

Separation axioms as lifting properties

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Abstract

We observe that many of the separation axioms of topology (including $T_0 - T_4$) can be expressed concisely and uniformly in terms of category theory as lifting properties (in the sense of Quillen model categories) with respect to (usually open) continuous maps of finite spaces (involving up to 4 points) and the real line.

1 Introduction

We observe that separation axioms of topology including $T_0 - T_4$ can be expressed concisely and uniformly in terms of category theory as Quillen lifting properties with respect to (usually open) continuous maps of finite spaces (involving up to 4 points) and the real line. To make the exposition as self-contained as possible, we took the Wikipedia page on the separation axioms and added there the lifting properties formulae expressing what is said there in words.

No attempt is made here to explore the expressive power of the Quillen lifting property; for example, the note leaves out a reformulation of compactness in terms of the Quillen lifting property and maps of finite topological spaces, as well as iterated lifting properties. See [G] for an attempt to suggest a context for these observations; in particular, [G] discusses this and other examples, e.g. a finite group being nilpotent, solvable, p-group, a module being projective, injective, and a map being injective, surjective.

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2 Preliminaries

2.1 Quillen lifting property

Definition 1. Let f and g be a pair of morphisms from a category. We say that “ f has the left lifting property wrt g ”, “ f is (left) orthogonal to g ”, and write $f \perp g$,

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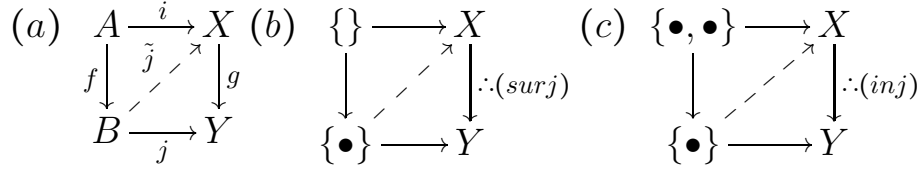


Figure 1: Lifting properties. (a) The definition of a lifting property $f \times g$. (b) $X \rightarrow Y$ is surjective (c) $X \rightarrow Y$ is injective

iff for each $i : A \rightarrow X$, $j : B \rightarrow Y$ such that $ig = fj$ (“the square commutes”), there is $j' : B \rightarrow X$ such that $fj' = i$ and $j'g = j$ (“there is a diagonal making the diagram commute”).

A useful intuition is to think that the property of left-lifting against each map in a class C is a kind of negation of the property of being in C , and that right-lifting is another kind of negation. For example, the Sierpinski space S consisting of one open point and one closed point, is perhaps the simplest counterexample to the separation axiom T_1 , and a space X satisfies T_1 iff

$$S \rightarrow pt \times X \rightarrow pt$$

where $S \rightarrow pt$, resp. $X \rightarrow pt$, denote the map sending S , resp. X , into the space consisting of a single point.

2.2 Notation for maps of finite topological spaces

A topological space comes with a *specialisation preorder* on its points: for points $x, y \in X$, $x \leq y$ iff $y \in clx$, or equivalently, a category whose objects are points of X and there is a unique morphism $x \searrow y$ iff $y \in clx$.

For a finite topological space X , the specialisation preorder or equivalently the category uniquely determines the space: a subset of X is closed iff it is downward closed, or equivalently, there are no morphisms going outside the subset.

The monotone maps (i.e. functors) are the continuous maps for this topology.

We denote a finite topological space by a list of the arrows (morphisms) in the corresponding category; $'\leftrightarrow'$ denotes an isomorphism and $'='$ denotes the identity morphism. An arrow between two such lists denotes a continuous map (a functor) which sends each point to the correspondingly labelled point, but possibly turning some morphisms into identity morphisms, thus gluing some points.

Thus, each point goes to “itself” and

$$\{a, b\} \rightarrow \{a \searrow b\} \rightarrow \{a \leftrightarrow b\} \rightarrow \{a = b\}$$

denotes

$$(\text{discrete space on two points}) \rightarrow (\text{Sierpinski space}) \rightarrow (\text{antidiscrete space}) \rightarrow (\text{single point})$$

In $A \rightarrow B$, each object and each morphism in A necessarily appears in B as well; we avoid listing the same object or morphism twice. Thus both

$$\{a\} \rightarrow \{a, b\} \text{ and } \{a\} \rightarrow \{b\}$$

denote the same map from a single point to the discrete space with two points. Both

$$\{a \not\leftarrow U \searrow x \not\leftarrow V \searrow b\} \longrightarrow \{a \not\leftarrow U = x = V \searrow b\} \text{ and } \{a \not\leftarrow U \searrow x \not\leftarrow V \searrow b\} \longrightarrow \{U = x = V\}$$

denote the morphism gluing points U, x, V .

In $\{a \searrow b\}$, the point a is open and point b is closed.

3 Separation Axioms

Let X be a topological space. Then two points x and y in X are *topologically distinguishable* iff the map $\{x \leftrightarrow y\} \longrightarrow X$ is not continuous, i.e. iff at least one of them has an open neighbourhood which is not a neighbourhood of the other.

Two points x and y are *separated* iff neither $\{x \searrow y\} \longrightarrow X$ nor $\{y \searrow x\} \longrightarrow X$ is continuous, i.e. each of them has a neighbourhood that is not a neighbourhood of the other; in other words, neither belongs to the other's closure, $x \notin cl\,y$ and $y \notin cl\,x$. More generally, two subsets A and B of X are *separated* iff each is disjoint from the other's closure, i.e. $A \cap cl\,B = B \cap cl\,A = \emptyset$. (The closures themselves do not have to be disjoint.) In other words, the map $i_{AB} : X \longrightarrow \{A \leftrightarrow x \leftrightarrow B\}$ sending the subset A to the point A , the subset B to the point B , and the rest to the point x , factors both as

$$X \longrightarrow \{A \leftrightarrow U_A \searrow x \leftrightarrow B\} \longrightarrow \{A \leftrightarrow U_A = x \leftrightarrow B\}$$

and

$$X \longrightarrow \{A \leftrightarrow x \not\leftarrow U_B \leftrightarrow B\} \longrightarrow \{A \leftrightarrow x = U_B \leftrightarrow B\}$$

here the preimage of x, B , resp. x, A is a closed subset containing B , resp. A , and disjoint from A , resp. B . All of the remaining conditions for separation of sets may also be applied to points (or to a point and a set) by using singleton sets. Points x and y will be considered separated, by neighbourhoods, by closed neighbourhoods, by a continuous function, precisely by a function, iff their singleton sets $\{x\}$ and $\{y\}$ are separated according to the corresponding criterion.

Subsets A and B are *separated by neighbourhoods* iff A and B have disjoint neighbourhoods, i.e. iff $i_{AB} : X \longrightarrow \{A \leftrightarrow x \leftrightarrow B\}$ factors as

$$X \longrightarrow \{A \leftrightarrow U_A \searrow x \not\leftarrow U_B \leftrightarrow B\} \longrightarrow \{A \leftrightarrow U_A = x = U_B \leftrightarrow B\}$$

here the disjoint neighbourhoods of A and B are the preimages of open subsets A, U_A and U_B, B of $\{A \leftrightarrow U_A \searrow x \not\leftarrow U_B \leftrightarrow B\}$, resp. They are *separated by closed neighbourhoods* iff they have disjoint closed neighbourhoods, i.e. i_{AB} factors as

$$X \longrightarrow \{A \leftrightarrow U_A \searrow U'_A \not\leftarrow x \searrow U'_B \not\leftarrow U_B \leftrightarrow B\} \longrightarrow \{A \leftrightarrow U_A = U'_A = x = U'_B = U_B \leftrightarrow B\}.$$

They are *separated by a continuous function* iff there exists a continuous function f from the space X to the real line \mathbb{R} such that $f(A) = 0$ and $f(B) = 1$, i.e. the map i_{AB} factors as

$$X \longrightarrow \{0'\} \cup [0, 1] \cup \{1'\} \longrightarrow \{A \leftrightarrow x \leftrightarrow B\}$$

where points $0', 0$ and $1, 1'$ are topologically indistinguishable, and $0'$ maps to A , and $1'$ maps to B , and $[0, 1]$ maps to x . Finally, they are *precisely separated by a*

continuous function iff there exists a continuous function f from X to \mathbb{R} such that the preimage $f^{-1}(\{0\}) = A$ and $f^{-1}(\{1\}) = B$. i.e. iff i_{AB} factors as

$$X \longrightarrow [0, 1] \longrightarrow \{A \leftrightarrow x \leftrightarrow B\}$$

where 0 goes to point A and 1 goes to point B .

These conditions are given in order of increasing strength: Any two topologically distinguishable points must be distinct, and any two separated points must be topologically distinguishable. Any two separated sets must be disjoint, any two sets separated by neighbourhoods must be separated, and so on.

The definitions below all use essentially the preliminary definitions above.

In all of the following definitions, X is again a topological space.

X is T0, or Kolmogorov, if any two distinct points in X are topologically distinguishable. (It will be a common theme among the separation axioms to have one version of an axiom that requires T0 and one version that doesn't.) As a formula, this is expressed as

$$\{x \leftrightarrow y\} \longrightarrow \{x = y\} \prec X \longrightarrow \{*\}$$

X is R0, or symmetric, if any two topologically distinguishable points in X are separated, i.e.

$$\{x \searrow y\} \longrightarrow \{x \leftrightarrow y\} \prec X \longrightarrow \{*\}$$

X is T1, or accessible or Frechet, if any two distinct points in X are separated, i.e.

$$\{x \searrow y\} \longrightarrow \{x = y\} \prec X \longrightarrow \{*\}$$

Thus, X is T1 if and only if it is both T0 and R0. (Although you may say such things as "T1 space", "Frechet topology", and "Suppose that the topological space X is Frechet", avoid saying "Frechet space" in this context, since there is another entirely different notion of Frechet space in functional analysis.)

X is R1, or preregular, if any two topologically distinguishable points in X are separated by neighbourhoods. Every R1 space is also R0.

X is weak Hausdorff, if the image of every continuous map from a compact Hausdorff space into X is closed. All weak Hausdorff spaces are T1, and all Hausdorff spaces are weak Hausdorff.

X is Hausdorff, or T2 or separated, if any two distinct points in X are separated by neighbourhoods, i.e.

$$\{x, y\} \hookrightarrow X \prec \{x \searrow X \swarrow y\} \longrightarrow \{x = X = y\}$$

Thus, X is Hausdorff if and only if it is both T0 and R1. Every Hausdorff space is also T1.

X is $T2\frac{1}{2}$, or Urysohn, if any two distinct points in X are separated by closed neighbourhoods, i.e.

$$\{x, y\} \hookrightarrow X \times \{x \searrow x' \swarrow X \searrow y' \swarrow y\} \longrightarrow \{x = x' = X = y' = y\}$$

Every $T2\frac{1}{2}$ space is also Hausdorff.

X is completely Hausdorff, or completely $T2$, if any two distinct points in X are separated by a continuous function, i.e.

$$\{x, y\} \hookrightarrow X \times [0, 1] \longrightarrow \{*\}$$

where $\{x, y\} \hookrightarrow X$ runs through all injective maps from the discrete two point space $\{x, y\}$.

Every completely Hausdorff space is also $T2\frac{1}{2}$.

X is regular if, given any point x and closed subset F in X such that x does not belong to F , they are separated by neighbourhoods, i.e.

$$\{x\} \longrightarrow X \times \{x \searrow X \swarrow U \searrow F\} \longrightarrow \{x = X = U \searrow F\}$$

(In fact, in a regular space, any such x and F will also be separated by closed neighbourhoods.) Every regular space is also $R1$.

X is regular Hausdorff, or $T3$, if it is both $T0$ and regular.[1] Every regular Hausdorff space is also $T2\frac{1}{2}$.

X is completely regular if, given any point x and closed set F in X such that x does not belong to F , they are separated by a continuous function, i.e.

$$\{x\} \longrightarrow X \times [0, 1] \cup \{F\} \longrightarrow \{x \searrow F\}$$

where points F and 1 are topologically indistinguishable, $[0, 1]$ goes to x , and F goes to F .

Every completely regular space is also regular.

X is Tychonoff, or $T3\frac{1}{2}$, completely $T3$, or completely regular Hausdorff, if it is both $T0$ and completely regular.[2] Every Tychonoff space is both regular Hausdorff and completely Hausdorff.

X is normal if any two disjoint closed subsets of X are separated by neighbourhoods, i.e.

$$\emptyset \longrightarrow X \times \{x \swarrow x' \searrow X \swarrow y' \searrow y\} \longrightarrow \{x \swarrow x' = X = y' \searrow y\}$$

In fact, by Urysohn lemma a space is normal if and only if any two disjoint closed sets can be separated by a continuous function, i.e.

$$\emptyset \longrightarrow X \times \{0'\} \cup [0, 1] \cup \{1'\} \longrightarrow \{0 = 0' \swarrow x \swarrow 1 = 1'\}$$

where points $0', 0$ and $1, 1'$ are topologically indistinguishable, $[0, 1]$ goes to x , and both $0, 0'$ map to point $0 = 0'$, and both $1, 1'$ map to point $1 = 1'$.

X is normal Hausdorff, or T_4 , if it is both T_1 and normal. Every normal Hausdorff space is both Tychonoff and normal regular.

X is completely normal if any two separated sets A and B are separated by neighbourhoods $U \supset A$ and $V \supset B$ such that U and V do not intersect, i.e.

$$\emptyset \longrightarrow X \times \{X \not\leftarrow A \leftrightarrow U \searrow U' \not\leftarrow W \searrow V' \not\leftarrow V \leftrightarrow B \searrow X\} \longrightarrow \{U = U', V' = V\}$$

Every completely normal space is also normal.

X is perfectly normal if any two disjoint closed sets are precisely separated by a continuous function, i.e.

$$\emptyset \longrightarrow X \times [0, 1] \longrightarrow \{0 \not\leftarrow X \searrow 1\}$$

where $(0, 1)$ goes to the open point X , and 0 goes to 0, and 1 goes to 1.

Every perfectly normal space is also completely normal.

X satisfies T_d iff each point x is the intersection of an open set U and a closed set Z , $\{x\} = U \cap Z$, i.e. each map the map $\{x\} \longrightarrow X$ has the left lifting property wrt

$$\begin{aligned} & \{U \bar{Z} \bar{x} \searrow U Z x \searrow \bar{U} Z x, U \bar{Z} x \leftrightarrow U \bar{Z} \bar{x} \searrow \bar{U} \bar{Z} x \leftrightarrow \bar{U} \bar{Z} \bar{x} \searrow \bar{U} Z x \leftrightarrow \bar{U} Z \bar{x}\} \\ & \longrightarrow \{U \bar{Z} \bar{x} = \bar{U} \bar{Z} \bar{x} = \bar{U} Z \bar{x} \leftrightarrow U Z x = \bar{U} Z x = U \bar{Z} x = \bar{U} \bar{Z} x\} \end{aligned}$$

X is extremally disconnected if the closure of every open subset of X is open, i.e.

$$\emptyset \longrightarrow X \times \{U \searrow Z', Z \not\leftarrow V\} \longrightarrow \{U \searrow Z' = Z \not\leftarrow V\}$$

or equivalently

$$\emptyset \longrightarrow X \times \{U \searrow Z', Z \not\leftarrow V\} \longrightarrow \{Z' = Z\}$$

It is not clear if the property of being sober can be expressed as a lifting property.

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