, there arrows have the lifting property wrt to $\tilde{A} \xrightarrow{(f)} B$ and $\tilde{B} \xrightarrow{(wf)} B$, i.e. $B \longrightarrow \tilde{A} \longrightarrow B$ and $B \longrightarrow \tilde{B} \longrightarrow B$ and therefore B is isomorphic to $\tilde{A}$ or $\tilde{B}$, and therefore $A \longrightarrow B$ satisfies items (wc) and (c), resp., of the claim.

As a corollary, we record that we proved axiom M2 of model categories, i.e. that every arrow $A \longrightarrow B$ decomposes as $A \overset{(wc)}{\dashrightarrow} A' \overset{(f)}{\dashrightarrow} B$ and $A \overset{(c)}{\dashrightarrow} A' \overset{(wf)}{\dashrightarrow} B$ (recall an arrow being dashed means an existential quantifier).

It is more convenient to prove (w) by pictures. First we give a picture representing the characterisation in item (w), and then we prove the item by pictures combining set-theoretic and categorical notations.

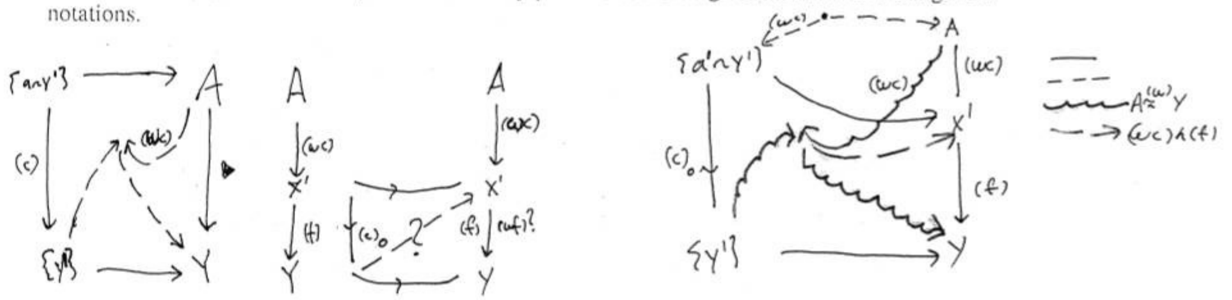


Fig. (w) (a) the characterisation of (w) (b) use M2; we need to prove that $X' \xrightarrow{(wf)} Y$ (c) add "arbitrary" black solid arrows; by M6 or the definition of the labelling, it is enough to show that we can always add the dashed black arrow. (d) add arrows successively following the legend on the right-hand side.

Example 4 *(M2) Lowenheim-Skolem theorem says that every morphism $\{A\} \longrightarrow \{M\}$ for a M a model, decomposes as $\{A\} \xrightarrow{(c)} M_A \xrightarrow{(wf)} \{M\}$ where M_A is a collection of models. This is a modification of Quillen's axiom M2.*

(M1) Fix a monster model $\mathcal{M}$ of a first-order theory. An elementary submodel $M_A \prec \mathcal{M}$ is prime and minimal over *a set $A \subset \mathcal{M}$ iff the arrow $\{A\} \longrightarrow \{M_A\} \ \rightthreetimes \ \{N\} \longrightarrow \{B\}$ for any (elementary sub)model $N \prec \mathcal{M}$ and any set $B \subset \mathcal{M}$. In words the lifting property means that for every elementary submodel $N \prec \mathcal{M}$, $A \subset N$ implies $M_A \subseteq N$, i.e. the definition of minimality. By Baldwin-Lachlan's theorem, these always exist under certain conditions. Thus, Baldwin-Lachlan's theorem reminds M1.*

(Adjoint) Baldwin-Lachlan's theorem is a typical claim about existence of a right adjoint functor. Consider a formula $\varphi : Naam(T) \longrightarrow Naam$ as a covariant functor from the category of models of a theory T (morphisms being inclusions) to the category Naam of sets (morphisms being inclusions). By definition, φ is right adjoint *to a functor $M_A \ : \ Naam \longrightarrow Naam(T)$ iff there is a bijection $Hom_{Naam}(A, \varphi(M)) \longrightarrow Hom_{Naam(T)}(M_A, M)$ for any set $A \in Naam$ and a model $M \in Naam(T)$, and, moreover, the bijection is functorial in A and M. However, $Hom_{Naam}(A, \varphi(M))$ has a unique element iff $A \subseteq \varphi(M)$, similarly for $Hom_{Naam(T)}(M, M_A)$;*

therefore the bijection is functorial if exists. That is, $A \subseteq \varphi(M)$ iff $M_A \subseteq M$ for any A, M. That is, the model M_A is prime and minimal over A.

($\bigcup X$) $\emptyset \longrightarrow \{\{a\}\} \not< X \longrightarrow \{Y\}$ *for all a iff* $Y = \cup X := \bigcup_{x \in X} x$

(lim) *Y is a directed limit/union of X iff* $\{A, B\} \longrightarrow \{A \cup B\} \not< X \longrightarrow \{Y\}$ *for all A, B*

($fini$) *for $A \subseteq B$, $B \setminus A$ is finite iff* $\{A\} \longrightarrow \{B\} \not< \{M_i\}_{i \in I} \longrightarrow \{\cup_{i \in I} M_i\}$ *for any increasing chain M_i (i.e. $M_i \subseteq M_j$ for $i \leqslant_I j$, $i, j \in I$).*

1.3.6. Do we have a model category? Axiom M5(2-out-of-3)

Our hero knows that axiom M5 are essentially the only ones he needs to check. As we saw, the axiom M6 is satisfied more-or-less by construction of a cofibrantly generated structure. In presence of M0 not only for finite diagrams, i.e., assuming existence of limits and colimits of infinite diagrammes, also axiom M2 is implied. (However, infinite limits and colimits may involve set-theoretic difficulties, and we avoid this argument.)

Axiom M5(2-out-of-3). The statement. Axiom M5 says that if two of the morphisms $g : X \longrightarrow Y, h : Y \longrightarrow Z, gh : X \longrightarrow Z$ are labelled (w), so is the third, and that any isomorphism is labelled (w). The second claim is by construction, and we only need to verify the 2-out-of-3 property.

Axiom M5. Analysis in StNaam. First he tries to check this for arrows labelled (wf) and (wc). So he draws

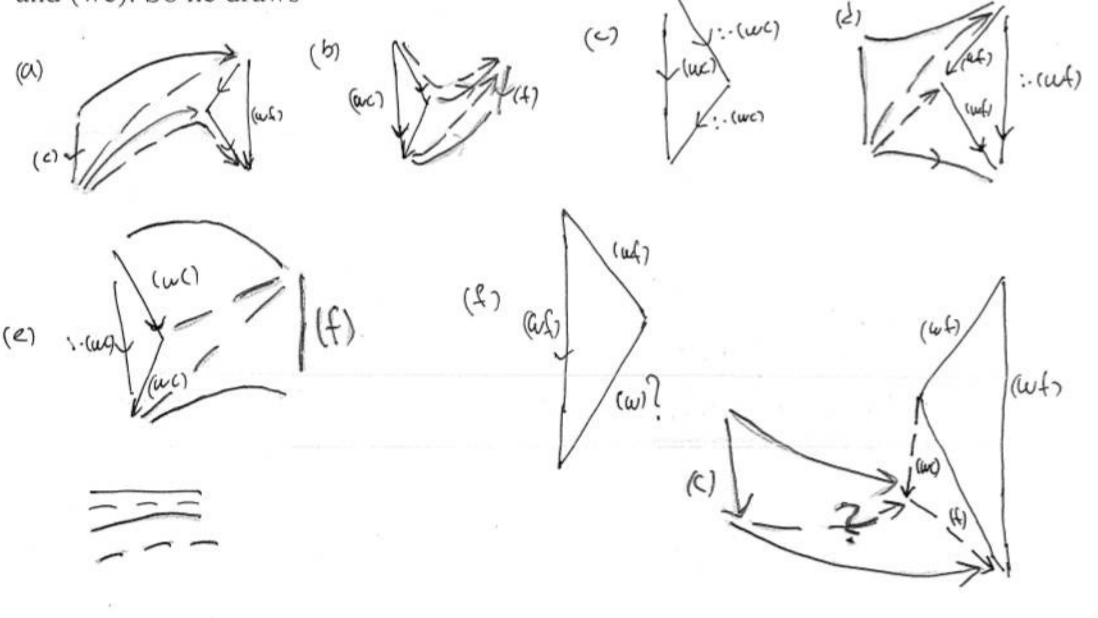


Fig. M5 (a) this proves that $fg \in (wf)$ implies $f \in (wf)$: add the red arrows and establish the lifting property. (b) an analogous "diagrammatic" proof of the fact $fg \in (wc)$ implies $g \in (wc)$. (c) follows from the set theoretic characterisation of (wc) in Claim 1. (d) this proves $f, g \in (wf)$ implies $fg \in (wf)$. (e) this proves $f, g \in (wc)$ implies $fg \in (wc)$. (f) our hero gets stuck proving this: if $g, fg \in (wf)$ then $f \in (w)$ (g) to prove (f), our hero needs to draw the dashed arrow

Then he looks at diagramme Fig. M5(g). After failing to prove it, he tries to construct an example of such a diagramme. The easiest way is perhaps to take the bottom arrow to be an identity and use the coproduct construction (Fig. M5(c×wf)(g:a)). Recall that in $StNaam$, the coproduct is just the union.

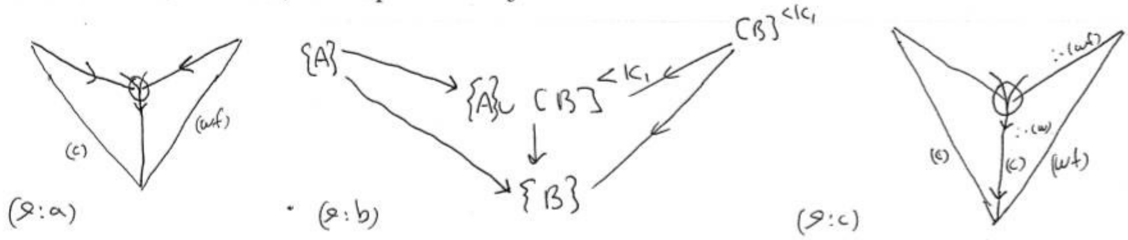


Fig. M5(c×wf). (g:a) take the ⚥-vertex to be the coproduct of the top left and top right vertices. (g:a') the previous diagramme expanded when the leftmost (c)-arrow is a $(c)_0$-arrow: card $B \leqslant$ card $A+\aleph_0$, $A \subseteq B$, and $[B]^{<\aleph_1}$ denotes the set of all countable subsets of B. (g:b) He observes that M5(2-out-of-3) implies that there in (g:a) and (g:a'), the bottom vertex is contained in the coproduct up to (wc), and that this is false in (g:a') unless either B and A differ by finitely many elements (i.e. $\{A\} \xrightarrow{(wc)} \{B\}$) or B is countable (i.e. $\emptyset \xrightarrow{(c)} \{B\}$).

Our hero observes that M5(2-out-of-3) implies that he needs to "forbid" arrows appearing in Fig. M5(c×wf)(g:a') *unless* either B and A differ by finitely many elements (i.e. $\{A\} \xrightarrow{(wc)} \{B\}$) or B is countable (i.e. $\emptyset \xrightarrow{(c)} \{B\}$).

How to do that? It is not option for him to use negation explicitly, it feels so unnatural to him; in particular, a definition using negation can hardly be used in diagramme chasing (and, we remind, our hero thinks by diagramme chasing). His first move is to use the lifting property: "forbid" these arrows by requiring the lifting property w.r.to them. This way, those arrows which are isomorphisms, are not "forbidden". We remark that this provides a useful way to use of negation in the category theory framework.

To summarise, our hero makes the following definition.

Definition 3 (a model category QtNaam) *The objects of the category $QtNaam$ are those $\widetilde{X} \in ObStNaam$ such that*

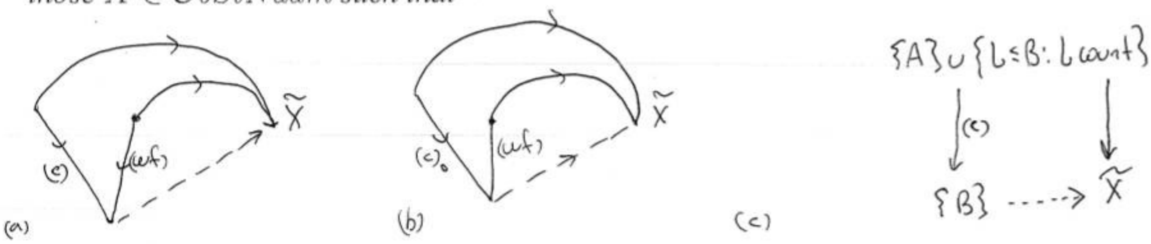


Fig. QtNaam (a) this is the "diagrammatic" definition (b) this is an equivalent definition (c) this is yet another equivalent definition expanded in set-theory notation

The structure of a model category on QtNaam is provided by the labelling inherited from StNaam.

Our hero ponders whether M5(2-out-of-3) holds for the category QtNaam. Perhaps he is able to prove this by diagramme chasing coupled with basic set theory; perhaps not. In any case, he shall not spend too much time trying to prove it; he shall *believe* in this unless he find a counterexample, and that won't happen, as the following lemma shows.

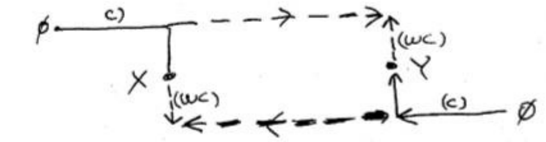
Lemma 5 *The category QtNaam satisfies the axioms of a Quillen's model category.*

We give a proof that clarifies the structure of $QtNaam$. It is enough to prove that QtNaam is a closed model category, i.e. satisfies M0, M2, M5 and M6. (It is known that axioms M3 and M4 are implied.) We verify M0, M1, M2, M6, M5 in that order.

$StNaam$ vs $QtNaam$.

Definition 4 *For $X, Y \in \mathcal{O}bStNaam$, define $X \approx Y$ iff*

(i) the picture holds:



(ii) for any countable set L (i.e. $0 \xrightarrow{(c)} \{L\}$), $\exists x \in X(L \setminus x$ is finite) iff $\exists y \in Y(L \setminus y$ is finite)

For an X, let $\tilde{X} := \bigcup\{X'' : X \longleftarrow X_0 \xrightarrow{(c)} X'' \xleftarrow{(wf)} X' \longrightarrow X\}$.

Claim 6 (QtNaam vs StNaam) *(a) $\tilde{X} \in \mathcal{O}bQtNaam$*

(b) $\tilde{X} \approx X$, $\{\} \xrightarrow{(c)} B \curlywedge X \longrightarrow \tilde{X}$

(c) $X \longrightarrow \tilde{X} \curlywedge Z \longrightarrow T$ for any arrow $Z \longrightarrow T$ in $QtNaam$.

Proof. Todo. Let us check only (a) and (b). First note $\{\} \xrightarrow{(c)} B \curlywedge X \longrightarrow \tilde{X}$ for all B implies $\tilde{X} \approx X$. This is done via pictures.

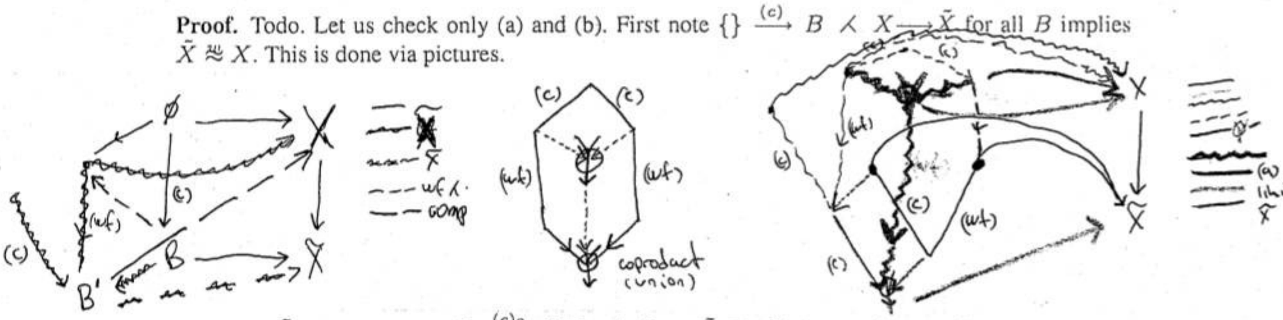


Fig.$\tilde{X}$. (a) this proves $\{\} \xrightarrow{(c)_0} \{D\} \curlywedge X \longrightarrow \tilde{X}$ (b) this lemma is proved by an easy set-theoretic argument (b) this proves $\tilde{X}$ is in $QtNaam$.

M0. Limits and colimits. For sets $X_1, ..., X_n$, let $X_\cup := X_1 \cup .. \cup X_n$, $X_{\tilde{\cup}} := \widetilde{X_\cup}$, $X_\cap := \{x_1 \cap x_2 \cap ... \cap x_n : x_1 \in X_1, ..., x_n \in X_n\}$. We remark that terminology seems somewhat confusing: a colimit (according to some) is a direct limit, and a colimit (according to some) is an inverse limit.

Claim 7 (M0) . *Let D be a finite commutative diagramme whose vertices are marked by objects $X_1, .., X_n \in \mathcal{O}bQtNaam$. (a) $X_\cup$ is the colimit of D in StNaam, and the coproduct of $X_1, ..., X_n$ in StNaam. (b) $X_\cap$ is the limit of D in StNaam, and the product of $X_1, ..., X_n$ in StNaam. (c) $\widetilde{X_\cup} \in QtNaam$ is the colimit of D in QtNaam, and the coproduct of $X_1, ..., X_n$ in QtNaam. . (b) $X_\cap \in QtNaam$ is the limit of D in QtNaam, and the product of $X_1, ..., X_n$ in QtNaam.*

Proof. An easy check. We only check that $X_\cap \in QtNaam$ and $\widetilde{X_\cup} \in QtNaam$ is the colimit of D in QtNaam.

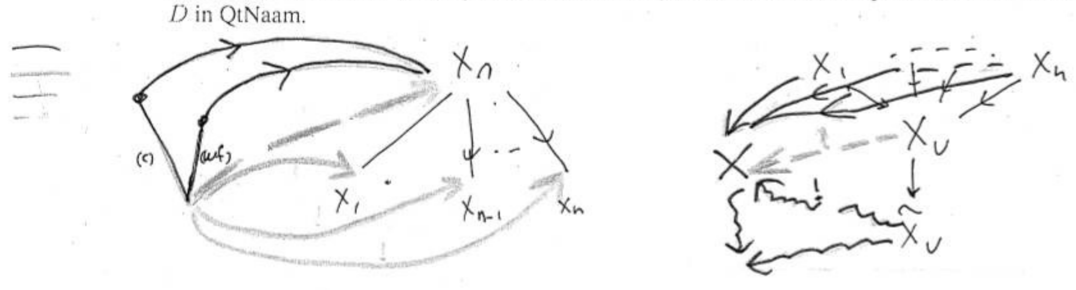


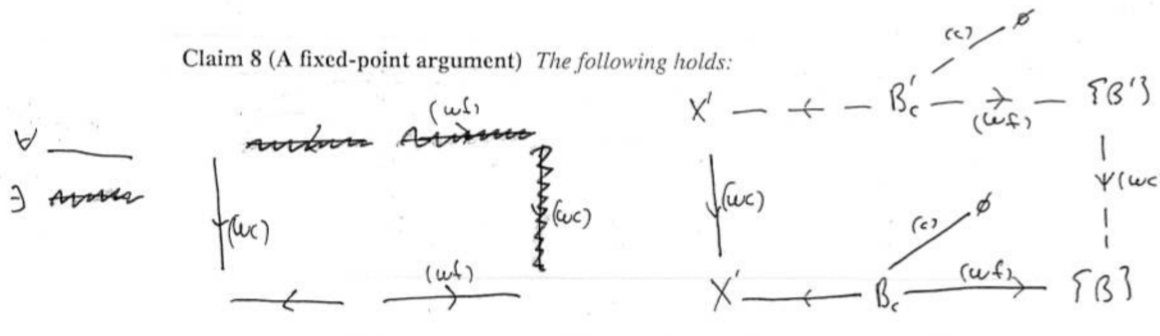
Fig.M0.(a) first construct the green arrows by $x_i \in QtNaam$, then use the limit universal property in StNaam (b) for on object $X \in ObQtNaam$, first construct the green arrow by the universal property of the colimit $X_\cup = X_1 \cup \ldots \cup X_n$ in StNaam, and then use the lifting property $X_\cup \longrightarrow \widetilde{X_\cup} \not\!\!\swarrow X \longrightarrow \{universe\}$ in $StNaam$ to construct the red arrow

Axiom M1 is inherited from StNaam. To prove Axiom M2 for QtNaam, we need a claim whose prove uses an important set-theoretic trick.

> Which alters when it alteration finds,
> Or bends with the remover to remove:
> O, no! it is an ever-fixed mark

A fixed-point argument.

Claim 8 (A fixed-point argument) *The following holds:*



In words, the (b) reads: if every countable subset of a set B is, up to finitely many elements, a subset of an element of a set X, then there is a subset of B' such that B and B' differ by finitely many elements, and every countable subset of B' is a subset of an element of X.

Proof. Pick, for every finite subset $a \subset B$, a countable set $B_a \subset B \setminus a$ such that $\{B_a\} \longrightarrow X$ does not hold. It may happen that $\{B_a \setminus b\} \longrightarrow X$ for some finite $b \subset B_a$. To avoid that, add B_b to B_a for every such b, i.e. consider $B_1 = B_a \cup \cup_{b \subset B_a \text{ finite}} B_b$. For every $b \subset B_a$ finite, $\{B_1 \setminus b\} \longrightarrow X$ does not hold but it may still hold for some $c \subset B_b$ finite. To avoid that, take $B_\omega = \cup_n B_n$ where $B_{n+1} = B_n \cup \cup_{b \subset B_n \text{ finite}} B_b$. The set B_ω is countable and contains B_b for every finite $b \subset B_\omega$. This implies that B_ω is not, up to finitely many elements, a subset of an element of X.

Axiom M2. Axiom M2(cfw) is simply by diagramme chasing; axiom M2(wcf) employs the fixed point argument.

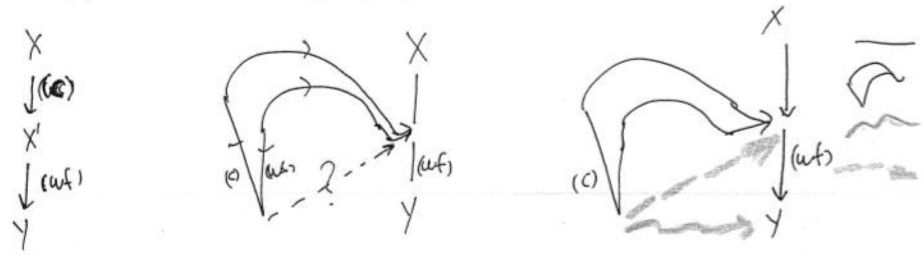


Fig. M2(QtNaam)(cfw). (a) we need to prove that $X' \in ObQtNaam$ (b) By definition of QtNaam, the following is enough. Add ("arbitrary") black arrows, and use model category axioms to "construct" the dashed black arrow. (c) apply the definition of QtNaam to construct the bottom red arrow, and apply M1(cfw) to construct the dashed arrow.

Note that in the previous figure, there was no real need to use letters X, X', Y. The argument to prove M2(cfw) is slightly more complicated, and we use set-theoretic notation to clarify the picture.

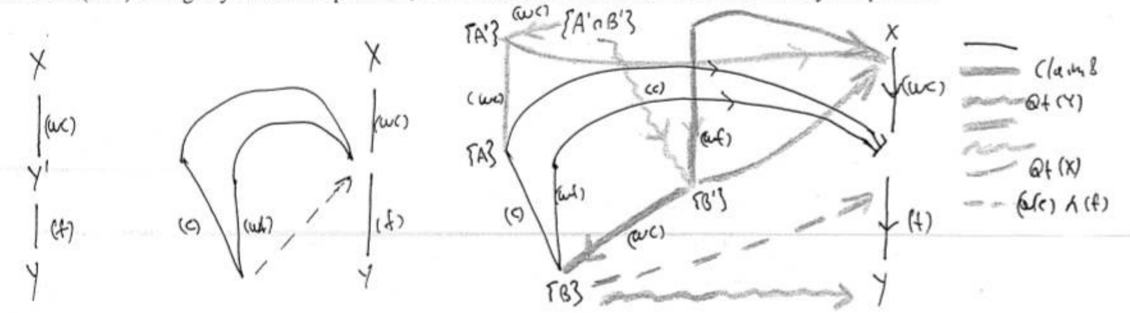


Fig. M2(QtNaam)(cfw). (a) we need to prove that $Y' \in ObQtNaam$ (b) By definition of QtNaam, the following is enough. Add ("arbitrary") black arrows, and use model category axioms to "construct" the dashed black arrow. (c) add arrows successively following the legend on the right-hand side;

Axiom M6 for QtNaam. To show M6 for $QtNaam$, we need to show that if an arrow in $QtNaam$ has the required lifting property to every arrow labelled by one of the four labels (c), (wc), (f), (wf), then it has the corresponding label in $QtNaam$ and therefore $StNaam$. It turns out that it is enough to test the arrow against its decomposition (in QtNaam) by M2. The following four pictures give the proof. They implicitly use the fact that, in $StNaam$, a decomposition $A \longrightarrow B \longrightarrow A$ implies $A \xrightarrow{(iso)} B \xrightarrow{(iso)} A$.

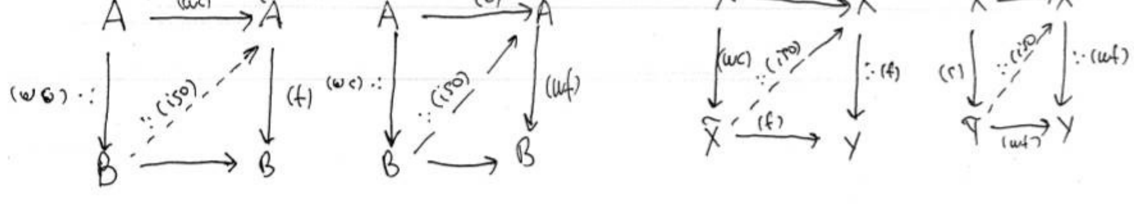


Fig. M6(QtNaam)

Axiom M5(2-out-of-3) for $QtNaam$.

Claim 9 *(a) $X \approx Y$ is an equivalence relation*

(b) in QtNaam, $X \xrightarrow{(w)} Y$ iff $X \longrightarrow Y$ and $X \approx Y$

(c) axiom M5(2-out-of-3) holds

Proof. (c) is implied by (a) and (b). (a) is by definition. (b) By the set-theoretic characterisation of an (w)-arrow we need to prove for any $B \subseteq y \in Y$, $A \subseteq x \in X$, and $L \subseteq B$ if $\mathrm{card}\, L \leqslant \mathrm{card}\,(B \cap A) + \aleph_0$ then there exists $x' \in X$ such that $L \setminus x'$ is finite. The definition of $X \overset{w}{\approx} Y$ requires the above holds for countable L only. For uncountable $L \subseteq B$, we know that for any countable $L' \subseteq L \subset_{\ast} B$, L' is, up to finitely many elements, a subset of an element of X (i.e. $0 \overset{(c)}{\dashrightarrow} B_c \overset{(wf)}{\dashrightarrow} \{B\}, B_c \dashrightarrow X' \overset{(wc)}{\dashleftarrow} X$). By the Claim (fixed point argument) above, there exists $B' \subseteq B$, $B \setminus B'$ is finite (i.e. $\{B'\} \xrightarrow{(wc)} \{B\}$) such that every countable subset of B' is a subset of an element of X (i.e. $B'_c \xrightarrow{(wf)} \{B'\}$ and $B'_c \longrightarrow X$). In notation, the latter sentence is: $0 \overset{(c)}{\dashrightarrow} B'_c \overset{(wf)}{\dashrightarrow} \{B'\} \overset{(wc)}{\dashrightarrow} \{B\}, B'_c \dashrightarrow X$.

Now apply the definition of QtNaam to conclude that every subset of B' of cardinality $\leqslant \mathrm{card}\,(B' \cap A) + \aleph_0 = \mathrm{card}\,(B \cap A) + \aleph_0$ is a subset of an element of X. That implies that every $\leqslant \mathrm{card}\,(B \cap A) + \aleph_0$ subset of B is a subset, up to finitely many elements, of an element of X, as required.

In pictures, the proof is as follows.

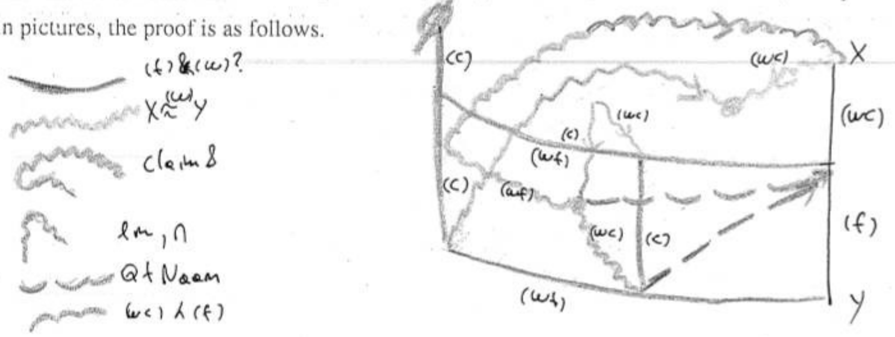


Fig. M5. Add arrows successively as shown on the legend.

The following lemmas require familiarity with notation in the book of Quillen.

In fact, by Quillen's Theorem 1' of §1.13 we can quite explicitly describe the homotopy category of QtNaam. Observe that a set/class X is cofibrant (meaning $\emptyset \xrightarrow{(c)} X$ is a cofibration) iff every element $x \in X$ is a countable set; the set/class X is fibrant (meaning $X \xrightarrow{(f)} \{universe\}$ is a fibration) iff for every $x \in X$ and a finite set a the union $x \cup a \subseteq y \in X$ for some $y \in X$. Let $\pi QtNaam_{cf}$ denote the full subcategory of all objects which are both fibrant and cofibrant.

Lemma 10 *The homotopy category $HoQtNaam$ exists and $\pi QtNaam_{cf} \longrightarrow HoQtNaam$ is an equivalence of categories. The objects of $HoQtNaam$ are same as that of $QtNaam$. There is an arrow $X \longrightarrow Y$ in $HoQtNaam$, necessarily unique, iff for every $y \in Y$ and a* countable *set $z \subseteq y \in Y$, there is $x \in X$ containing z* up to finitely many elements. *Objects X and Y are isomorphic in $HoQtNaam$ iff for every countable set L, L is contained up to finitely many elements in an element of X iff it is, up to finitely many elements, contained in an element of Y.*

Proof. Implied by Theorem 1' of Quillen. See appendix and his book for precise definitions.

We take a chance to remind the construction of the *category* $C^{\downarrow}_A = C/A$ we need later. By definition, for an object $A \in QtNaam$, let the objects of $QtNaam^{\downarrow}_A = QtNaam/A$ be arrows $A \longrightarrow X$, and morphisms are arrows $X \longrightarrow Y$ making the obvious triangular diagramme commutative. In our case, of course, $QtNaam^{\downarrow}_A = QtNaam/A$ is a full subcategory of $QtNaam$.

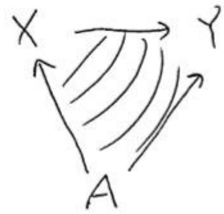


Fig. $(QtNaam/A)$

1.4. Deriving functors. Recovering the covering number. Homotopy Continuum Hypothesis

A model category is a tool to derive functors. It is known that calculating (co)homology groups is often quite meaningful. In the formalism of model categories, instead of homology groups one looks at *derived functors*; a (co)homology group $H^q(X, A)$ is then a group of homotopy classes of maps defined by a formula like

$$H^q(X, A) = [\mathbb{L}_{ab}(X) \longrightarrow \Omega^{q+N}\Sigma^N A]$$

where $\mathbb{L}_{ab}$ is some derived functor, Ω and Σ are so-called *loop* and *suspension* functors, $\mathbb{L}_{ab}(X) \longrightarrow \Omega^{q+N}\Sigma^N A]$ is the group of homotopy classes of maps from $\mathbb{L}_{ab}(X)$ to $\Omega^{q+N}\Sigma^N A$, where $N \geqslant 0$ is an integer with $q + N \geqslant 0$; see §5.2-5.6 of Quillen (1967) for more explanations.

Accordingly, model categories are sometimes thought of as a mere tool to define and calculate homotopy-invariant *derived functors*.

We remark/remind in passing that the notion of the model category itself is considered not homotopy-invariant, and this makes some people to consider it as a *wrong* notion needed to be improved or replaced.

Standard homotopy tools degenerate. We remark that that all the standard tools of model categories degenerate in QtNaam, for the simple reason that the category is a quasi partially ordered set: every two morphisms $X \longrightarrow Y$ are left and right homotopic (being identical), and a path and cylinder object of any object X is always X, as these by definition factor as $X \longrightarrow X^I \longrightarrow X \oplus X$ and $X \otimes X \longrightarrow X \times I \longrightarrow X$ and $X \oplus X = X \otimes X = X$ (as arrows are unique). Further, the category is not pointed so (co)fibration sequences(analogues of long exact sequences of (co)homology), loop and suspension functors etc do not make sense. It appears that *every* computation in the Quillen's book involves some of these notions, and, accordingly, most(all?) theorems in the book are trivial to prove when specialised to our example QtNaam.

Still, as we shall soon see, the notion of a *left derived functor* and *cofibrant replacement* does not trivialise, at least after a slight generalisation.

Derived functors. Partially ordered sets. Encountering a non-homotopy invariant functor is a typical situation in homotopy theory, and the solution is to find a a functor *closest from the left* which *factors* through the homotopy category; closest from the left means a universality property; these functors are known as (left) *derived* functors. What does it mean in our case?

Our categories $StNaam$, $QtNaam$ are quasi partially ordered sets/classes, in the following sense, and for such categories the necessary notions of category and homotopy theory may be described quite explicitly. We remark that we slightly extend the definitions to be able to derive not necessarily functors.

A category A carries canonically the structure of a quasi partially ordered set $(A, \leqslant_A)$: for $\bullet_1, \bullet_2 \in ObA$, $\bullet_1 \leqslant_A \bullet_2$ iff there is a morphism from $\bullet_1$ to $\bullet_2$. Conversely, every quasi partially ordered set $(P, \leqslant)$ can be canonically considered as a category: $ObA = P$, and there is a *unique* morphism $\bullet_1 \longrightarrow \bullet_2$ iff $\bullet_1 \leqslant_P \bullet_2$. There is no morphism $\bullet_1 \longrightarrow \bullet_2$ for $\bullet_1 \not\leqslant_P \bullet_2$; the composition of morphisms $\bullet_1 \longrightarrow \bullet_2$ and $\bullet_2 \longrightarrow \bullet_3$ is the unique morphism $\bullet_1 \longrightarrow \bullet_3$.

For quasi partially ordered sets/classes A, B considered as categories, a functor $F : A \longrightarrow B$ is a *monotonic function*, and a *covariant functor* is a non-decreasing function. For covariant functors $F, G : A \longrightarrow B$, there exists a natural transformation taking F into G iff $\forall X \in \mathcal{O}bA(F(X) \leqslant_A G(X))$; such a natural transformation is necessarily unique if exists. The functors $F, G : A \longrightarrow B$ are naturally equivalent iff $F(\bullet) \leqslant G(\bullet) \leqslant_B F(\bullet)$ for every object $\bullet \in \mathcal{O}bA$.

For quasi-partially ordered sets/classes A, A', B as categories, and covariant functors $\gamma : A \longrightarrow A'$ and $F : A \longrightarrow B$, the left derived functor $\mathbb{L}^\gamma F : A' \longrightarrow B$ with respect to $\gamma : A \longrightarrow A'$ is "the functor from A' to B such that $\mathbb{L}^\gamma F \circ \gamma$ is closest to F from the left" in the following precise sense. By definition $\mathbb{L}^\gamma F : A' \longrightarrow B$ is a covariant functor, i.e. an order-preserving function from A' to B, such that (i) firstly, $\forall \bullet \in ObA(\mathbb{L}^\gamma F \circ \gamma(\bullet) \leqslant_B F(\bullet))$, and (ii) secondly, for every non-decreasing function (functor) $G : A' \longrightarrow B$ such that $\forall \bullet \in ObA(G \circ \gamma(\bullet) \leqslant_B F(\bullet))$, it holds $\forall \bullet \in ObA'(G(\bullet) \leqslant_B \mathbb{L}^\gamma F(\bullet))$. Similarly we may define the right-defined (covariant) functor inverting the direction of all the inequalities in the above formulae.

For $B = On$ a well-ordered set, the left-derived functor always exists and

$$\mathbb{L}^\gamma F(\bullet') = \min\{F(\bullet) : \bullet' \leqslant_{A'} \gamma(\bullet),\ \bullet \in ObA\}$$

Note that the formula defines, up to natural equivalence, a functor, i.e. an order-preserving function, satisfying (i) and (ii) for for $F : A \longrightarrow B$ an arbitrary function not necessarily order-preserving (i.e. functorial).

"We shall be concerned only with the case where A is a model category and γ is the localisation functor $\gamma : A \longrightarrow HoA$. Recall that by construction of the homotopy category we outlined in §1.2, $\mathcal{O}bA = \mathcal{O}bA' = \mathcal{O}bHoA$, and a morphism in $A' = HoA$ from X to Y is (an equivalence class represented by) a chain of morphisms in A of the following form :

$$X \longrightarrow X_1 \xleftarrow{(w)} X_2 \longrightarrow X_3 \xleftarrow{(w)} \ldots \longrightarrow X_n \longrightarrow Y,$$

the localisation $\gamma : A \longrightarrow HoA$ is the identify on objects and (almost) morphisms (a morphism is taken into the equivalence class of itself).

Thus, in this case, $\gamma : A \longrightarrow HoA$, for $F : A \longrightarrow On$ a functor and $\mathbb{L}^\gamma F : HoA \longrightarrow On$,

$$\mathbb{L}^\gamma F(X) = \min\{F(Y) : X \longrightarrow X_1 \xleftarrow{(w)} X_2 \longrightarrow X_3 \xleftarrow{(w)} \ldots \longrightarrow X_n \longrightarrow Y,\\ Y, X_1, \ldots, X_n \in \mathcal{O}bA\}$$

In homotopy theory it is often useful to require certain maps to be cofibrations; e.g. we may require $\emptyset \longrightarrow Y$ to be a cofibration:

$$\mathbb{L}^\gamma_c F(X) = \min\{F(Y) : X \longrightarrow X_1 \xleftarrow{(w)} X_2 \longrightarrow X_3 \xleftarrow{(w)} \ldots \longrightarrow X_n \longrightarrow Y \xleftarrow{(c)} \emptyset,\\ Y, X_1, \ldots, X_n \in \mathcal{O}bA\}$$

Note that both these formulae are defined for an arbitrary *function* $F : A \dashrightarrow On$, not necessary a functor. In particular, they are well-defined for card $: QtNaam \dashrightarrow On$.

The derived functor $\mathbb{L}^{\gamma}\mathrm{card}$ *of cardinality* $\mathrm{card}\ : QtNaam \dashrightarrow On$. Let us look at what our hero has been doing while we have been busy explaining the basics about derived functors.

Meanwhile, our hero was looking for functors to derive. In all the journey so far s/he has encountered only one notion that reminds him of a functor—the notion of *cardinality of a set*. For example, it is the only notion he encountered that you can plug into an object and get something rather different. Moreover, it was definitely the only notion he has used as he would use a (forgetful) functor: to plug in a set and get something simpler (his definition of a (c)-arrow worked this way). So he looks closely at the cardinality and notices that, regrettably, it is not homotopy-invariant (and that's something very important to him/her!): the arrow $\{\{\bullet_1\},\{\bullet_1,\bullet_2\}\} \xrightarrow{(w)} \{\{\bullet_1,\bullet_2\}\}$ is a weak equivalence but the arrow $2 = \mathrm{card}\,\{\{\bullet_1\},\{\bullet_1,\bullet_2\}\} > \mathrm{card}\,\{\{\bullet_1,\bullet_2\}\} = 1$ is not an isomorphism. His prejudices do not allow him to notice that cardinality is not even a functor from $QtNaam$, as these two sets $\{\{\bullet_1\},\{\bullet_1,\bullet_2\}\}$ and $\{\{\bullet_1,\bullet_2\}\}$ are in fact isomorphic.

Besides, it's not like that he has a choice, he has not yet seen anything else he could possibly derive...

And so he defines:

$$\mathbb{L}\mathrm{card}\,(X) = \min\{\mathrm{card}\,(Y) : X \longrightarrow X_1 \xleftarrow{(w)} X_2 \longrightarrow X_3 \xleftarrow{(w)} \ldots \longrightarrow X_n \longrightarrow Y,$$

$$Y, X_1, \ldots, X_n \in \mathcal{O}bQtNaam\}$$

Then he plugs as Y the terminal object $\{universe\}$ of QtNaam, and observes that for any X $\mathbb{L}\mathrm{card}\,(X) = 1$ as $\mathrm{card}\,\{universe\} = 1$. How boring!

Lost, he turns for advice of a seasoned topologist.

Cofibrant replacement. "Faced with something like this, a seasoned topologist would probably [..] invoke the philosophy that to give [...] homotopy significance the maps involved should be replaced if necessary by cofibrations. In fact, it becomes clear [...] that this philosophy is no different from the philosophy in homological algebra that a cautious practitioner should usually replace a module by a projective resolution before, for instance, tensoring it with something. (In model categories, taking projective resolution of an X corresponds to decomposing $X \xrightarrow{(wc)} X' \xrightarrow{(f)}$ terminal object or initial object $\xrightarrow{(c)} X' \xrightarrow{(wf)} X$).

Grothendieck teaches (todo: find quote) us that one should work with morphisms and not objects; and, particularly, that it is good/useful to identify/think of an object as the unique morphism from the initial object to the object. So think of the object Y as the unique morphism $\emptyset \longrightarrow Y$ from the initial object. Now we do have a map/morphism which we replace, as necessary, by cofibrations.

So we write instead of $(*)$

$$\mathbb{L}_c\mathrm{card}\,(X) = \min\{\mathrm{card}\,(Y) : X \longrightarrow X_1 \xleftarrow{(w)} X_2 \longrightarrow X_3 \xleftarrow{(w)} \ldots \longrightarrow X_n \longrightarrow Y \xleftarrow{(c)} \{\},$$

$Y, X_1, ..., X_n \in \mathcal{O}bQtNaam\}(**)$

We compute $\mathbb{L}_c\mathrm{card}\,(X)$ for singletons X: for $X = \{\aleph_0\}$ $\mathbb{L}_c(\{\aleph_0\}) = \mathbb{L}_c(2^{\aleph_0}) = 1$, for $X = \{\aleph_n\}, n > 0$ finite, $\mathbb{L}_c\mathrm{card}\,(\{\aleph_n\}) = \mathbb{L}_c\mathrm{card}\,(2^{\aleph_n}) = \aleph_n$. However, we get stuck computing $\mathbb{L}_c\mathrm{card}\,(\{\aleph_\omega\}) = \mathbb{L}_c\mathrm{card}\,(2^{\aleph_\omega})$. Further we notice that $\mathbb{L}_c\mathrm{card}\,(\{\aleph_\alpha\}) = \mathbb{L}_c\mathrm{card}\,(2^{\aleph_\alpha})$. Interesting! (We state these bounds in Theorem 13 below..)

The derived functor $\mathbb{L}_c\mathrm{card}$ *as a covering number of PCF.* In fact, the numbers $\mathbb{L}_c(\{\aleph_\alpha\})$ are well-known in set theory under the name of covering numbers in PCF. Further, a celebrated theorem of Shelah in set theory provides a bound on $\mathbb{L}_c\mathrm{card}\,(2^{\aleph_\omega}) = \mathbb{L}_c\mathrm{card}\,(\{\aleph_\omega\}) < \aleph_{\omega_4}$ in ZFC.

By definition the covering number[15]

$$\mathrm{cov}(\lambda, \kappa, \theta, \sigma)$$

is the *least size of a family* $X \subseteq [\lambda]^{<\kappa}$ *of subsets of* λ *of cardinality less than* κ*, such that every subset of* λ *of cardinality less than* θ*, lies in a union of less than* σ *subsets in* X.

The following is proven by unwinding definitions.

Lemma 11 (the covering number as a derived functor) *For* $\{\lambda\} \in QtNaam$,

$$\mathbb{L}_c\mathrm{card}\,(\{\lambda\}) = cov(\lambda, \aleph_1, \aleph_1, 2)$$

Proof. Call $X \in \mathcal{O}bStNaam$ a (wc)-*covering family* of λ iff every countable subset of λ is a subset, up to finitely many elements, of an element of X. Prove by induction on n that each X, $X_1, .., X_n, Y$ is a (wc)-covering family for λ. For $X = \{\lambda\}$ this is obvious; for X_1 this is immediate by the definition of a morphism $X \longrightarrow X_1$, for X_2 this is immediate by the definition of a weak equivalence $X_1 \overset{(w)}{\longleftarrow} X_2$, etc. Thus Y is a (wc)-covering family for λ; the condition $\{\} \overset{(c)}{\longrightarrow} Y$ implies that every element of Y is countable. In notation, $\emptyset \overset{(c)}{\longrightarrow} Y \overset{(wc)}{\longrightarrow} \{\lambda\}$; apply Claim(Fixed point argument) to find Λ such that $\emptyset \overset{(c)}{\dashrightarrow} Y' \overset{(wf)}{\dashrightarrow} \{\Lambda\} \overset{(wc)}{\dashrightarrow} \{\lambda\}, Y' \dashrightarrow Y$. This shows $\mathbb{L}_c\mathrm{card}\,(\{\lambda\}) \leqslant cov(\lambda, \aleph_1, \aleph_1, 2)$. Conversely, for Y a covering family, take $n = 2$, $X_1 = X, X_2 = Y$, then by the definition of a covering family $\{\lambda\} = X_1 \overset{(w)}{\longleftarrow} X_2 = Y \overset{(c)}{\longleftarrow} \emptyset$.

More covering numbers as values of derived functors. We explained that the definition of $\mathbb{L}_c\mathrm{card}\,(X)$ is natural and straightforward in homotopy theory and particularly in Quillen's formalism of model categories. The following two modifications are seemingly minor and not entirely unnatural from the homotopy point of view:

$\mathbb{L}_c^{Qt_{/A}}\mathrm{card}\,(A \longrightarrow X) =$

$= \min\{\mathrm{card}\, X' : A \overset{(c)}{\longrightarrow} X' \longleftarrow X_1 \overset{(w)}{\longrightarrow} X_2 \longleftarrow ... \overset{(w)}{\longrightarrow} X_n \longleftarrow X,$

[15] The 4-parameter notation $\mathrm{cov}(\lambda, \lambda, \kappa, 2)$ is standard and follows [Shelah, Cardinal Arithmetic], p. ?? (Appendix S). We refer to two expository papers [Shelah, Cardinal arithmetics for skeptics] and [Kojman, 2001](PCF Theory) that use slightly different 2-parameter notation for the covering number $\mathrm{cov}(\lambda, \kappa) := \mathrm{cov}(\lambda, \lambda, \kappa, 2)$ (Shelah) and $\mathrm{cov}(\lambda, \omega) := \mathrm{cov}(\lambda, \aleph_1, \aleph_1, 2)$ (Kojman). In [Shelah, Cardinal arithmetics for skeptics], Theorem 5.7 identifies $\mathrm{pp}_\kappa(\lambda)$ as $\mathrm{pp}_\kappa(\lambda) = \mathrm{cov}(\lambda, \kappa) = \mathrm{cov}(\lambda, \lambda, \kappa, 2)$ for $\mathrm{cf}\lambda \leqslant \kappa < \lambda$ and $\lambda \neq \aleph_\lambda$. It is not known whether it is consistent with ZFC that $\mathrm{pp}_\kappa(\lambda) \neq \mathrm{cov}(\lambda, \lambda, \kappa, 2)$ for some λ. Theorem 6.3 [ibid.] is what we call Homotopy Generalised Continuum Hypothesis. The relevant two pages of the paper are in the appendix; at a later stage we shall provide references to the book "Cardinal Arithmetic" which contains proofs.

$A{\longrightarrow}X', A{\longrightarrow}X_1, ..., A{\longrightarrow}X_n \in QtNaam\}$ $\qquad (***)$

and

$\mathbb{L}_c^{Qt}\mathrm{card}\,(A{\longrightarrow}X) =$

$$= \min\{\mathrm{card}\, X' : A \xrightarrow{(c)} X' \longleftarrow X_1 \xrightarrow{(w)} X_2 \longleftarrow ... \xrightarrow{(w)} X_n \longleftarrow X, \\ X', X_1, ..., X_n \in QtNaam\} \qquad (****)$$

Lemma 12 *For a cardinal $\aleph_\alpha \in QtNaam$,*

(i)

$$\mathbb{L}_c\mathrm{card}\,(\{\aleph_\alpha\}) = \mathbb{L}_c^{Qt}\mathrm{card}\,(\{\}{\longrightarrow}\{\aleph_\alpha\}) = \mathbb{L}_c^{St}\mathrm{card}\,(\{\}{\longrightarrow}\{\aleph_\alpha\}) = \mathrm{cov}(\aleph_\alpha, \aleph_1, \aleph_1, 2).$$

In particular, (i')

$$\mathbb{L}_c\mathrm{card}\,(\{\aleph_\omega\}) = \mathbb{L}_c^{Qt}\mathrm{card}\,(\{\}{\longrightarrow}\{\aleph_\omega\}) = \mathbb{L}_c^{St}\mathrm{card}\,(\{\}{\longrightarrow}\{\aleph_\omega\}) = \mathrm{cov}(\aleph_\omega, \aleph_1, \aleph_1, 2) = \mathrm{pp}(\aleph_\omega).$$

(ii)

$$\mathbb{L}_c^{Qt/\aleph_\alpha}(\aleph_\alpha{\longrightarrow}\{\aleph_\alpha\}) = \mathrm{cov}(\aleph_\alpha, \aleph_\alpha, \aleph_\alpha, 2)$$

(iii)

$$\mathbb{L}_c^{Qt}(\aleph_\alpha{\longrightarrow}\{\aleph_\alpha\}) = \mathrm{cov}(\aleph_\alpha, \aleph_\alpha, \aleph_1, 2)$$

Proof. (i) has the same proof as previous Lemma. (iii) Take $n = 1$, $X_1 := [\aleph_\alpha]^{\leqslant\omega}$ be the set of countable subsets, X' be a covering family as in the definition of $\mathrm{cov}(\aleph_\alpha, \aleph_\alpha, \aleph_1, 2)$. (ii) As in the previous Lemma, we know that every countable subset of $\aleph_\alpha$ is a subset of an an element of every X_i, up to finitely many elements. However, we also know that $X_i \in QtNaam$ and there is an arrow $\aleph_\alpha{\longrightarrow}X_i$, for each X_i. Use the lifting property (Fig. QtNaam) in the definition of QtNaam to show that in fact X_i covers every subset of $\aleph_\alpha$ of cardinality less than $\aleph_\alpha$, e.g. by taking $A = \aleph_\mu$, $\mu < \alpha$, and $B = \aleph_\mu \cup B'$ where B' is arbitrary such that $\mathrm{card}\, B' \leqslant \aleph_\mu$.

The reader would have little trouble giving other examples, e.g. by replacing the arrow $\aleph_\alpha{\longrightarrow}\{\aleph_\alpha\}$ by $[\aleph_\alpha]^{<\aleph_\alpha}{\longrightarrow}\{\aleph_\alpha\}$ to get rid of assumption $\aleph_\alpha \in QtNaam$.

Est' ljudi, dlja kotoryh teorema wierna.

There are men, for whom the/a? theorem is true

PCF as a homotopy theory. Generalised Homotopy Continuum Hypothesis. We summarise some of what is known in our notation. As explained in the introduction, Shelah ([Shelah, Cardinal Arithmetic], [Shelah, Logical Dreams]) views these bounds as answers to the *right* questions. Note that analogously, from the homotopic point of view, these are answers to natural homotopy-invariant questions. In the introduction we say more on PCF as a homotopy-invariant theory. We note that passing to homotopy-invariant/PCF questions avoids independence of ZFC.

Theorem 13 (Shelah; bounds towards HGCH) *There are following bounds on the values of the derived functors* $\mathbb{L}_c$ *and* $\mathbb{L}_c^{Qt}, \mathbb{L}_c^{St}$.

(i) if $\mathrm{cf}\aleph_\alpha = \aleph_\alpha$ *a regular cardinal, then*

$$\mathbb{L}_c(\{\aleph_\alpha\}) = \mathbb{L}_c(2^{\aleph_\alpha}) = \mathrm{cov}(\aleph_\alpha, \aleph_1, \aleph_1, 2) = \aleph_\alpha$$

(ii) $\mathbb{L}_c(\{\aleph_\omega\}) = \mathbb{L}_c(2^{\aleph_\omega}) = \mathrm{cov}(\aleph_\omega, \aleph_1, \aleph_1, 2) < \aleph_{\omega_4}$

(iii) if δ *is a limit ordinal,* $\mathrm{cf}\delta = \omega$, *and* $\delta < \aleph_{\alpha+\delta}$, $\alpha + \delta < \aleph_{\alpha+\delta}$, *then*

$$\mathbb{L}_c^{Qt}(\aleph_{\alpha+\delta} \longrightarrow \{\aleph_{\alpha+\delta}\}) = \mathrm{cov}(\aleph_{\alpha+\delta}, \aleph_{\alpha+\delta}, \aleph_1, 2) < \aleph_{\alpha+\delta+4}$$

Proof. Todo: give references to [Shelah, Cardinal Arithmetics].

Note that we do not say anything about the *fixed points* $\alpha = \aleph_\alpha$ of $\aleph_\bullet$-function. (TODO: Is there an explanation)

In other words, Generalised Homotopy Continuum Hypothesis holds for regular cardinals, and there are non-trivial bounds in ZFC for most of the limit cardinals. Arguably, we may say that PCF solves the Generalised Continuum Hypothesis by replacing it with a better, homotopy-invariant question.

Some remarks. We leave the following as an exercise to our reader.

Exercise. Call a set A is *closed under homotopy countable unions* iff for every countable family of sets $a_1, a_2, ... \in A$ there exists $a \in A$ such that each a_i is contained in a *up to finitely many elements*.

(0) Check that this notion is homotopy-invariant, i.e. if $X \xrightarrow{(w)} Y$ and either of X or Y is closed under homotopy countable unions, then both X and Y are closed under homotopy countable unions.

(1) prove that objects of QtNaam closed under homotopy countable unions, form a model category.

(2) Calculate, in the subcategory,

$$\mathbb{L}_c\mathrm{card}\,(\{\aleph_1\} \longrightarrow \{\kappa\}) := \min\{\mathrm{card}\, X' : \{\aleph_1\} \xrightarrow{(c)} X' \xleftarrow{(w)} X_1 \xrightarrow{(w)} X_2 \xleftarrow{(w)} ... \xrightarrow{(w)} X_n \xleftarrow{(w)} X\} \quad (\cdot$$

(Answer: this is $cov(\kappa, \aleph_1, \aleph_1, \omega)$)

Task. Play more with this formalism...

2. Journey's end. Directions for further research

We take the liberty of offering a few questions.

Perhaps some of the questions are quite straightforward to do, perhaps others are not.

The order of questions is insignificant.

Question 1(i). Find a natural (e.g. in homotopy theory) characterisation/axiomatisation of QtNaam (or StNaam), possibly adding more structure, e.g. that of infinity-category. For example, characterise/axiomatise QtNaam as (a) a labelled category up to isomorphism, or (b) as a model category, e.g. up to Quillen's equivalence of model categories, or perhaps (c) offer an interesting and relevant notion of equivalence.

Question 1(ii) Rewrite first-order axioms of (a large fragment) of ZFC in terms of arrows, lifting properties, commutative diagrammes in StNaam, or, better, QtNaam, preferably in the spirit of homotopy theory, e.g. Quillen's model category book. If necessary, find and add more structure to QtNaam/StNaam to axiomatise the whole of ZFC, e.g. something of higher category structure. Does this clarify any issues in ZFC ? Does this reformulation makes ZFC easier to appreciate or use by a non-specialist mathematician? How much is lost?

Question 1(iii). Use methods of set theory, possibly also model theory, to suggest a natural notion extending in some way the notion of a model category. Does it make sense in the context of homotopy theory?

Question 2(i). Use methods of stability/classification/model theory (of mathematical logic) to study the structure on the homotopy category induced by the model category, even if in our rather degenerate setting. We already saw that there is somewhat of a non-trivial connection to set theory. As the main topic of interest of both authors is model theory, we cannot resist asking whether methods of model theory can contribute to the study of model categories, e.g. the category QtNaam. For example, in the explanatory exposition we said "axioms of a model category require that the labelling induces no further structure on the homotopy category". Do this words admit an interpretation that certain structure in stably/conservatively embedded in the labelled category as a structure, here all structures as in the sense of logic?

The following two questions are perhaps slightly unrelated to the current work.

Question 2(ii). (a) In general, can logic say anything to explain the "unreasonable power" of algebraic topology language, arguably the language that shapes/used a substantial part of mathematics. This seems as a natural question for a logician, although perhaps not necessarily a model theorist. I am unaware of *any* study of algebraic topology or model categories by logicians, particularly model theorists. There are studies in the opposite direction, e.g. to apply methods of algebraic topology to study of o-minimal structures. Still, as far as I am aware, the words "model category" occur in model theory literature only once, in a paper by Artur Piekosz [Piekosz, 2009](O-minimal homotopy and generalized (co)homology,2006).

Question 2(iii). In model theory, inside of the subfield of o-*minimality*, once known as *tame topology* of Grothendieck, there are many studies developing *homotopy theory* inside of an o-*minimal* model. Is it possible to construct a *model category* inside an o-*minimal* structure? This question belongs to Artur Piekosz [Piekosz, 2009] who asked it in a slightly different setting.

As now, these studies usually follow old-fashioned expositions of algebraic topology, instead of trying to set up the general setup of a model category, and then apply the general machinery of model categories to develop homotopy theory, e.g. the theory of fundamental groups within a structure. In fact, the only exception I know is the paper by Artur Piekosz just mentioned.

The following question has rather technical motivation. We stress that at the moment, we see no "conceptual" motivation for the next question.

Question 3. Observe that most of common computation tools of algebraic topology, e.g. fibration and cofibrations sequences, loop or suspension objects, path spaces, maps spaces degenerate in our setting. Try to enlarge the category by adding either new morphisms or objects. One way to start is to add all injective maps as morphisms, "quantise" to add formal limits, new path and map objects as necessary. And iterate this step countably many times.

Question 4. Elaborate explicitly our hero's strategy in the context of Gromov's Ergosystems. Is it really as simple and automatic as our exposition seem to suggest?

Question 4'. Develop better notation so that *everything* our hero does, becomes a calculation on rather simple marked graphs or surfaces. E.g. use N.Durov's idea to consider the dual of the commutative diagramme and his observation that (todo: state the observation).

Question 5. Construct a model category whose objects are (some) families of models of an excellent abstract elementary class, e.g. an uncountably categorical first-order theory in a countable language a quasi-minimal excellent class of Zilber. Is the expressive power of the "homotopy" language of category theories, sufficient to develop the theory or at least state its main results and lemmas ? If not, is it possible to enrich it while keeping the "homotopic" and category-theoretic character of the exposition? Does this allow a an exposition of the theory of AEC or its results easier to a non-specialist?

One way to start is to consider the full subcategory of StNaam

$$StNaam(\mathcal{M}) := \{\mathbb{M} : \forall M (M \in \mathbb{M} \implies M \prec \mathcal{M})\}$$

consisting only of families of elementary submodels of a fixed monster model $\mathcal{M}$ of the class we are interested in. Label an arrow (c) or (wf) iff it carries the same label in $StNaam$. Rest of labelling is already not entirely clear: for most AEC, no arrow but identity in $StNaam(\mathcal{M})$ may inherit (wc)-label.

Intuition may suggest the following conditions to place on families.

A topologist may imagine every model $M \in \mathbb{M}$ as a *simplex* in a *simplicial set* $\mathbb{M}$ and $\{K\} \prec \{M\}$ as being *faces*, *subsimplices* of simplex $M \in \mathbb{M}$. It may be reasonable to place finiteness restrictions on families $\mathbb{M}$, e.g. requiring $\mathbb{M}$ to be (w-o) *well-founded* there is no strictly decreasing infinite chain $... \prec M_{n+1} \prec M_n \prec ... \prec M_0$ in $\mathbb{M}$ or (Δ-fini)("*that a simplex has finitely many faces*") for every $M \in \mathbb{M}$ there exists finitely many *faces* $M_1, ..., M_n \in \mathbb{M}$, $M_1, ..., M_n \prec M$, $M_1, ..., M_n \neq M$, such that for every $M' \in \mathbb{M}$ either $M \prec M'$ or $M' \cap M \prec M_1$ or ... or $M' \cap M \prec M_n$ (every $M' \cap M$ either is the whole *simplex* M or lies in one of its finitely many *faces* $M_1, ..., M_n$). This condition appears in the definition of a *good system* of the first page of [Shelah, 1973], and in fact was a starting point of this research.

A logician may imagine an inductive construction (or proof of something about) a large model U, and that a family $\mathbb{M}$ is *a stage of induction*, a collection of models already constructed at an (infinite) inductive step, or perhaps the models the inductive hypothesis says something about at a step.

Topology /To*pol"o*gy/, n. [Gr. ? place + -logy.] The art of, or method for, assisting the memory by associating the thing or subject to be remembered with some place. [R.]

Question 6. A 'functorial' definition of a topological space. Given a compact nice (sequential, Hausdorff, etc) topological space T, consider a pair of functors:

$$\mathsf{acc}_T : StNaam(T) \longrightarrow T$$

$$\{\cdot\}_T : T \longrightarrow StNaam(T)$$

where $StNaam(T)$ is the full subcategory of StNaam, $\mathcal{O}bStNaam(T) := 2^{2^T}$ with induced structure, for a subset $Z \subseteq T$ $\{U\}_T = \{U\}$, and

$$\mathsf{acc}_T(\mathcal{X}) := \cup_{X\in\mathcal{X}}\{t : t \text{ is an accumulation point of } X \subseteq T\}.$$

Observe that for T compact, a subset $U \subseteq T$ is *open* iff for any $\mathcal{X} \in StNaam(T)$, any arrow $\mathsf{acc}_T(\mathcal{X})\longrightarrow U$ "lifts" to an arrow $\mathcal{X} \overset{Ho}{\dashrightarrow} \{U\}_T$ in the homotopy category (equivalently, to $\mathcal{X} \overset{(wc)}{\dashleftarrow} X' \dashrightarrow \{U\}_T$).

Furthermore, it seems that such pairs of functors could be characterised in a reasonable functorial manner by properties like

(i) $\mathcal{X} \overset{(w)}{\longrightarrow} \mathcal{Y}$ implies $\mathsf{acc}_T(\mathcal{X}) = \mathsf{acc}_T(\mathcal{Y})$

(ii) $\mathsf{acc}_T\mathcal{X} \overset{(iso)}{\longrightarrow} \emptyset$ implies $\mathcal{X} \overset{(w)}{\longleftarrow} \emptyset$

(iii) if $\mathsf{acc}_T(\mathcal{X})\longrightarrow U \longleftarrow V$, then there exists $\mathcal{Y} \in \mathcal{O}bStNaam(T), \mathcal{Y} \dashrightarrow \mathcal{X}, \mathsf{acc}_T(\mathcal{Y}) \dashrightarrow V$

Is this characterisation useful for anything ? Does it give rise to a nice category of topological spaces ? Does it generalise, e.g. if we take $StNaam(\mathcal{M})$ instead of a topological space T ? Can one nicely define the unit interval $[0,1]$ in this way ?

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3. ThanX

I thank my Mother and Father for support, patience and more.

I also thank Artem Harmaty for attention to this work, and encouraging conversations. I thank Boris Zilber; dept to him and his ideas is obvious and large. I thank Martin Bays for reading this draft at a very early stage and at a very late stage, and pointing out many a mistake.

I would like thank St.Petersburg Steklov Mathematical Institute (PDMI RAS) and N.Durov&A.Smirnov seminar for hospitality and insight in model categories. I also thank my girlfriends for not being there when I did not need them.

We plagiarised[16] [Gromov, Ergobrain] and [Quillen, 1967](Homotopical Algebra) and also [Dwyer, Spalinski; 1995](Homotopy theories and Model categories) to improve our exposition; though, arguably and applicable to our case, an exposition of a mathematical idea is the idea itself. We learned of PCF from the Kojman's note [Kojman, 2001](PCF Theory). [Shelah, 1983](The classification theory of abstract elementary classes), especially a combinatorial condition on the first page, gave us a push and importantly hinted on connection to simplicial objects and therefore model categories. Notation $(wf), (wc), ...$ belongs to [N.Durov,2007](A new approach to Arakelov geometry); we use $(wf), (wc)$ instead of his $(af), (ac)$.

Since late in his DPhil an author of this paper had a running in-joke that it would be very punny to prove a theorem *model stable category is a stable categorical model* or at least that *a model category is a categorical model*. (todo: find an exact quote) and it would nicely fit in Zilber's programme, as the notion of a model category is a widely used notion in mathematics . It took her/him few years to take her/himself seriously. Shelah's goodness condition in [Shelah, 1983] was an important push to make him take herself seriously.

One of the authors thanks Marie Curie grant (Berlin grant), Skirball foundation and his current grant; I also thank A.Baudisch for the freedom I enjoyed at the early stages of this research.

Full-text searching in mathematical books was essential in this research.

More tanks!!!

4. Bibliography

We quoted freely from the papers [Gromov, 2009+](Structures, Learning and Ergosystems), [Quillen, 1967](Homotopy Algebra), [Shelalh, 1994](Cardinal Arithmetic) and [Dwyer,Spalinski; 1995](Homotopy Theories and Model Categories). Our brief

[16]A common scenario in the mathematical community is as follows. X gives a lecture, while (preferably young) Y is in the audience but who understands nothing of the lecture; every word of the lecture is forgotten next day. (This is what normally happens when you attend a mathematics lecture with new ideas.) A year later, Y writes an article essentially reproducing the subject matter of the lecture with full conviction that he/she has arrived at the idea by himself/herself.

Unsurprisingly, X is unhappy. He/she believes that Y could not arrive at the idea(s) by himself/herself, since Y has no inkling of how the idea came up to him/her, while X is well aware when, why and how he/she started developing the idea. (A similar "structure recall" is common in solving non-mathematical problems, such as "egg riddle" in 3.3, for example.) [Gromov, Ergobrain] exposition of category theory consists mostly of quotes from Gromov's paper; the reader is advised to read [Gromov,2009+,§3.5] for more. Our debt to [Quillen, 1967] is obvious. The "seasoned topologist" is taken from [Dwyer,Spalinski;1995; §10,p.46]. All these sources are available online.

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5. Appendix

5.1. Notions of category theory for preorders

5.2. Definition of a derived functor

5.3. A construction of the localisation of a model category

5.4. Axioms of a model category

5.4.1. Quillen's original exposition

5.4.2. Our diagrammatic rendering

5.4.3. Exposition by Dwyer and Spalinski

5.5. An exposition on cardinal arithmetic

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3. Interpret "functor" in the following special types of categories: (a) A functor between two preorders is a function T which is monotonic (i.e., $p \leqq p'$ implies $Tp \leqq Tp'$). (b) A functor between two groups (one-object categories) is a morphism
4. For functors $S, T: C \to P$ where C is a category and P a preorder, show that there is a natural transformation $S \dot\to T$ (which is then unique) if and only if $Sc \leqq Tc$ for every object $c \in C$.

In a preorder P, a least upper bound $a \cup b$ of two elements a and b, if it exists, is an element $a \cup b$ with the properties (i) $a \leqq a \cup b$, $b \leqq a \cup b$; and (ii) if $a \leqq c$ and $b \leqq c$, then $a \cup b \leqq c$. These properties state exactly that $a \cup b$ is a coproduct of a and b in P, regarded as a category.

5. Adjoints for Preorders

Recall that a preorder P is a set $P = \{p, p', \ldots\}$ equipped with a reflexive and transitive binary relation $p \leqq p'$, and that preorders may be regarded as categories so that order-preserving functions become functors. An order-reversing function $\bar{L}$ on P to Q is then a functor $L: P \to Q^{op}$.

Theorem 1. *(Galois connections are adjoint pairs). Let P, Q be two preorders and $L: P \to Q^{op}$, $R: Q^{op} \to P$ two order-preserving functions. Then L (regarded as a functor) is a left adjoint to R if and only if, for all $p \in P$ and $q \in Q$,*

$$Lp \geqq q \text{ in } Q \text{ if and only if } p \leqq Rq \text{ in } P. \tag{1}$$

When this is the case, there is exactly one adjunction φ making L the left adjoint of R. For all p and q, $p \leqq RLp$ and $LRq \geqq q$; hence also

$$Lp \geqq LRLp \geqq Lp, \quad Rq \leqq RLRq \leqq Rq. \tag{2}$$

What is a monad in a preorder P? A functor $T: P \to P$ is just a function $T: P \to P$ which is monotonic ($x \leqq y$ in P implies $Tx \leqq Ty$); there are natural transformations η and μ as in (1) precisely when

$$x \leqq Tx, \quad T(Tx) \leqq Tx \tag{3}$$

for all $x \in P$; the diagrams (2) then necessarily commute because in a preorder there is at most one arrow from here to yonder. The first equation of (3) gives $Tx \leqq T(Tx)$. Now suppose that the preorder P is a partial order ($x \leqq y \leqq x$ implies $x = y$). Then the Eqs. (3) imply that $T(Tx) = Tx$. Hence a monad T in a partial order P is just a *closure operation* t in P; that is, a monotonic function $t: P \to P$ with $x \leqq tx$ and $t(tx) = tx$ for all $x \in P$.

We leave the reader to describe a morphism $\langle T, \mu, \eta \rangle \to \langle T', \mu', \eta' \rangle$ of monads (a suitable natural transformation $T \dot\to T'$) and the category of all monads in a given category X.

Fig. 2: Notions of category theory for preorders. We quote from Maclane, Categories for working mathematician

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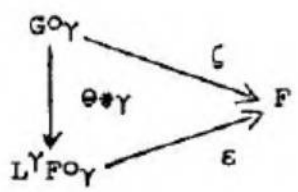
<u>Definition 1</u>: Let $\gamma: \underline{A} \longrightarrow \underline{A}'$ and $F: \underline{A} \longrightarrow \underline{B}$ be two functors. By the <u>left-derived functor</u> of F with respect to γ we mean a functor $L^{\gamma}F: \underline{A}' \longrightarrow \underline{B}$ with a natural transformation $\varepsilon: L^{\gamma}F \circ \gamma \longrightarrow F$ having the following universal property: Given any $G: \underline{A}' \longrightarrow B$ and natural transformation $\zeta: G \circ \gamma \longrightarrow F$ there is a unique natural transformation $\theta: G \longrightarrow L^{\gamma}F$ such that

(1) 

commutes.

<u>Remarks</u>: 1. $L^{\gamma}F$ is the functor from $\underline{A}'$ to $\underline{B}$ such that $L^{\gamma}F \circ \gamma$ is closest to F from the left. Similarly we may define the <u>right-derived functor</u> of F with respect to γ to be "the" functor $R^{\gamma}F: \underline{A}' \longrightarrow \underline{B}$ with a natural transformation $\eta: F \longrightarrow R^{\gamma}F \circ \gamma$ which is closest to F from the right.

2. The terminology left-derived functor comes from Verdier's treatment of homological algebra.[17] In that case $\underline{A}$ is the category K(A), where A is an abelian category, γ is the localization $K(A) \longrightarrow D(A)$, $F: K(A) \longrightarrow \underline{B}$ is a cohomological functor from K(A) to an abelian category $\underline{B}$, and $L^{\gamma}F$, $R^{\gamma}F$ are what Verdier calls the left and right

~~derived functors of F.~~

Fig. 3: Definition of a derived functor

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3. We shall be concerned only with the case where $\underline{A}$ is a model category $\underline{C}$ and γ is the localization functor $\gamma: \underline{C} \longrightarrow Ho\underline{C}$. In this case we will write just LF.

4. If $\underline{C}$ is a model category and $F: \underline{C} \longrightarrow \underline{B}$ is a functor then it is clear that $\varepsilon: LF \circ \gamma \longrightarrow F$ is an isomorphism if and only if F carries weak equivalences in $\underline{C}$ into isomorphisms in $\underline{B}$. In this case we may assume that LF is induced by F in the sense that LF is the unique functor $Ho\underline{C} \longrightarrow \underline{B}$ with $LF \circ \gamma = F$. Moreover RF=LF.

Fig. 4: Definition of a derived functor, continued

2. A Simple Proof of the Existence: Localization of a Category

Let $\mathcal{B}$ be an arbitrary category and S an arbitrary class of morphisms in $\mathcal{B}$. We show that there exists a universal functor transforming elements of S into isomorphisms. More precisely, we construct a category $\mathcal{B}[S^{-1}]$ and a "localization by S" functor $Q : \mathcal{B} \to \mathcal{B}[S^{-1}]$ with the universality property similar to that of III.2.1b) above.

To do this we set first $\operatorname{Ob}\mathcal{B}[S^{-1}] = \operatorname{Ob}\mathcal{B}$ and define Q to be identity on objects.

To construct morphisms in $\mathcal{B}[S^{-1}]$ we proceed in several steps.

a) Introduce variables x_s, one for every morphism $s \in S$.

b) Construct an oriented graph Γ as follows

> vertices of Γ = objects of $\mathcal{B}$;
> edges of $\Gamma = \{\text{morphisms in } \mathcal{B}\} \cup \{x_s, s \in S\}$;
> the edge $X \to Y$ is oriented from X to Y;
> the edge x_s has the same vertices as the edge s but the opposite orientation.

c) A *path* in Γ is a finite sequence of edges such that the end of any edge coincides with the beginning of the next one.

d) A morphism in $\mathcal{B}[S^{-1}]$ is an equivalence class of paths in Γ with the common beginning and the common end. Two paths are equivalent if they can be joined by a chain of elementary equivalences of the following type:

– two consecutive arrows in a path can be replaced by their composition;
– arrows $X \xrightarrow{s} Y \xrightarrow{x_s} X$ (resp. $Y \xrightarrow{x_s} X \xrightarrow{s} Y$) can be replaced by $X \xrightarrow{\text{id}} X$ (resp. $Y \xrightarrow{\text{id}} Y$).

Finally, the composition of two morphisms is induced by the conjuction of paths and the functor $Q : \mathcal{B} \to \mathcal{B}[S^{-1}]$ maps a morphism $X \to Y$ into the class of corresponding path (of length 1). For any $s \in S$ the morphism $Q(s)$ is clearly an isomorphism in $\mathcal{B}[S^{-1}]$, the inverse being the class of the path x_s.

For another functor $\mathcal{B} \to \mathcal{B}'$ transforming morphisms from S into quasi-isomorphisms the functor $G : \mathcal{B}[S^{-1}] \to \mathcal{B}'$ with the condition $F = G \circ Q$ is constructed as follows:

$$G(X) = F(X), X \in \operatorname{Ob}\mathcal{B} = \operatorname{Ob}\mathcal{B}[S^{-1}];$$

$$G(f) = F(f), f \in \operatorname{Mor}\mathcal{B},$$

$$G(\text{class of } x_s) = F(s^{-1}), s \in S.$$

The reader can easily verify that all definitions are unambiguous and that the functor G is unique.

Lecture Notes in Mathematics

A collection of informal reports and seminars

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Homotopical Algebra

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Homotopical Algebra

Daniel G. Quillen[1]

Homotopical algebra or non-linear homological algebra is the generalization of homological algebra to arbitrary categories which results by considering a simplicial object as being a generalization of a chain complex. The first step in the theory was presented in [5], [6], where the derived functors of a non-additive functor from an abelian category $\underline{A}$ with enough projectives to another category $\underline{B}$ were constructed. This construction generalizes to the case where $\underline{A}$ is a category closed under finite limits having sufficiently many projective objects, and these derived functors can be used to give a uniform definition of cohomology for universal algebras. In order to compute this cohomology for commutative rings, the author was led to consider the simplicial objects over $\underline{A}$ as forming the objects of a homotopy theory analogous to the homotopy theory of algebraic topology, then using the analogy as a source of intuition for simplicial objects. This was suggested by the theorem of Kan [10] that the homotopy theory of simplicial groups is equivalent to the homotopy theory of connected pointed spaces and by the derived category ([9], [19]) of an abelian category. The analogy turned out to be very fruitful, but there were a large number of arguments

[1]Supported in part by the National Science Foundation under grant GP 6166.

which were formally similar to well-known ones in algebraic topology, so it was decided to define the notion of a homotopy theory in sufficient generality to cover in a uniform way the different homotopy theories encountered. This is what is done in the present paper; applications are reserved for the future.

The following is a brief outline of the contents of this paper; for a more complete discussion see chapter introductions. Chapter I contains an axiomatic development of homotopy theory patterned on the derived category of an abelian category. In Chapter II we give various examples of homotopy theories that arise from these axioms, in particular we show that the category of simplicial objects in a category $\underline{A}$ satisfying suitable conditions gives rise to a homotopy theory. Also in §5 we give a uniform description of homology and cohomology in a homotopy theory as the "linearization" or "abelianization" of the non-linear homotopy situation, and we indicate how in the case of algebras this yields a reasonable cohomology theory.

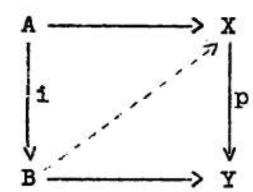
The author extends his thanks to S. Lichtenbaum and M. Schlesinger who suggested the original problem on commutative ring cohomology, to Robin Hartshorne whose seminar [9] on Grothendieck's duality theory introduced the author to the derived category, and to Daniel Kan for many conversations during which the author learned about simplicial methods and formulated many of the ideas in this paper.

Contents

Chapter I. Axiomatic Homotopy Theory

Introduction

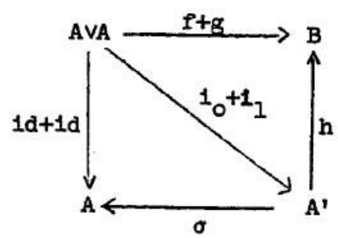
Chapter I is an attempt to define what is meant by a "homotopy theory" in a way sufficiently general for various applications. The basic definition is that of a model category which is a category endowed with three distinguished families of maps called cofibrations, fibrations, and weak equivalences satisfying certain axioms, the most important being the following two: M1. Given a commutative solid arrow diagram



where i is a cofibration, p is a fibration, and either i or p is also a weak equivalence, there exists a dotted arrow such that the total diagram is commutative. M2. Any map f may be factored $f = pi$ and $f = p'i'$ where i, i' are cofibrations where p, p' are fibrations, and where p and i' are also weak equivalences. It should be noticed that we do not assume the existence of a path or cylinder functor; in fact the homotopy relation for maps may be recovered as follows: Call an object X cofibrant if the map $\emptyset \to X$ is a cofibration (hence in the category of simplicial groups the cofibrant objects are the free

~~simplicial groups) and fibrant if the map $X \to e$ is a fibration~~ (hence in the category of simplicial sets the fibrant objects are the Kan complexes). Then two maps f, g from a cofibrant object A to a fibrant object B are said to be homotopic if there exists a commutative diagram

(1)



where $\vee$ denotes direct sum, $f+g$ is the map with components f and g, and where σ is a weak equivalence.

Given a model category $\underline{C}$, the homotopy category $\text{Ho}\,\underline{C}$ is obtained from $\underline{C}$ by formally inverting all the weak equivalences. The resulting "localization" $\gamma: \underline{C} \to \text{Ho}\,\underline{C}$ is in general not calculable by left or right fractions [7] but is rather a mixture of both. The main result of §1 is that $\text{Ho}\,\underline{C}$ is equivalent to the category $\pi\underline{C}_{cf}$ whose objects are the cofibrant and fibrant objects of $\underline{C}$ and whose morphisms are homotopy classes of maps in $\underline{C}$. If $\underline{C}$ is a pointed category then, in §§2-3 we construct the loop and suspension functors and the families of fibration and cofibration sequences in the homotopy category. If one defines a cylinder object for a cofibrant object A to be an object A'

~~together with a cofibration i_o+i_1 and a weak equivalence σ as~~ in diagram (1), then the constructions are the same as in ordinary homotopy theory except that, since a cylinder object of A is neither unique nor functorial in A, one has to be careful that things are well-defined. This is done by defining operations in two ways using the left (cofibration) structure and the right (fibration) structure, and showing that the two definitions coincide.

The term "model category" is short for "a category of models for a homotopy theory", where the homotopy theory associated to a model category $\underline{C}$ is defined to be the homotopy category $\text{Ho}\,\underline{C}$ with the extra structure defined in §§2-3 on this category when $\underline{C}$ is pointed. The same homotopy theory may have several different models, e.g. ordinary homotopy theory with basepoint is ([10], [15]) the homotopy theory of each of the following model categories: 0-connected pointed topological spaces, reduced simplicial sets, and simplicial groups. In section 4 we present an abstract form of this result which asserts that two model categories have the same homotopy theory provided there are a pair of adjoint functors between the categories satisfying certain conditions.

This definition of the homotopy theory associated to a model category is obviously unsatisfactory. In effect, the loop and suspension functors are a kind of primary structure on $\text{Ho}\,\underline{C}$, and the families of fibration and cofibration sequences are a kind of secondary structure since they determine the Toda bracket (see §3) and are equivalent to the Toda bracket when Ho C is additive. (This last remark is a result of Alex Heller.) Presumably there is higher order structure ([8], [17]) on the homotopy category which forms part of the homotopy theory of a model category, but we have not been able to find an inclusive general definition of this structure with the property that this structure is preserved when there are adjoint functors which establish an equivalence of homotopy theories.

In section 5 we define a closed model category which has the desirable property that a map is a weak equivalence if and only if it becomes an isomorphism in the homotopy category.

Chapter I. Axiomatic Homotopy Theory.

§1. The Axioms.

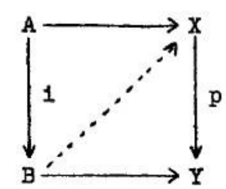
All diagrams are assumed to be commutative unless stated otherwise.

Definition 1: By a model category we mean a category together with three classes of maps in $\mathcal{C}$, called the fibrations, cofibrations, and weak equivalences, satisfying the following axioms.

M0. $\mathcal{C}$ is closed under finite projective and inductive limits.

M1. Given a solid arrow diagram

(1)



where i is a cofibration, p is a fibration, and where either i or p is a weak equivalence, then the dotted arrow exists.

M2. Any map f may be factored $f = pi$ where i is a cofibration and weak equivalence and p is a fibration. Also $f = pi$ where i is a cofibration and p is a fibration and weak equivalence.

M3. Fibrations are stable under composition, base change, and any isomorphism is a fibration.

Cofibrations are stable under composition, co-base change, and any isomorphism is a cofibration.

M4. The base extension of a map which is both a fibration and a weak equivalence is a weak equivalence. The co-base extension of a map which is both a cofibration and a weak equivalence is a weak equivalence.

M5. Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be maps in $\mathcal{C}$. Then if two of the maps f, g, and gf are weak equivalences, so is the third. Any isomorphism is a weak equivalence.

Examples. A. Let $\mathcal{C}$ be the category of topological spaces and continuous maps. Let fibrations in $\mathcal{C}$ be fibrations in the sense of Serre, let cofibrations be maps having the lifting property of Axiom M1 whenever p is both a Serre fibration and a weak homotopy equivalence, and finally let weak equivalences in $\mathcal{C}$ be weak homotopy equivalences (maps inducing isomorphisms for the functions $[K,\cdot]$ where K is a finite complex). Then the axioms are satisfied. (This is proved in Chapter II, §3.)

B. Let $\mathcal{Q}$ be an abelian category with sufficiently many projectives and let $\mathcal{C} = C_+(\mathcal{Q})$ be the category of complexes $K = \{K_q, d: K_q \to K_{q-1}\}$ of objects of $\mathcal{Q}$ which are bounded below ($K_q = 0$ if $q << 0$). Then $\mathcal{C}$ is a model category where weak equivalences are maps inducing isomorphisms on homology, where fibrations are the epimorphisms in $\mathcal{C}$, and where cofibrations are maps i which are injective and such that Coker i is a complex having a projective object of $\mathcal{Q}$ in each dimension.

C. Let $\mathcal{C}$ be the category of semi-simplicial sets and let fibrations in $\mathcal{C}$ be Kan fibrations, cofibrations be injective maps, and let the weak equivalences be maps which become homotopy equivalences when the geometric realization functor is applied. Then $\mathcal{C}$ is a model category (Ch II, §3).

For the rest of this section $\mathcal{C}$ will denote a fixed model category.

Definition 2: Let $\emptyset$ (resp. e) denote "the" initial (resp. final) object of the category $\mathcal{C}$. (These exist by M0.) An object X will be called cofibrant if $\emptyset \to X$ is a cofibration and fibrant if $X \to e$ is a fibration. A map which is both a fibration (resp. cofibration) and a weak equivalence will be called a trivial fibration (resp. trivial cofibration.)

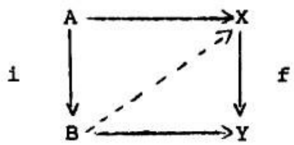
Remark: In example A every object is fibrant and the class of cofibrant objects includes CW complexes, and more generally any space that constructed by a well ordered succession of attaching cells. In example B every object is fibrant and the cofibrant objects are the projective complexes (that is, complexes consisting of projective objects--these are not projective objects in $C_+(\mathcal{Q})$). In example C every object is cofibrant and the fibrant objects are those s.s. sets satisfying the extension condition.

Before stating the next definition we recall some standard notation concerning fibre products and introduce some not-so-standard notation for cofibre products. Given a diagram

5. Closed Model Categories

We will say that a map i: A --> B has the <u>left lifting property</u> with respect to a class S of maps in a category $\underline{C}$ if the dotted arrow exists in any diagram of the form

(1)



where f is in the class S. Similarly f has the <u>right lifting property</u> with respect to S if the dotted arrow exists in any diagram of the form (1) where i is in S.

<u>Definition 1</u>: A model category $\underline{C}$ is said to be <u>closed</u> if it satisfies the axiom

<u>M6</u>: Any two of the following classes of maps in $\underline{C}$ – the fibrations, cofibrations, and weak equivalences – determine the third by the following rules:

a) A map is a fibration <==> it has the right lifting property with respect to the maps which are both cofibrations and weak equivalences

b) A map is a cofibration <==> it has the left lifting property with respect to the maps which are both fibrations and weak equivalences.

c) A map f is a weak equivalence <==> f = uv where v has the left lifting property with respect to the class of fibrations and u has the right lifting property with respect to the class of cofibrations.

<u>Remarks</u>: 1. It is clear that M6 implies M1, M3, and M4. Hence a closed model category may be defined using axioms M0, M2, M5, and M6.

2. Examples A, B, and C of §1 are all closed model categories (see proposition 2 below). Model categories which are not closed may be constructed by reducing the class of cofibrations but keeping M2, M3, and M4 valid. For example, take example B, §1, where $\underline{A}$ is the category of left R modules, R a ring, and define cofibrations to be injective maps f in $C_+(\underline{A})$ such that Coker f is a complex of <u>free</u> R modules.

In the following $\underline{C}$ is a fixed model category and we retain the notations of the previous sections.

<u>Lemma 1</u>: Let p: X --> Y be a fibration in $\underline{C}_{cf}$. The following are equivalent.

(i) p has the right lifting property with respect to the cofibrations.

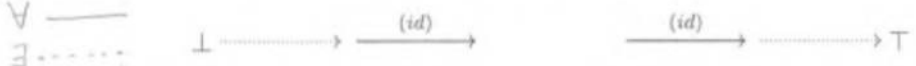
(ii) p is the dual of a strong deformation retract map in the following precise sense: there is a map t: Y --> X with $pt = id_Y$ and there is a homotopy h: XxI --> X from tp to id_X with ph = pσ.

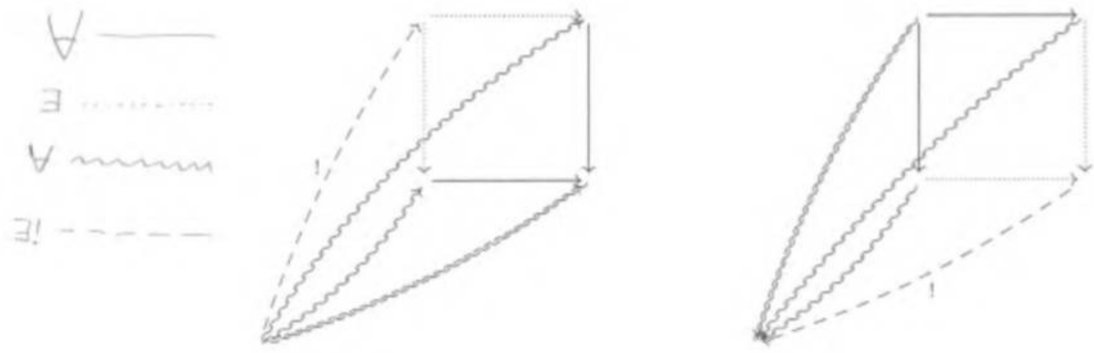
Axioms of a model category in labelled commutative diagrammes notation.

We state the axioms of Quillen of a model category in their original form. In particular, we follow the axiom numeration of Quillen(Homotopical Algebra).

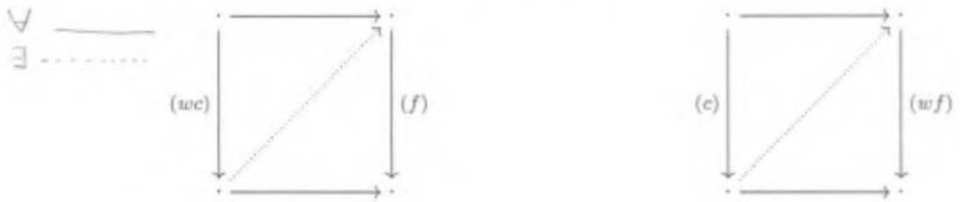
Notation (Commutative diagrammes). *Commutative diagrammes will be used systematically throughout this note. Most importantly, diagrammes will be used to introduce new definitions. We introduce our notation for commutative diagrams. The properties defined are always properties of arrows. To distingish the arrows in the diagrammes which are the object of the definition we will denote them by ◄ or ►. We will mostly use commutative diagrammes to introduce ∀∃-definitions. In such cases solid arrows will be universally quantified and dashed arrows will be existentially quantified. Whenever definitions involving higher quantifier depth (such as in Figure) a legend will be provided. As in Figure 1, we will use the notation* $X \xrightarrow{\because(\cdot)} Y$ *to mean "if the commutative diagram is true, then* $X \longrightarrow Y$ *is labeled* $(\cdot)$*". Notation* $X \xrightarrow{!} Y$ *indicates uniqueness. A legend on the right might be used to indicate the quantifiers and their order (from top to bottom). Unless stated otherwise, solid arrows are quantified universally, and dotted arrows are quantified existentially.*

Axiom ($M0$). *The category C is closed under finite projective and injective limits. It is known that it is enough to require existence of initial objects, terminal objects and pullbacks and pushouts.*

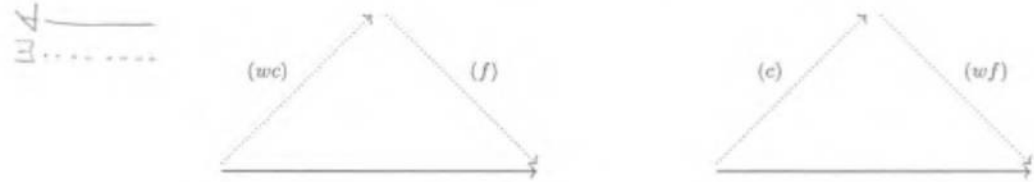




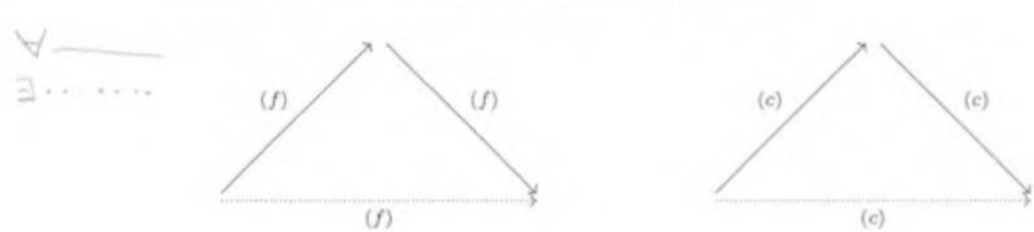
Axiom (M1). *The two following lifting properties for labeled arrows hold:*



Axiom (M2). *The following two ∀∃-diagrams hold:*



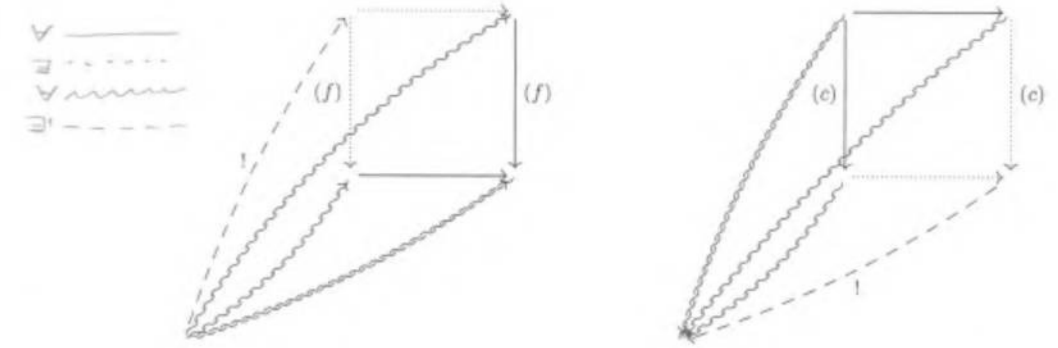
Axiom (M3(ccc,fff)). *Fibrations and cofibrations are stable under compositions. Namely, the following two ∀∃-diagrams hold:*



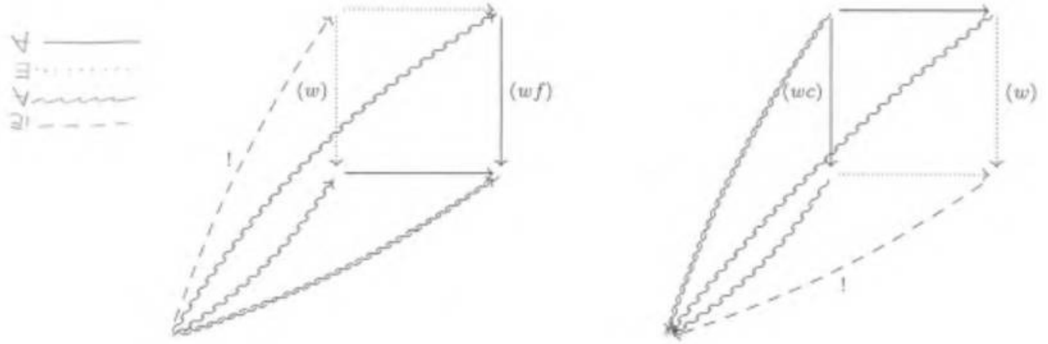
Axiom (M2(cwf)). *Isomorphisms are fibrations, co-fibrations and weak equivalences:*

Figure 1: The figure reads: if the commutative ∀∃-diagramme is true then *the left arrow is labeled (wcf).*

Axiom (M3($f \leftarrow f, c \rightarrow c$)). *Fibrations and cofibrations are stable under base change and co-base change respectively. I.e. the following diagrammes are true:*



Axiom (M4($wf \leftarrow w, wc \rightarrow w$)). *The base extension of an arrow labeled (wc) and the co-base extension of an arrow labeled (wf) are both labeled (w):*



The last axiom assures that weak equivalence is close enough to being transitive:

Axiom (M5, Two out of three). *In a triangluar diagram, if any two of the arrows are labeled (w) so is the third*