STANDARD CONJECTURES IN MODEL THEORY, AND CATEGORICITY OF COMPARISON ISOMORPHISMS. A MODEL THEORY PERSPECTIVE.

by

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Abstract. — We formulate two conjectures about étale cohomology and fundamental groups motivated by categoricity conjectures in model theory.

One conjecture says that there is a unique $\mathbb{Z}$-form of the étale cohomology of complex algebraic varieties, up to $Aut(\mathbb{C})$-action on the source category; put differently, each comparison isomorphism between Betti and étale cohomology comes from a choice of a topology on $\mathbb{C}$.

Another conjecture says that each functor to groupoids from the category of complex algebraic varieties which is similar to the topological fundamental groupoid functor π_1^{top}, in fact factors through π_1^{top}, up to a field automorphism of the complex numbers acting on the category of complex algebraic varieties.

We also try to present some evidence towards these conjectures, and show that some special cases seem related to Grothendieck standard conjectures and conjectures about motivic Galois group.

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1. Introduction

We consider the following question as it would be understood by a model theorist

Question. — *Is there a purely algebraic definition of the notion of singular (Betti) cohomology or the topological fundamental groupoid of a complex algebraic variety?*

and formulate precise conjectures proposing that the comparison isomorphism of étale cohomology/fundamental groupoid admits such a purely algebraic definition (characterisation).

Note that an algebraic geometer might interpret the question differently from a logician and in that interpretation, the answer is well-known to be negative.

For a logician this is a question about the desirable property of *categoricity*: a theory (i.e. a collection of properties expressed in a language) of a model is called *categorical* iff any two models of the theory (i.e. any two structures satisfying the properties) are necessarily isomorphic. In the approach of logically perfect structures [MoralesZilber], [Zilber] constructs a number of categorical theories associated with complex analytic functions and particularly exponentiation [BaysKirby], or rather their conjectural properties related to the field structure, such as Schanuel-type conjectures on transcendence of values, Kummer theory and Galois ℓ-adic representations (seen as being about the field automorphism action on the sequences of form

$e^z, e^{z/2}, e^{z/3}, ...$). In algebraic geometry, this kind of conjectures occurs in the conjectural theory of the motivic Galois group and the conjectural theory of pure motives enabled by Grothendieck Standard Conjectures, e.g. Mumford-Tate and Grothendieck periods conjecture. Hence "standard conjectures in model theory" in the title refer both to the Grothendieck Standard Conjectures and to the Zilber's standard conjectures in the approach of logically perfect structures claiming categoricity of certain theories associated with complex analytic functions.

We now explain our motivation in two essentially independent ways. §1.1 explains how a model theorist would interpret the question above; §1.2 views these conjectures as continuation of work in model theory on the complex field with pseudoexponentiation [Zilber, Bays-Zilber, Bays-Kirby. Manin-Zilber] and its main goal is to make the reader aware of the possibilities offered by methods of model theory.

1.1. How to interpret the question.— Let us now explain the difference between how an algebraic geometer and a model theorist might interpret the question.

Let H_{top} be a functor defined on the category $\mathcal{V}ar$ of algebraic varieties (say, separated schemes of finite type) over the field $\mathbb{C}$ of complex numbers; we identify this category with a subcategory of the category of topological spaces. We shall be interested in the case when H_{top} is either the functor $H_{\text{sing}} : \mathcal{V}ar \longrightarrow Ab$ of singular cohomology or the fundamental groupoid functor $\pi_1^{top} : \mathcal{V}ar \longrightarrow Groupoids$.

An algebraic geometer might reason as follows. A purely algebraic definition applies both to H_{top} and $H_{\text{top}} \circ \sigma$ where $\sigma : \mathbb{C} \longrightarrow \mathbb{C}$ is a field automorphism. Hence, to answer the question in the negative, it is enough to find a field automorphism $\sigma : \mathbb{C} \longrightarrow \mathbb{C}$ such that H_{top} and $H_{\text{top}} \circ \sigma$ differ. And indeed, [Serre, Exemple] constructs an example of a projective algebraic variety X and a field automorphism σ such that $X(\mathbb{C})$ and $X^\sigma(\mathbb{C})$ have non-isomorphic fundamental groups.

A model theorist might reason as follows. A purely algebraic definition applies both to H_{top} and $H_{\text{top}} \circ \sigma$ where $\sigma : \mathbb{C} \longrightarrow \mathbb{C}$ is a field automorphism. Hence, we should try to find purely algebraic description (possibly involving extra structure) of H_{top} which fits precisely functors of form $H_{\text{top}} \circ \sigma, \sigma \in Aut(\mathbb{C})$ with the extra structure. We say that *such a purely algebraic description determines* H_{top} *(with the extra structure) uniquely up to an automorphism of* $\mathbb{C}$.

For $H_{\text{top}} = H_{\text{sing}}$ the singular (Betti) cohomology theory, a model theorist might continue thinking as follows. The singular (Betti) cohomology theory admits a comparison isomorphism to a cohomology theory defined purely algebraically, say ℓ-adic étale cohomology theory. This is an algebraic description in itself. However, note that it considers the ℓ-adic étale cohomology theory and the comparison isomorphism as part of structure. Thus an appropriate conjecture (see §2) gives a purely algebraic description of the family of comparison isomorphisms coming from a choice of topology on $\mathbb{C}$

$$H_{\text{sing}}(X(\mathbb{C})_{top}, \mathbb{Z}) \otimes \mathbb{Z}_l \xrightarrow{\sigma} H_{et}(X(\mathbb{C}), \mathbb{Z}_l), \quad \sigma \in Aut(\mathbb{C})$$

For $H_{\text{top}} = \pi_1^{top}$ a model theorist might continue thinking as follows. The profinite completion of the topological fundamental groupoid functor is the étale fundamental groupoid defined algebraically. This is an algebraic property of the topological fundamental groupoid on which we can base our purely algebraic description if we include the étale fundamental groupoid as part of structure. Essentially, this describes subgroupoids of the étale fundamental groupoids. Category theory suggests to consider a related universality property (see §3): *up to* $Aut(\mathbb{C})$ *action on the category of complex algebraic varieties*, there is a universal functor among those whose profinite completion embeds into the étale fundamental groupoid, and it is the topological fundamental groupoid. Some technicalities may be necessary to ignore non-residually finite fundamental groups.

1.1.1. Structure of the paper.— In the introduction we explain our motivation in two essentially independent ways. §1.1 explains how a model theorist would interpret the question above; §1.2 views these conjectures as a continuation of work in model theory on the complex field with pseudoexponentiation [Zilber, Bays-Zilber, Bays-Kirby], see also [Manin-Zilber, Ch. X.6.11-6.19] and its main goal is to make the reader aware of the possibilities offered by methods of model theory. In §2.1 we begin by formulating two conjectures on étale cohomology. Conjecture 2.1 says there is a unique there is a unique $\mathbb{Z}$-form of the étale cohomology of complex algebraic varieties, up to $Aut(\mathbb{C})$-action on the source category; put differently, each comparison isomorphism between Betti and étale cohomology comes from a choice of topology on $\mathbb{C}$. Conjecture 2.2 states a similar property for the Weil cohomology theory restricted to a category generated by a single pure motive. In §2.2 Proposition 2.3 and Statement 2.4 give examples of similar properties which hold for a subcategory generated by a single abelian variety. §2.3 contains a concise exposition of several conjectures on the motivic Galois group which appear related to our conjectures; it follows [Serre,1.3]. §2.4 contains some speculations about the model theoretic point of view on these conjectures. §3 states analogous conjectures for the etale fundamental group. In §3.1 we define the notion of a π_1-like functor and state two conjectures saying there is a universal π_1-like functor up to a field automorphism. In §3.2 we state two conjectures which we hope to be within reach of current methods. In §3.3 we list several partial positive results which are implicit in model theoretic literature. We end by §3.4 which mentions mathematical facts which go into the proofs of these results.

1.2. Pseudo-exponentiation, Schanuel conjecture and categoricity theorems in model theory.— Complex topology allows to construct a number of objects with good algebraic properties e.g. a group homomorphism $\exp : \mathbb{C}^+ \longrightarrow \mathbb{C}^*$, singular (Betti) cohomology theory and the topological fundamental groupoid of varieties of complex algebraic varieties.

A number of theorems and conjectures says that such an object constructed topologically or analytically is "free" or "generic", for lack of better term, in the sense that it satisfies algebraic relations only, or mostly, for "obvious" reasons of algebraic nature.

Sometimes such a conjecture is made precise by saying that a certain automorphism group is as large as possible subject to some "obvious obstructions or relations" imposed by functoriality and/or homotopy theory. Such an automorphism group may involve values of functions or spaces defined analytically or topologically.

A natural question to ask is whether these conjectures are "consistent" in the sense that there do exist such "free" objects with the conjectured properties, not necessarily of analytic or topological origin.

Methods of model theory allow to build such objects by an elaborate transfinite induction. In what follows we shall sketch results of [Zilber, Bays-Kirby] which does this for the complex exponential function and Schanuel conjecture.

Let us now explain what we mean by showing how to view Kummer theory, Hodge conjecture, conjectural theory of the motivic Galois group, and Schanuel conjecture in this way, i.e. as saying certain objects satisfies algebraic relations only, or mostly, for "obvious" reasons of algebraic nature.

1.2.1. Grothendieck Standard Conjectures.— In words of [Grothendieck], '[these conjectures] would form a basis of the so-called theory of "pure motives" which is a systematic theory of "arithmetic properties" of algebraic varieties, as embodied in their group of classes of cycles for numerical equivalence.' For the purposes of this paper, it is important that these conjectures imply that there are many automorphisms of the Betti cohomology theory of the complex algebraic varieties, and are needed to formulate the conjectural theory of the motivic Galois group $Aut^{\otimes}(H_{\mathrm{sing}_\sigma})$, cf. §1.2.4.

A logician would wish to interpret the words of Grothendieck as follows. View a cohomology theory as a language which extends the usual theory ACF of algebraically closed fields by being able to talk about classes of algebraic cycles up to numerical equivalence; Grothendieck says this language is also able to talk about arithmetical properties of algebraic varieties. In model theory it is natural to assume (or wish) that such a structure is homogeneous and perhaps also categorical, which is a much stronger property. In this way, it is tempting to think of the motivic Galois group as the automorphism group of such a structure, and therefore quite large. However, it is unclear how to make this point of view precise.

In this paper we do not need the precise statements of the Standard Conjectures; see [Grothendieck, Kleiman] for details.

1.2.2. Kummer theory.— An "obvious way" to make $e^{\alpha_1/N}, ..., e^{\alpha_n/N}$, $N > 0$ satisfy a polynomial relation is is to pick $\alpha_1, ..., \alpha_n$ such that they satisfy a $\mathbb{Q}$-linear relation over $2\pi i$, which is preserved by exp, or such that that $e^{\alpha_1/M}, ..., e^{\alpha_n/M}$ satisfy a polynomial relation for some other M.

Kummer theory tells you these are the only reasons for polynomial relations between these numbers. This is stated precisely in terms of automorphisms groups as follows:

For any $\mathbb{Q}$-linearly independent numbers $\alpha_1, ..., \alpha_n \in \mathbb{C}$ there is $N > 0$ such that for any $m > 0$ it holds

$$\mathrm{Gal}(\mathbb{Q}(e^{\frac{\alpha_1}{mN}}, ..., e^{\frac{\alpha_n}{mN}}, e^{2\pi i\mathbb{Q}})/\mathbb{Q}(e^{\frac{\alpha_1}{N}}, ..., e^{\frac{\alpha_n}{N}}, e^{2\pi i\mathbb{Q}})) \approx (\mathbb{Z}/m\mathbb{Z})^n$$

1.2.3. Hodge conjecture. — Consider the Hodge theory of a non-singular complex projective manifold $X(\mathbb{C})$. By Chow theory we know that X is in fact a complex algebraic variety and an easy argument using harmonic forms shows that an algebraic subvariety $Z(\mathbb{C})$ defines an element of $H(X,\mathbb{Q}) \cap H^{(p,p)}(X,\mathbb{C})$ where $H^{(p,p)}(X,\mathbb{C})$ is a certain linear subspace of $H^{2p}(X,\mathbb{C})$ defined analytically.

A topological cycle in $X(\mathbb{C})$ defines an element of $H(X,\mathbb{Q})$ which may lie in $H^{(p,p)}(X,\mathbb{C})$. An "obvious reason" for this is that it comes from an algebraic subvariety, or a $\mathbb{Q}$-linear combination of such. Hodge conjecture tells you that this is the only reason why it could happen.

1.2.4. ℓ-adic Galois representations and motivic Galois group $Aut^{\otimes}(H_{sing_\sigma})$. — Remarks below are quite vague but we hope some readers might find them helpful. In §2.3 we sketch several definitions and conjectures in the conjectural theory of motivic Galois group following [Serre].

We would like to think that these conjectures say that the singular (Betti) cohomology theory of complex algebraic varieties is "free" in the sense that it satisfies algebraic relations only, or mostly, for "obvious" reasons of algebraic nature. The theory of the motivic Galois group assumes that there are many automorphisms of the singular cohomology theory of complex algebraic varieties, and they form a pro-algebraic, in fact pro-reductive ([Serre, Conjecture 2.1?], group. Conjectures on ℓ-adic Galois representations, e.g. [Serre, Conjecture 3.2?,9.1?], describe the image of Galois action as being dense or open in a certain algebraic group defined by the cohomology classes which the Galois action has to preserve (or is conjectured to preserve).

Let us very briefly sketch some details.

The conjectural theory of the motivic Galois group [Serre], also cf. §2.3, assumes that the following is a well-defined algebraic group:

$$G_E = Aut^{\otimes}\left(H_{\mathrm{sing}_\sigma} : \langle E\rangle \longrightarrow \mathbb{Q}\text{-Vect}\right)$$

Here $\sigma : k \longrightarrow \mathbb{C}$ is an embedding of a number field k into the field of complex numbers, E is a pure motive in the conjectural category $\mathcal{M}ot_k$ of pure motives defined over k, and $\langle E\rangle$ is the least Tannakian subcategory of $\mathcal{M}ot_k$ containing E, and H_{sing_σ} is the fibre functor on $\langle E\rangle$ corresponding to the singular cohomology of complex algebraic varieties and embedding $\sigma : k \longrightarrow \mathbb{C}$. This is well-defined if we assume certain conjectures, e.g. Standard Conjectures and Hodge conjecture [Serre, Grothendieck, Kleiman].

[Serre, Conjecture 3.1?] says that G_E is the subgroup of $GL(H_{\mathrm{sing}_\sigma}(E))$ preserving the tensors corresponding to morphisms $\mathbf{1} \longrightarrow E^{\otimes r} \otimes E^{\vee\otimes s}$, $r,s \geq 0$. Think of these tensors as "obvious relations" which have to be preserved.

[Serre, Conjecture 3.2? and Conjecture 9.1?] describe the image of ℓ-adic Galois representations in $G_E(\mathbb{Q}_l)$.

Both say it is dense or open in the group of ℓ-adic points of a certain algebraic subgroup of GL_N; we think of this subgroup as capturing "obvious obstructions or relations" imposed by functoriality of H_{sing}.

1.2.5. Schanuel conjecture: questions. — Schanuel conjecture says that for $\mathbb{Q}$-linearly independent $x_1,...,x_n \in \mathbb{C}$, the transcendence degree of $x_1,...,x_n,e^{x_1},...,e^{x_n}$ is at least n:

$$\mathrm{tr.deg.}_{\mathbb{Q}}(x_1,...,x_n,e^{x_1},...,e^{x_n}) \geq \mathrm{lin.deg.}_{\mathbb{Q}}(x_1,...,x_n) \qquad \text{(SC)}$$

The bound becomes sharp if we use surjectivity to pick $x_2 = e^{x_1}$, ..., $x_{i+1} = e^{x_i}, ..., x_n = e^{x_{n-1}}$ and $e^{x_n} \in \mathbb{Q}$:

$$\text{tr.deg.}_{\mathbb{Q}}(x_1, e^{x_1}, e^{e^{x_1}}, ..., x_n, e^{x_1}, e^{e^{x_1}}, ..., e^{x_n}) \leq n$$

Here "an algebraic relation" is a polynomial relation between $x_1, ..., x_n, e^{x_1}, ..., e^{x_n}$; an obvious way to make these numbers satisfy such a relation is to pick x_i such that either $x_i = a_1x_1 + ... + a_{i-1}x_{i-1}$ or $e^{x_i} = a_1x_1 + ... + a_{i-1}x_{i-1}$ or $x_i = e^{a_1x_1+...+a_{i-1}x_{i-1}}$ where $a_1, ..., a_i \in \mathbb{Q}$ are rational.

Is Schanuel conjecture "consistent" in the sense that there is a *pseudo-exponentiation*, i.e. a group homomorphism $\text{ex} : \mathbb{C}^+ \longrightarrow \mathbb{C}^*$ satisfying conjectural properties of complex exponentiation, in particular Schanuel conjecture? Does there exist such a "free" pseudo-exponentiation $\text{ex} : \mathbb{C}^+ \longrightarrow \mathbb{C}^*$, e.g. such that a system of exponential-polynomial equations has a zero only iff it does not contradict Schanuel conjecture? Can we build such an algebraic "free" object without recourse to topology?

Does every such "free" object come from a choice of topology on $\mathbb{C}$, i.e. is the complex exponential $\exp : \mathbb{C}^+ \longrightarrow \mathbb{C}^*$ up an automorphism of $\mathbb{C}$?

Note that the last question is the only one which mentions topology. It turns out this difference is crucial: model theory says nothing about this question while giving fairly satisfactory positive answers to the previous ones.

1.2.6. Schanuel conjecture and pseudoexponentiation: answers. — The following theorem of [Zilber] provides a positive answer for $\exp : \mathbb{C}^+ \longrightarrow \mathbb{C}^*$. For a discussion of the theorem and surrounding model theory see [Manin-Zilber, 6.16]; for a proof, detailed statements and generalisations to other analytic functions see [Bays-Kirby, Thm. 1.2,Thm. 1.6; Thm. 9.1; also Thm. 8.2; Thm. 9.3] and references therein.

***Theorem 1.1* (Zilber).** — *Let K be an uncountable algebraically closed field of characteristic* 0.

Up to $Aut(K)$, there is a unique surjective group homomorphism $\text{ex} : K^+ \longrightarrow K^*$

(SK) *(Standard Kernel)* $\text{Ker}\,\text{ex}$ *is the infinite cyclic group generated by a transcendental element*

(SC) *(Schanuel Property)* $\text{ex} : K^+ \longrightarrow K^*$ *satisfies Schanuel conjecture*

(SEAC) *(Strong exponential-algebraic closedness) any system of n independent exponential-polynomial equations in n variables that does not directly contradict Schanuel conjecture has a regular zero, but not more than countably many*

Call this unique group homomorphism $\text{ex} : K^+ \longrightarrow K^*$ *pseudoexponentiation* defined on field K.

In somewhat more detail, this can also be expressed as follows.

Let K and K' be two uncountable algebraically closed fields of characteristic 0, and let $\text{ex} : K^+ \longrightarrow K^*$ and $\text{ex}' : K'^+ \longrightarrow K'^*$ be group homomorphisms satisfying the properties above.

Then if there is a bijection $\sigma_0 : K \overset{\approx}{\longrightarrow} K'$, then there is a bijection $\sigma : K \overset{\approx}{\longrightarrow} K'$ preserving +, ·, and ex, i.e. such that for each $x, y \in K$ it holds

$$\sigma(x + y) = \sigma(x) + \sigma(y), \quad \sigma(xy) = \sigma(x)\sigma(y), \quad \sigma(\text{ex}(x)) = \text{ex}'(\sigma(x))$$

***Conjecture 1.2* (Zilber).** — *If* $\operatorname{card} K = \operatorname{card} \mathbb{C}$*, then* $(K, +, \cdot, \mathrm{ex})$ *is isomorphic to* $(\mathbb{C}, +, \cdot, \exp)$.

Our conjectures are direct analogues of the Theorem and Conjecture above stated in the language of functors. Instead of the complex exponentiation we consider the comparison isomorphisms between topological and étale cohomology, resp. fundamental groupoid functor. We hope that model theoretic methods used by [Zilber] may be of use in proving these conjectures.

1.2.7. Pseudoexponentiation: automorphisms groups. — It is known that certain automorphisms groups associated with pseudoexp are largest possible in the following sense. The purpose of this section is to give a quick summary for the reader who might want to follow up our references to literature in model theory.

We need some preliminary definitions. We say that tuples a and b in K have *the same quantifier-free type*, write $\mathrm{qftp}(a) = \mathrm{qftp}(b)$, iff they satisfy the same exponential-polynomial equations, and, moreover, the same exponential-polynomial equations with coefficients from a, resp. b, have a solution; see [Bays-Kirby, §6, Def. 6.7] for details. Note that for a finite tuple a in K, there is a minimal $\mathbb{Q}$-linear vector subpace $A \supset a$ such that $A \leq_\delta K$ and this A determines $\mathrm{qftp}(a)$ (see below for the definition of $\leq_\delta$).

We quote from [Bays-Kirby, Def. 6.1, Proposition 6.5].

Fact 1.3. — *Let K be a field with pseudoexponentiation as defined above.*

QM4. *(Uniqueness of the generic type) Suppose that* $C, C' \subset M$ *are countable closed subsets, enumerated such that* $\mathrm{qftp}(C) = \mathrm{qftp}(C')$*. If* $a \in M \setminus C$ *and* $a' \in M \setminus C'$ *then* $\mathrm{qftp}(C, a) = \mathrm{qftp}(C', a')$ *(with respect to the same enumerations for C and C').*

QM5. *($\aleph_0$-homogeneity over closed sets and the empty set) Let* $C, C' \subset K$ *be countable closed subsets or empty, enumerated such that* $\mathrm{qftp}(C) = \mathrm{qftp}(C')$*, and let* b, b' *be finite tuples from K such that* $qftp(C, b) = \mathrm{qftp}(C', b')$*, and let* $a \in cl(C, b)$*. Then there is* $a' \in K$ *such that* $qftp(C, b, a) = \mathrm{qftp}(C', b', a')$.

QM5a. *($\aleph_0$-homogeneity over the empty set) If a and b are finite tuples from K and* $\mathrm{qftp}(a) = \mathrm{qftp}(b)$ *then there is a field automorphism* $\theta : K \longrightarrow K$ *preserving* $\mathrm{ex} : K^+ \longrightarrow K^*$ *such that* $\theta(a) = b$.

Note that it is an open problem to construct a non-trivial automorphism of $(\mathbb{C}, +, \cdot, \exp)$ which is not the complex conjugation. An easy argument shows that any continuous or measurable field automorphism of $\mathbb{C}$ is in fact either trivial or the complex conjugation.

Note that the field with the pseudoexponentiation is highly homogeneous and has a large group of automorphisms.

1.2.8. Remarks about the proof.— We adapt [Manin-Zilber, 6.11-6.16]; see also [Bays-Kirby] for a detailed exposition in a more general case using different terminology. Pseudoexponentiation is constructed by an elaborate transfinite induction. We start with an algebraically closed field $K_{\mathrm{base}} \subset K$ and a partial group homomorphism

$\mathrm{ex} : K^+_{\mathrm{base}} \dashrightarrow K^*_{\mathrm{base}}$ and try to extend the field and the group homomorphism such that it is related to the field in as free a way as possible.

Informally the freeness condition is described as follows:

(Hr) the number of independent explicit basic dependencies *added* to a subset $X \cup \mathrm{ex}(X)$ of K by the new structure is at most the dimension of $X \cup \mathrm{ex}(X)$ in the old structure.

This is made precise in the following way.

The new structure is the group homomorphism $\mathrm{ex} : K^+ \longrightarrow K^*$; *explicit basic dependencies in* $X \cup \mathrm{ex}(X)$ *added by the new structures* are defined as as equations $\mathrm{ex}(x) = y$ where $x \in X$. For example, for $X = \{x\}$ where $\mathrm{ex}(\mathrm{ex}(x)) = x$, we do not regard $\mathrm{ex}(\mathrm{ex}(x)) = x$ as a explicit basic dependency in $X \cup \mathrm{ex}(X) = \{x, \mathrm{ex}(x)\}$.

The number of independent basic explicit dependencies is the $\mathbb{Q}$-linear dimension $\mathrm{lin.dim.}_{\mathbb{Q}}(x_1, ..., x_n)$; *the dimension of* X *in the old structure* is its transcendence degree which is equal to $\mathrm{tr.deg.}(x_1, ..., x_n, \mathrm{ex}(x_1), ..., \mathrm{ex}(x_n))$.

With this interpretation, (Hr) becomes Schanuel conjecture (SC).

Define *Hrushovski predimension* $\delta(X) := \mathrm{tr.deg.}(X \cup \mathrm{ex}(X)) - \mathrm{lin.dim.}_{\mathbb{Q}}(X)$. Say a partial group homomorphism $\mathrm{ex} : K^+ \dashrightarrow K^*$ satisfies *Hrushovski inequality with respect to Hrushovski predimension* δ iff for any finite $X \subset K$ it holds $\delta(X) \geq 0$. An extension $(K, \mathrm{ex}_K) \subset (L, \mathrm{ex}_L)$ of fields equipped with partial group homomorphisms is *strong*, write $K \leq_\delta L$, iff all dependencies between elements of K occurring in L can be detected already in K, i.e. for every finite $X \subset K$,

$$\min\{\, \delta(Y) : Y \text{ finite, } X \subset Y \subset K \,\} = \min\{\, \delta(Y) : Y \text{ finite, } X \subset Y \subset L \,\}$$

We then build a countable algebraically closed field $(K_{\aleph_0}, \mathrm{ex}_{K_{\aleph_0}})$ by taking larger and larger strong extensions $K_{\mathrm{base}} \leq_\delta K_1 \leq_\delta K_2 \leq_\delta ...$ of finite degree. If we do this with enough care, we obtain a countable algebraically closed field $K_{\aleph_0} = \cup K_n$ and a group homomorphism $\mathrm{ex}_{K_{\aleph_0}} : K^+_{\aleph_0} \longrightarrow K^*_{\aleph_0}$ defined everywhere which satisfies (SC) and other conditions of Theorem 1.1. For details see [Bays-Kirby, §5] where it is described in terms of taking Fraisse limit along a category of strong extensions.

Building an uncountable model requires deep model theory; see [Bays-Kirby, §6] and [BH^{2}K^{2}14]. Let us say a couple of words about this. In the inductive construction above, being countable is essential: if we start with an uncountable field, we can no longer hope to obtain an algebraically closed field after taking union of countably many extensions of finite degree. Very roughly, it turns out that we can construct composites of countable linearly disjoint algebraically closed fields this way, and this helps to build an uncountable field with pseudoexponentiation and prove it is unique in its cardinality up to isomorphism.

1.2.9. Generalisations and Speculations.— [Bays-Kirby] generalises the considerations above in a number of ways. In particular, they construct pseudo-exponential maps of simple abelian varieties, including pseudo-$\wp$-functions for elliptic curve. [Proposition 10.1, §10, ibid.] relates the Schanuel property of these to the André-Grothendieck conjecture on the periods of 1-motives. They suspect that for abelian varieties the predimension inequality $\delta(X) \geq 0$ also follows from the André-Grothendieck periods conjecture, but there are more complications because the Mumford-Tate group plays a role and so have not been able to verify it. [§9.2, ibid.] says it is possible to construct a pseudoexponentiation incorporating a counterexample to Schanuel conjecture, by suitably modifying the Hrushovski predimension and thus the inductive assumption (Hr). [§9.7, also Thm. 1.7, ibid.] considers differential equations.

We intentionally leave the following speculation vague.

Speculation 1.4. — *Can one build a pseudo-singular, or pseudo-de Rham cohomology theory, or a pseudo-topological fundamental group functor of complex algebraic varieties, or an algebra of pseudo-periods which satisfies a number of conjectures such as the Standard Conjectures, the conjectural theory of the motivic Galois group, the conjectures on the image of ℓ-adic Galois representations, André-Grothendieck periods conjecture, Mumford-Tate conjecture, etc.?*

1.3. A glossary of terminology in model theory.— We give a very quick overview of basic terminology used in model theory, to help the reader follow up references to model theoretic literature. See [Tent-Ziegler; Manin-Zilber] for an introduction into model theory.

In logic, a property is called *categorical* iff any two structures (models) satisfying the property are necessarily isomorphic. A *structure* or *a model* is usually understood as a set X equipped with names for certain distinguished subsets of its finite Cartesian powers X^n, $n > 0$, called *predicates*, and also equipped with names for certain distinguished functions between its finite Cartesian powers. Names of predicates and functions form a *language. First order formulas in language* L is a particular class of formulas which provide names for subsets obtained from the L-distinguished subsets by taking finitely many times intersection, union, completion, and projection onto some of the coordinates; a formula $\varphi(x_1, .., x_n)$ defines the subset $\varphi(M^n)$ of M^n consisting of tuples satisfying the formula. A *theory in language* L is a collection of formulas in language L. *A model* of a theory T in language L is a structure in language L such that for each $\varphi \in T$ $\varphi(M^n) = M^n$ where n is the arity of φ.

The first order theory of a structure consists of all possible names (formulas) for the subsets $M^n, n \geq 0$, i.e. formulas φ such that $\varphi(M^n) = M^n$.

A *categoricity* theorem in model theory usually says that any two models of a first order theory of the same uncountable cardinality are necessarily isomorphic, i.e. if there is a bijection between (usually assumed uncountable) models M_1 and M_2 of the theory, then there is a bijection which preserves the distinguished subsets and functions. A theory is *uncountably categorical* iff it has a unique model, up to isomorphism, of each uncountable cardinality.

The *type* $\mathrm{tp}(a_1,...,a_n) = \{\varphi(x_1,..,x_n) : \varphi(a_1,...,a_n)$ *holds in* $M\}$ *of a tuple* $(a_1, .., a_n) \in M^n$ is the collection of all formulas satisfied by the tuple $(a_1, .., a_n)$. A *type in a theory* is the type of a tuple in a model of the theory. The *type* $\mathrm{tp}(a_1, ..., a_n) = \{ \varphi(x_1, .., x_n) :$ $\varphi(a_1, ..., a_n)$ *holds in* $M\}$ *of a tuple* $(a_1, .., a_n) \in M^n$ *with parameters in subset* $A \subset M$ is the collection of all formulas with parameters in A satisfied by the tuple $(a_1, .., a_n)$. A *type in a theory* is the type of a tuple in a model of the theory. Informally, the type of a tuple is a syntactic notion playing the role of an orbit of $Aut^L(M)$ on M^n, e.g. in a situation when we do not yet know whether non-trivial automorphisms of M exist, for example because M has not been completely constructed yet.

In an uncountably categorical first order theory with finitely many predicates and functions the number of types is at most countable, and the number of types with parameters in a subset A has cardinality at most $\operatorname{card} A + \aleph_0 = \max(\operatorname{card} A, \aleph_0)$.

2. Uniqueness property of comparison isomorphism of singular and étale cohomology of a complex algebraic variety

2.1. Statement of the conjectures. — A $\mathbb{Z}$*-form* of a functor $H_l : \mathcal{V} \longrightarrow \mathbb{Z}_l$-Mod is a pair (H, τ) consisting of a functor $H : \mathcal{V} \longrightarrow \mathbb{Z}$-Mod and an isomorphism

$$H \otimes_{\mathbb{Z}} \mathbb{Z}_l \xrightarrow{\tau} H_l$$

of functors.

An example of a $\mathbb{Z}$-form we are interested in is given by the comparison isomorphism between étale cohomology and Betti cohomology, see [SGA 4, XVI, 4.1], also [Katz,p.23] for the definitions and exact statements.

Let $H_{et} : Schemes \longrightarrow \mathbb{Z}$-Mod be the functor of ℓ-adic étale cohomology, and let $H_{sing} : Top \longrightarrow \mathbb{Z}$-Mod be the functor of singular cohomology. For X a separated $\mathbb{C}$-scheme of finite type there is a canonical *comparison isomorphism*

$$H_{sing}(X(\mathbb{C}), \mathbb{Z})) \otimes \mathbb{Z}_l \approx H_{et}(X, \mathbb{Z}_l).$$

This defines a $\mathbb{Z}$-form of the functor of ℓ-adic étale cohomology $H_{et}(-, \mathbb{Z}_l)$ restricted to the category of separated $\mathbb{C}$-schemes of finite type.

Let K be an algebraically closed field, let $\mathcal{V}_K$ be a category of varieties over K. A field automorphism $\sigma : K \longrightarrow K$ acts $X \mapsto X^\sigma$ on the category $\mathcal{V}_K$ by automorphisms. Moreover, for each variety X defined over K, a field automorphism σ defines an isomorphism $\sigma_X : X \longrightarrow X^\sigma$ of schemes (over $\mathbb{Z}$ or $\mathbb{Z}/p\mathbb{Z}$), and hence

$$H_{et}(X, \mathbb{Z}_l) \xrightarrow{\sigma_*} H_{et}(X^\sigma, \mathbb{Z}_l).$$

This defines an action of $Aut(K)$ on the $\mathbb{Z}$-forms of H_l:

$$(H, \tau) \longmapsto (H \circ \sigma, \tau \circ \sigma_*^{-1})$$

$$H(X^\sigma) \xrightarrow{\tau \circ \sigma_*^{-1}} H_{et}(X, \mathbb{Z}_l).$$

We conjecture that, up to action of $Aut(\mathbb{C})$ defined above, the comparison isomorphism between singular and ℓ-adic cohomology of is the only $\mathbb{Z}$-form of the ℓ-adic cohomology theory $H_{et}(-, \mathbb{Z}_l)$:

Conjecture 2.1 ($Z(H_{sing}, H_l)$). — *Up to* Aut($\mathbb{C}$) *action, there is a unique* $\mathbb{Z}$*-form of the* ℓ*-adic cohomology theory functor* $H_{et}(-, \mathbb{Z}_l)$ *restricted to the category of separated* $\mathbb{C}$*-schemes of finite type which respects the cycle map and Kunneth decomposition.*

In other words, every comparison isomorphism of a $\mathbb{Z}$- and the ℓ-adic cohomology theory of separated $\mathbb{C}$-schemes of finite type is, up to a field automorphism of $\mathbb{C}$, the standard comparison isomorphism

$$H_{sing}(X(\mathbb{C}),\mathbb{Z})\otimes\mathbb{Z}_l = H_{et}(X,\mathbb{Z}_l)$$

The conjecture is intended to be too optimistic; it is probably more reasonable to conjecture uniqueness of $\mathbb{Z}$-form of the *torsion-free* part of the ℓ-adic cohomology.

Assume Grothendieck Standard Conjectures and that, in particular the ℓ-adic cohomology theory factors via the category $\mathcal{M}ot_k$ of pure motives over a field k. Then, a Weil cohomology theory (cf. [Kleiman]) corresponds to a tensor fibre functor from the category of pure motives, and we may ask how many $\mathbb{Z}$-forms does have the fibre functor corresponding to the ℓ-adic cohomology theory. Moreover, we may formulate a "local" version of the conjecture restricting the functor to a subcategory generated by a single motive.

Conjecture 2.2 **($Z(H_{et},\langle E\rangle_k)$).** — *Let k be a number field. Assume Grothendieck Standard Conjectures and that, in particular, ℓ-adic cohomology factors via the category of pure numerical motives $\mathcal{M}ot_k$ over k.*

Let E be a motive and let $\langle E\rangle$ be the subcategory of $\mathcal{M}ot_k$ generated by E, i.e. the least Tannakian subcategory of $\mathcal{M}ot_k$ containing E. Up to $Aut(\bar{k}/k)$-action, the functor $H_{et}(-\otimes\bar{\mathbb{Q}},\mathbb{Z}_l):\langle E\rangle_k\longrightarrow\mathbb{Z}_l$-Mod has at most finitely many $\mathbb{Z}$-forms.

Moreover, if E has finitely many $\mathbb{Z}$-forms [Serre,10.2?], then the functor $H_{et}(-\otimes\bar{\mathbb{Q}},\hat{\mathbb{Z}}):\langle E\rangle_k\longrightarrow\hat{\mathbb{Z}}$-Mod has at most finitely many $\mathbb{Z}$-forms.

2.2. An example: an Abelian variety.— Let us give an example of a particular case of the conjecture which is easy to prove.

Proposition 2.3. — *Let A be an Abelian variety defined over a number field k. Assume that the Mumford-Tate group of A is the maximal possible, i.e. the symplectic group $MT(A)=GSp_{2g}$ where $\dim A=g$, and that the image of Galois action on the torsion has finite index in the group $GSp_{2g}(\mathbb{Z}_l)$ of $\mathbb{Z}_l$-points of the symplectic group.*

Then there are at most finitely many $\mathbb{Z}$-form of the ℓ-adic cohomology theory $H_{et}(-\otimes\bar{k},\mathbb{Z}_l)$ restricted to the category $\langle A\rangle_k$, up to $Aut(\bar{k}/k)$.

Proof (sketch). The Weil pairing corresponds to the divisor corresponding to an ample line bundle over A, and by compatibility with the cycle class map of a $\mathbb{Z}$-form and $H_l=H_{et}(-\otimes\bar{k},\mathbb{Z}_l)$ the non-degenerate Weil pairing

$$\omega:(H^1_l(A))^2\longrightarrow H^0_l(A)=\mathbb{Z}_l$$

restricts to a pairing

$$\omega:(H^1(A))^2\longrightarrow H^0(A)\approx\mathbb{Z},$$

which is easily seen to be non-degenerate.

Now let H_i be a $\mathbb{Q}$-form for $i=1,2$.

Let $(x^1_i,...x^g_i,y^1_i,...,y^g_i)$ be a symplectic basis for $H^1_i(A)$. Then each is also a symplectic basis for $H^1_{et}(A)$, and so some $\sigma\in GSp(H^1_{et}(A),\omega)$ maps $H^1_1(A)$ to $H^1_2(A)$. The assumption on the Mumford-Tate group precisely means that such a σ extends to $\sigma \in Aut(H_l|_{\langle A\rangle})$, and it follows from the fact that the cohomology of an Abelian variety is generated by H^1 that $\sigma(H_1) = H_2$.

Finally, use the assumption on the Galois representation to see that there are at most finitely many $\mathbb{Z}$-forms. □

It might be true that a reader familiar with the theory of the motivic Galois group may find that a straightforward generalisation of the argument above leads to the following.

***Statement 2.4* (a generalisation of the example).** — *Let A be a motive of a smooth projective variety defined over a number field k. Assume the Mumford-Tate group $G = Aut^{\otimes}(H_{et}(-\otimes \bar{k}, \mathbb{Z}_l)_{|\langle A\rangle})$ has the following property:*

> *if V_1 and V_2 are abelian subgroups of $H_{et}(A \otimes \bar{k}, \mathbb{Z}_l)$ which are both dense and of the same rank and such that*
>
> *$GL(V_i) \cap G(\mathbb{Z}_l)$ is dense in $G(\mathbb{Z}_l)$ for i=1,2,*
>
> *then there is a $g \in G(\mathbb{Z}_l)$ such that $gV_1 = V_2$ (setwise).*

Then the conjectures [2.1?,3.1?,3.2?,9.1?] of [Serre] imply that there are at most finitely many $\mathbb{Z}$-forms of $H_{et}(-\otimes \bar{k}, \mathbb{Z}_l)_{|\langle A\rangle}$.

Conjectures [2.1?,3.1?,3.2?,9.1?] have analogues the cohomology theories with coefficients in the ring of finite adeles $\mathbb{A}^f$, cf. [Serre, 11.4?(ii), 11.5?], cf. also [Serre, 10.2?, 10.6?].

2.3. Standard Conjectures and motivic Galois group. — Now we try to give a self-contained exposition of several conjectures on motivic Galois group which appear related to our conjectures. Our exposition follows [Serre,§1,§3]

Let k be a field of characteristic 0 which embeds into the field $\mathbb{C}$ of complex numbers; pick an embedding $\sigma : k \longrightarrow \mathbb{C}$.

Assume Standard Conjectures and Hodge conjecture [Grothendieck, Kleiman]. Let $\mathcal{M}ot_k$ denote the category of pure motives over k defined with the help of numerical equivalence of algebraic cycles (or the homological equivalence, which should be the same by Standard Conjectures). $\mathcal{M}ot$ is a semi-simple category.

Let $E \in Ob\mathcal{M}ot$ be a motive; let $\langle E\rangle$ denote the least Tannakian subcategory of $\mathcal{M}ot_k$ containing E.

A choice of embedding $\sigma : k \longrightarrow \mathbb{C}$ defines an exact *fibre functor* $\mathcal{M}ot_k \longrightarrow \mathbb{Q}$-Vect corresponding to *the Betti realisation*

$$H_{\mathrm{sing}_\sigma} : \mathcal{M}ot_k \longrightarrow \mathbb{Q}\text{-Vect}, \quad E \mapsto H_{\mathrm{sing}}(E_\sigma(\mathbb{C}), \mathbb{Q}).$$

The scheme of automorphisms $MGal_{k,\sigma} = Aut^{\otimes}(H_{\mathrm{sing}_\sigma} : \mathcal{M}ot \longrightarrow \mathbb{Q}\text{-Vect})$ of the functor preserving the tensor product is called *motivic Galois group of k*. It is a linear proalgebraic group defined over $\mathbb{Q}$. Its category of $\mathbb{Q}$-linear representations is equivalent to $\mathcal{M}ot$. The group depends on the choice of σ.

The motivic Galois group of a motive E is $Aut^{\otimes}(H_{\mathrm{sing}_\sigma} : \langle E\rangle \longrightarrow \mathbb{Q}\text{-Vect})$.

We now list several conjectures from [Serre].

***Conjecture* (2.1?).** — *The group $Aut^{\otimes}(H_{sing_\sigma} : \mathcal{M}ot_k \longrightarrow \mathbb{Q}\text{-}Vect)$ is proreductive, i.e. a limit of liner reductive $\mathbb{Q}$-groups.*

Let **1** denote the trivial morphism of rank 1, i.e. the cohomology of the point *Spec k*.

***Conjecture* (3.1?).** — *The group* $Aut^{\otimes}(H_{sing_\sigma} : \langle E\rangle \longrightarrow \mathbb{Q}\text{-}Vect)$ *is the subgroup of* $GL(H_{sing_\sigma}(E))$ *preserving the tensors corresponding to morphisms* $\mathbf{1} \longrightarrow E^{\otimes r} \otimes E^{\vee\otimes s}$, $r, s \geq 0$.

It is also conjectured that this group is reductive. Via the comparison isomorphism of étale and singular cohomology,

$$H_{\text{sing}}(E(\mathbb{C}), \mathbb{Q}) \otimes \mathbb{Q}_l = H_{et}(E \otimes \bar{\mathbb{Q}}, \mathbb{Q}_l),$$

the $\mathbb{Q}_l$-points of $Aut^{\otimes}(H_{\text{sing}_\sigma} : \langle E\rangle \longrightarrow \mathbb{Q}\text{-Vect})(\mathbb{Q}_l)$ act on the étale cohomology $H_{et}(E \otimes \bar{\mathbb{Q}}, \mathbb{Q}_l)$. On the other hand, the Galois group $Gal(\bar{\mathbb{Q}}/k)$ acts on $\bar{\mathbb{Q}}$ and therefore on $E \otimes \bar{\mathbb{Q}}$. By functoriality, the Galois group acts by automorphisms of the functor of étale cohomology. Hence, this gives rise to *ℓ-adic representation associated to E*

$$\rho_{k,l} : Gal(\bar{\mathbb{Q}}/k) \longrightarrow Aut^{\otimes}(H_{\text{sing}_\sigma} : \langle E\rangle \longrightarrow \mathbb{Q}\text{-Vect})(\mathbb{Q}_l).$$

***Conjecture* (3.2?).** — *Let k be a number field. The image of the ℓ-adic representation associated with E is dense in the group* $Aut^{\otimes}(H_{sing_\sigma} : \langle E\rangle \longrightarrow \mathbb{Q}\text{-}Vect)(\mathbb{Q}_l)$ *in the Zariski topology.*

***Conjecture* (9.1?).** — *Let k be a number field. The image*

$$Im(\rho_{k,l} : Gal(\bar{\mathbb{Q}}/k) \longrightarrow Aut^{\otimes}(H_{sing_\sigma} : \langle E\rangle \longrightarrow \mathbb{Q}\text{-}Vect)(\mathbb{Q}_l))$$

is open in $Aut^{\otimes}(H_{sing_\sigma} : \langle E\rangle \longrightarrow \mathbb{Q}\text{-}Vect)(\mathbb{Q}_l)$.

***Conjecture* (9.3?).** — *Let k be a number field.* $H_{et}(E \otimes \bar{\mathbb{Q}}, \mathbb{Q}_l)$ *is semi-simple as a* $Gal(\bar{\mathbb{Q}}/k)$*-module.*

We suggest that the conjectures [2.1?,3.1?,3.2?,9.1?,10.2?,10.3?.10.4?,10.7?,10.8?] may be interpreted as saying there are only finitely many $\mathbb{Z}$-forms of the étale cohomology $H_{et}(-, \mathbb{Z}_l) : \langle E\rangle \otimes_k \bar{\mathbb{Q}} \longrightarrow \mathbb{Q}$-Vect, up to Galois action. There are similar conjectures for finite adeles instead of $\mathbb{Q}_l$, cf. [Serre, 11.4?(ii), 11.5?], also [Serre, 10.2?, 10.6?].

2.4. Speculations and remarks.— Standard conjectures claim there are algebraic cycles corresponding to various cohomological constructions. Model-theoretically it should mean that something is definable in ACF and it is natural to expect that such properties be useful in a proof of categoricity, i.e. in the characterisation of the $\mathbb{Q}$-forms of étale cohomology theory.

We wish to specifically point out the conjectures and properties involving smooth hyperplane sections, namely *weak and strong Lefschetz theorems* and *Lefschetz Standard Conjecture*, cf. [Kleiman,p.11,p.14]. Weak Lefschetz theorem describes part of the cohomology ring of a smooth hyperplane section of a variety. Perhaps such a description can be useful in showing that a $\mathbb{Q}$-form extends uniquely to Mot/K from the subcategory $\mathcal{M}ot/\bar{\mathbb{Q}}$. An analogue of the weak Lefschetz theorem for the fundamental group was used in a similar way in [GavrDPhil, Lemma V.III.3.2.1], see

3.4.3 for some details. Namely, as is well-known, the fundamental group of a smooth hyperplane section of a smooth projective variety is essentially determined by the fundamental group of the variety. [GavrDPhil, III.2.2] extends this to a somewhat technical weaker statement about arbitrary generic hyperplane sections. An arbitrary variety can be represented as a generic hyperplane section of a variety defined over $\bar{\mathbb{Q}}$ and this implies that, in some sense, the fundamental groupoid functor on the subcategory of varieties defined over $\bar{\mathbb{Q}}$ "defines" its extension to varieties defined over larger fields. The word "defines" is used in a meaning similar to model theoretic meaning of one first-order language definable in another.

Question 1. — *Find a characterisation of the following families of functors:*

$$H_{sing}(X(K_\tau), \mathbb{Q}) : \mathcal{V}ar/K \longrightarrow \mathbb{Q}\text{-}Vect,$$

$$H_{sing}(X(K_\tau), \mathbb{C}) : \mathcal{V}ar/K \longrightarrow \mathbb{Q}\text{-}Hodge$$

where τ varies though isomorphisms of K to $\mathbb{C}$, or, almost equivalently, though locally compact locally connected topologies on K.

Note that Zilber [Zilber] *unconditionally* constructs a pseudo-exponential map $ex : \mathbb{C}^+ \longrightarrow \mathbb{C}^*$ which satisfies the Schanuel conjecture. Of course, this map is not continuous (not even measurable). Hence we ask:

Question 2. — *Construct a pseudo-singular cohomology theory which satisfies an analogue of the Schanuel conjecture and some other conjectures.*

2.5. Model theoretic conjectures. — Define model theoretic structures corresponding to the cohomology theories.

Conjecture 2.5. — *The field is purely embedded into the structures corresponding to functors*

(i) $H_{sing} : Var/\bar{\mathbb{Q}} \longrightarrow \mathbb{Q}\text{-}Vect$

(ii) $H_{sing}(-, \mathbb{Q}) : Var/\mathbb{C} \longrightarrow \mathbb{Q}\text{-}Vect$

(iii) $H_{sing}(-, \mathbb{C}) : Var/\mathbb{C} \longrightarrow \mathbb{Q}\text{-}Hodge$

Moreover, the structure (ii) is an elementary extension of (i) and the cohomology ring $H_{sing}(V, Q)$ is definable for every variety over C.

Several of the Standard Conjectures [Kleiman, §4,p.11/9] claim that certain cohomological cycles (construction) correspond to algebraic cycles. This feels related to many of the conjectures above, in particular to the purity conjectures.

Problem 1. — 1. *Define a model-theoretic structure and language corresponding to the notion of a Weil cohomology theory, and formulate a categoricity conjecture hopefully related to the Standard Conjectures ([Grothendieck, Kleiman]) and conjectures on the motivic Galois Group and related Galois representations [Serre].*

2. *Do the same in the language of functors, namely:*
   2.1. *Consider the family of cohomology theories on $\mathcal{V}ar/K$ coming from a choice of isomorphism $K \approx \mathbb{C}$.*

2.2. *Define a notion of isomorphism of these/such cohomology theories, and what it means to a "purely algebraic" property of such a theory.*
3.3. *Find a characterisation of that family up to that notion of isomorphism by such properties. Or rather, show existence of such a characterisation is equivalent to a number of well-known conjectures such as the Standard Conjectures etc.*

3. Uniqueness properties of the topological fundamental groupoid functor of a complex algebraic variety

3.1. Statement of the conjectures. — Let $\mathcal{V}$ be a category of varieties over a field K, let π be a functor to groupoids such that $\mathrm{Ob}\,\pi(X) = X(K)$ is the functor of K-points. For $\sigma \in Aut(K)$, define $\sigma(\pi)$ by

$$\mathrm{Ob}\,\sigma(\pi) = \mathrm{Ob}\,\pi(X) = X(K), \quad \sigma(\mathrm{Mor}\,(x,y)) = \mathrm{Mor}\,(\sigma(x),\sigma(y)),$$

$$\mathrm{source}(\gamma) = \sigma(\mathrm{source}(\gamma)), \mathrm{target}(\gamma) = \sigma(\mathrm{target}(\gamma)),$$

For $K = \mathbb{C}$, an example of such a functor is the topological fundamental groupoid functor $\pi_1^{top}(X(\mathbb{C}))$ of the topological space of complex points of an algebraic variety, and $\{\sigma(\pi_1^{top}) : \sigma \in Aut(\mathbb{C})\}$ is the family of all the topological fundamental groupoid functors associated with different choices of a locally compact locally connected topology on $\mathbb{C}$. (Such a topology determines a field automorphism, uniquely up to conjugation).

$Aut(K)$ acts by automorphisms of the source category, hence all these (possibly non-equivalent!) functors have the same properties in the language of functors, in particular

(0) Ob $\pi(X) = X(K)$ is the functor of K-points of an algebraic variety X
(1) preserve finite limits, i.e. $\pi(X \times Y) = \pi(X) \times \pi(Y)$
(2) $\pi(X)$ is connected if X is geometrically connected (i.e. the set of points $X(K)$ equipped with Zariski topology is a connected topological space)
(3) for $\tilde{X} \xrightarrow{f} X$ étale, the map $\pi(\tilde{X}) \xrightarrow{\pi(f)} \pi(X)$ of groupoids has the path lifting property of topological covering maps, namely
    for $x = f(\tilde{x}), \tilde{x} \in \tilde{X}(K)$, for every path $\gamma \in \pi(X)$ starting at x, there is a unique path $\tilde{\gamma} \in \pi(\tilde{X})$ such that $\mathrm{source}(\gamma) = \tilde{x}$ and $(\pi(f))(\tilde{\gamma}) = \gamma$.
(4) whenever Y is connected, $f : X \longrightarrow Y$ is a proper and separable morphism with geometrically connected fibres satisfying assumptions of Corollary 1.4 [SGA1, X], there is a short exact sequence
    $pi_1(X_x, x, x) \longrightarrow pi_1(X, x, x) \longrightarrow \pi_1(Y, f(x), f(x))$ where X_x is the geometric fibre over a geometric point x of X.

Definition 3.1. — *A π_1-like functor over a field K* is a functor from a category of varieties to the category of groupoids satisfying (0-4) above. A π_1-like functor is a π_1-like functor over some field.

Note that by (0) a π_1-functor comes equipped with a forgetful natural transformation to the functor of K-points.

Conjecture 3.2 ($Z(\pi_1^{top}, \mathcal{V}ar_{\mathbb{C}})$). — *Each π_1-like functor on the category of smooth quasi-projective complex varieties factors through the topological fundamental groupoid functor, up to a field automorphism.*

In detail: Let $\mathcal{V}ar_{\mathbb{C}}$ be the category of smooth quasi-projective varieties over the field of complex numbers $\mathbb{C}$. For each π_1-like functor $\pi : \mathcal{V}ar_{\mathbb{C}} \longrightarrow Groupoids$ there is a field automorphism $\sigma : \mathbb{C} \longrightarrow \mathbb{C}$ and a natural transformation $\varepsilon : \pi_1^{top} \Longrightarrow \pi^\sigma$ such that the induced natural transformation $\mathrm{Ob}\,\pi_1^{top} \Longrightarrow \mathrm{Ob}\,\pi^\sigma$ on the functor of $\mathbb{C}$-points is identity.

Conjecture 3.3 ($Z(\pi_1, K, \mathcal{V}ar_K)$). — *Let K be an algebraically closed field. Let $\mathcal{V}ar_K$ be the category of smooth quasi-projective varieties over K.*

There is a functor $\pi_1 : \mathcal{V}ar_K \longrightarrow Groupoids$ such that for each π_1-like functor $\pi : \mathcal{V}ar_K \longrightarrow Groupoids$ there is a field automorphism $\sigma : K \longrightarrow K$ and a natural transformation $\varepsilon : \pi_1 \Longrightarrow \pi^\sigma$ such that the induced natural transformation $\mathrm{Ob}\,\pi_1 \Longrightarrow \mathrm{Ob}\,\pi^\sigma$ on the functor of K-points is identity.

These conjectures are direct analogues of categoricity conjectures in model theory, hence we shall refer to these conjectures as *categoricity conjectures for the fundamental groupoid functor*.

Remark 3.4. — As stated, these conjectures are likely too optimistic. To get more plausible and manageable conjectures, replace $\mathcal{V}ar_K$ by a smaller category and add additional conditions on the π_1-like functors. The conclusion can also be weakened to claim there is a finite family of functors, rather than a single functor, through which π_1-like functors factor up to field automorphism.

It may also be necessary to put extra structure on the fundamental groupoids.

Remark 3.5. — In model theory, it is more convenient to work with universal covering spaces rather than fundamental groupoids. Accordingly, model theoretic results are stated in the language of universal covering spaces, sometimes with extra structure.

The conjectures above are motivated by questions and theorems about categoricity of certain structures.

Remark 3.6. — It is tempting to think that the right generalisation of the conjectures above should make use of the short exact sequence of étale fundamental groups (see [SGA1, XIII.4.3;XII.4.4])

$$1 \longrightarrow \pi_1^{alg}(X \times_{\mathrm{Spec} k} \mathrm{Spec} k^{\mathrm{sep}}, x) \longrightarrow \pi_1^{alg}(X, \bar{x}) \longrightarrow \pi_1(\mathrm{Spec}\, k, \mathrm{Spec} k^{\mathrm{sep}}) = \mathrm{Gal}(k^{\mathrm{sep}}/k) \longrightarrow 1$$

where X is a scheme over a field k, k^{sep} is a separable closure of k, and $x : \mathrm{Spec}\, k^{\mathrm{sep}} \longrightarrow X \times_{\mathrm{Spec} k} \mathrm{Spec} k^{\mathrm{sep}}$ is a geometric point of $X \times_{\mathrm{Spec} k} \mathrm{Spec} k^{\mathrm{sep}}$, and $\bar{x} : \mathrm{Spec}\, k^{\mathrm{sep}} \longrightarrow X \times_{\mathrm{Spec} k} \mathrm{Spec} k^{\mathrm{sep}} \longrightarrow X$ is the corresponding geometric point of X.

In fact such a sequence could be associated with a morphism $X \longrightarrow S$ admitting a section and satisfying certain assumptions [SGA 1, XIII.4].

These short exact sequences comes from pullback squares

$$\begin{array}{ccc} X \times_{\mathrm{Spec}k} \mathrm{Spec}k^{\mathrm{sep}} & \longrightarrow & \mathrm{Spec}\, k^{\mathrm{sep}} \\ \downarrow & & \downarrow \\ X & \longrightarrow & \mathrm{Spec}\, k \end{array} \qquad \begin{array}{ccc} X_s & \longrightarrow & s \\ \downarrow & & \downarrow \\ X & \longrightarrow & S \end{array}$$

3.2. Conjectures within reach.— In this section we give two conjectures which we hope to be within reach. Their proof requires a combination of methods of model theory and algebraic geometry. In the next subsection we list several partial positive results towards conjectures above which are implicit in model theoretic literature.

We find the following conjecture plausible and hope its statement clarifies the arithmetic nature of our conjectures. It is perhaps the simplest conjecture not amendable to model theoretic analysis because it uses bundles, a notion from geometry rather than model theory.

For a variety X, let $\langle X \rangle_K$ denote the category whose objects are the finite Cartesian powers of X, and morphisms are morphisms of algebraic varieties defined over K; we let X^0 denote a variety consisting of a single K-rational point.

Conjecture 3.7 ($Z(\pi_1, L_A^*)$). — *Let K be an algebraically closed field of zero characteristic, A an Abelian variety defined over a number field k. Let L_A be an ample line bundle over A and L_A^* be the corresponding $\mathbb{G}_m$-bundle. Further assume that the Mumford-Tate group of A is the maximal possible, i.e. the general symplectic group,*

$$MT(A) = \mathrm{GSp}_{\mathbb{Z}}$$

and that the image of Galois action on the torsion has finite index in the group of $\hat{\mathbb{Z}}$-points of the symplectic group.

Then there is a finite family of π_1-like functors Π_1 such that each π_1-like functor on the full subcategory $\langle L_A^* \rangle_K$ consisting of the Cartesian powers of the $\mathbb{G}_m$-bundle L_A^*, factors via an element of Π_1.

These functors in Π_1 correspond to different embeddings of the field of definition of A into the field of complex numbers.

The following conjecture is probably within reach, at least if we replace the fundamental groupoid functor by its residually finite part.

Model theoretic methods of [BH^{2}K^{2}14] are likely enough to show that it is enough to prove this conjecture for a countable algebraically closed subfield. Methods of [GavrDPhil,III.5.4.7], cf. §3.4.3 and particularly Statement 3.13, are likely enough to reduce the remaining part of the conjecture to certain properties of complex analytic topology and normalisation of varieties.

Conjecture 3.8 ($Z(\pi_1, \bar{\mathbb{Q}} \subset \mathbb{C}, \mathcal{V}ar_{\bar{\mathbb{Q}}} \subset \mathcal{V}ar_{\mathbb{C}})$). — *Let $\mathcal{V}ar_{\mathbb{C}}$ be the category of smooth quasi-projective varieties over $\mathbb{C}$, and let $\mathcal{V}ar_{\bar{\mathbb{Q}}}$ be its subcategory consisting of varieties and morphisms defined over $\bar{\mathbb{Q}}$.*

Assume that $\pi : \mathcal{V}ar_{\mathbb{C}} \longrightarrow Groupoids$ is a π_1-like functor which coincides with the topological fundamental groupoid π_1^{top} for varieties and morphisms defined over

$\bar{\mathbb{Q}}$, i.e. for each variety V in $\mathcal{V}ar_{\bar{\mathbb{Q}}}$ and each morphism $V \xrightarrow{f} W$ in $\mathcal{V}ar_{\bar{\mathbb{Q}}}$ it holds $\pi(V) = \pi_1^{top}(V)$ and $\pi(f) = \pi_1^{top}(f)$.

Then there exist a field automorphism $\sigma \in Aut(\mathbb{C}/\bar{\mathbb{Q}})$ such that $\pi \circ \sigma$ and π_1^{top} are equivalent.

There are a number of theorems and conjectures which can be seen as saying that, up to finite index, the Galois action is described by geometric, algebraic or topological structures; our conjectures can also be seen in this way.

3.3. Partial positive results. — These conjectures are closely related to *categoricity* theorems in model theory, and this has led to several partial positive results uniqueness of π_1-like functor restricted to certain full subcategories of the category of varieties over an algebraically closed field: the full subcategories $\langle K^*\rangle$ of algebraic tori in arbitrary characteristic, $\langle E\rangle$ powers of an elliptic curve over a number field, a weaker result about $\langle A\rangle$ powers of an Abelian variety over a number field, a still weaker result about $\langle V\rangle$ powers of a smooth projective variety whose fundamental group satisfies a group theoretic property of being subgroup separable, a strengthening of residually finite.

Note that the first three categories are linear in the sense that the the groups $Aut(K^*)$, $Aut_{End\,E\text{-}mod}(E(K))$, and $Aut_{End\,A\text{-}mod}A(K)$ act on the set of π_1-like functors on the respective categories $\langle K^*\rangle$, $\langle E\rangle_K$, and $\langle A\rangle_K$. This is so because these groups act on these categories.

Let us now list several statements translated from categoricity theorems available in model theory literature. We give references but do not explain the translation.

A reader familiar with algebraic geometry may find that there is a better way to formulate these results. In the case of a countable field K, their proofs do not require elaborate techniques of model theory, and such a reader may find it easier to reprove these results in a familiar language while only drawing inspiration from the proofs in the literature.

For a variety V over a field K, let $\langle V\rangle_K$ denote the full subcategory of the category of varieties defined over K whose objects are all finite Cartesian powers of V and a K-rational point.

Statement 3.9 is a particular case of Conjecture 3.3($Z(\pi_1, K, \mathcal{V}ar_K)$) when we take $\mathcal{V}ar_K = \langle K^*\rangle_K$ to be the category of algebraic tori. Note this implies Conjecture 3.2($Z(\pi_1^{top}, \langle\mathbb{C}^*\rangle_{\mathbb{C}})$).

***Statement 3.9* (BaysZilber,Th.2.1;GavrK).** — *Let K be an algebraically closed field of* $\operatorname{char} K = 0$. *Then there is a π_1-like functor π over K defined on the category $\langle K^*\rangle_K$ of algebraic tori such that for any π_1-like functor π' over K defined on the same category there is a field automorphism $\sigma : K \longrightarrow K$ and a natural transformation $\pi \Longrightarrow \sigma(\pi')$ such that the induced natural transformation $Ob\pi \Longrightarrow Ob\sigma(\pi')$ is identity.*

Similarly, Statement 3.10 is a particular case of Conjecture 3.3($Z(\pi_1, K, \langle E\rangle_K)$) when we take $\mathcal{V}ar_K = \langle E\rangle_K$ to be the category generated by an elliptic curve defined over a number field, and weaken the conclusion to require finitely many rather than unique. This is necessary, as there are finitely many embedding of the number field into the field of complex numbers, and they give rise to non-equivalent π_1-like functors. Note this implies Conjecture 3.2($Z(\pi_1^{top}, \langle E\rangle_{\mathbb{C}})$) weakened in a similar way.

Statement 3.10 **(BaysDPhil, Th.4.4.1; GavrK, Prop.2)**
Let K be an algebraically closed field of $\operatorname{char} K = 0$*, and let E be an elliptic curve defined over a number field k with a k-rational point $0 \in E(k)$.*

Then there are finitely many π_1-like functors $\pi^1, ..., \pi^n$ on the category $\langle E\rangle_K$ such that for any functor π such that

– $\pi(E, 0, 0) \approx \mathbb{Z}^2$

there is a field automorphism $\sigma : K \longrightarrow K$ and a natural transformation $\pi \Longrightarrow \sigma(\pi^i)$ for some i such that the induced natural transformation $Ob\pi \Longrightarrow Ob\sigma(\pi^i)$ is identity.

Statement 3.11 is a particular case of Conjecture 3.3($Z(\pi_1, K, \langle E\rangle_K)$) when we take K to be an algebraically closed field of prime characteristic, $\mathcal{V}ar_K = \langle K^*\rangle_K$ to be the category of algebraic tori, and weaken the conclusion by placing additional restrictions on π_1-functors.

These restrictions are necessary because there are too few automorphisms of the the algebraic closure $\bar{\mathbb{F}}_p$ of the field with p elements. However, we may view this as a hint towards Remark 3.6 suggesting we should replace below $Aut(K/\bar{\mathbb{F}}_p)$ by the etale fundamental group of an appropriate scheme over $\mathbb{F}_p$.

Statement 3.11 **(BaysZilber,Th.2.2)**. — *Let K be an algebraically closed field of* $\operatorname{char} K = p$*. Then there is a π_1-like functor $\pi_1 : \langle K^*\rangle \longrightarrow Groupoids$ such that each π_1-like functor $\pi : \langle K^*\rangle \longrightarrow Groupoids$ factors via $\pi_1 : \langle K^*\rangle \longrightarrow Groupoids$ up to $Aut(K/\bar{\mathbb{F}}_p)$ provided*

– $\pi(K^*, 1, 1) \approx \mathbb{Z}[1/p]$
– *the restrictions $\pi_{1|\bar{\mathbb{F}}_p}$ and $\pi_{|\bar{\mathbb{F}}_p}$ to $\bar{\mathbb{F}}_p$-rational points coincide:*

$$\pi_{|\bar{\mathbb{F}}_p} = \pi'_{|\bar{\mathbb{F}}_p}$$

Statement 3.12 is a particular case of Conjecture 3.3($Z(\pi_1, K, \langle A\rangle_K)$), for A is an abelian variety over a number field. weakened by requiring the fundamental *group* (rather than groupoid) functors of the π_1-like functors to coincide.

Statement 3.12 **(BaysDPhil, Th.4.4.1)**. — *Let K be an algebraically closed field of* $\operatorname{char} K = 0$*. Let A be an Abelian variety defined over a number field k with a k-rational point $0 \in A(k)$.*

Then there is a π_1-like functor $\pi_1 : \langle A\rangle \longrightarrow Groupoids$ such that each π_1-like functor $\pi : \langle A\rangle \longrightarrow Groupoids$ factors via $\pi_1 : \langle K^*\rangle \longrightarrow Groupoids$ up to $Aut(K/\mathbb{Q}(A_{tors}))$ whenever

– $\pi(A, 0, 0) = \mathbb{Z}^{2\dim A}$
– *for any two functors π, π' in $\mathcal{F}$, the corresponding fundamental group functors coincide*

$$\pi(A, 0, 0) = \pi'(A, 0, 0)$$

and further, for $p : \tilde{A} \longrightarrow A$ *is étale,* $\gamma \in \pi(A, 0, 0) = \pi'(A, 0, 0)$, $\pi(\gamma_\pi) = \pi'(\gamma_{\pi'}) = \gamma$, *it holds that*

$$\mathrm{source}(\gamma_\pi) = \mathrm{source}(\gamma_{\pi'}) \quad \text{implies} \quad \mathrm{target}(\gamma_\pi) = \mathrm{target}(\gamma_{\pi'})$$

Statement 3.13 is a particular case of Conjecture 3.8($Z(\pi_1, \bar{\mathbb{Q}} \subset \mathbb{C}, \mathcal{V}ar_{\bar{\mathbb{Q}}} \subset \mathcal{V}ar_{\mathbb{C}})$) when $\mathcal{V}ar_{\bar{\mathbb{Q}}} = \langle V\rangle_{\bar{\mathbb{Q}}}$ and $\mathcal{V}ar_{\mathbb{C}} = \langle V\rangle_{\mathbb{C}}$ are the subcategories generated by a variety with certain properties.

Recall a group G is subgroup separable iff for each finitely generated subgroup $H < G$ and $h \notin H$ there is a morphism $f : G \longrightarrow G_H$ into a finite group G_H such that $h \notin f(H)$.

***Statement 3.13* (GavrDPhil,III.5.4.7).** — *Let K be an algebraically closed field of* $\operatorname{char} K = 0$ *and of the least uncountable cardinality $\aleph_1$. Let V be an smooth projective variety defined over a number field k with a k-rational point $0 \in V(k)$ such that the universal covering space of $V(\mathbb{C})$ is holomorphically complex, for some embedding $K \hookrightarrow \mathbb{C}$, and its fundamental groups $\pi_1(V(\mathbb{C}), 0, 0)^n$ are subgroup separable for each $n > 0$.*

Then there is a π_1-like functor $\pi_1 : \langle V\rangle \longrightarrow Groupoids$ such that a π_1-like functor $\pi : \langle V\rangle \longrightarrow Groupoids$ factors via $\pi_1 : \langle V\rangle \longrightarrow Groupoids$ up to $Aut(K/\bar{\mathbb{Q}})$ provided

- $\pi(V, 0, 0) \approx \pi_1^{top}(V(\mathbb{C}), 0, 0)$
- *the restrictions $\pi_{1|\bar{\mathbb{Q}}}$ and $\pi_{|\bar{\mathbb{Q}}}$ to $\bar{\mathbb{Q}}$-rational points coincide:*

$$\pi_{|\bar{\mathbb{Q}}} = \pi'_{|\bar{\mathbb{Q}}}$$

We wish to mention the work of [HarrisDPhil, DawHarris] and particularly [Eterovich] on Shimura curves, which does not quite fit in our framework. To interpret their results, one needs to consider π_1^{top} as a functor to groupoids *with extra structure.*

Remark 3.14. — The conjectures on independence of Galois representations of non-isogenious curves likely imply our conjectures for the full subcategory $\langle E_1 \times \ldots \times E_n\rangle_K$ generated by a finite product of elliptic curves $E_1, \ldots, E_n$ over a number field.

Consider the family of π_1-like functors with Abelian fundamental groups.

Is it true that each such "abelianised" π_1-like functor factors via π_1^{top} up to a field automorphisms? Is this equivalent to Conjecture 3.2($Z(\pi_1^{top}, AbVar)$) for the subcategory $AbVar$ of all abelian varieties?

3.4. Mathematical meaning of the conjectures. Elements of proof of the conjectures. — Here we try to explain the arithmetic and geometric meaning of the conjectures. In a sense, the conjectures say that $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ and $Aut(K/\mathbb{Q})$ are large enough. We try to show below in what sense, by showing possible obstructions/difficulties in proof.

3.4.1. Galois action on roots of unity and Kummer theory. — Consider the infinite sequence $exp(2\pi i/n)$ of roots of unity. This sequence can be obtained topologically: take the loop γ generating $\pi(\mathbb{C}^*, 1, 1) \approx \mathbb{Z}$, the étale morphism $z^n : \mathbb{C}^* \longrightarrow \mathbb{C}^*$ and lift γ uniquely to a path $\tilde{\gamma}_n$ starting at $1 \in \mathbb{C}^*$. Then $exp(2\pi/n)$ is the end-point of γ_n. This construction shows that a π_1-like functor on the category $\langle K^*\rangle$ determines a distinguished sequence $\xi_n, n \geq 0$ of roots of unity. Hence, our conjectures require that the Galois group acts transitively on the set of sequences of roots of unity associated with π_1-like functors.

Consider a π_1-like functor on the category $\langle K^* \rangle$. As noted above, group automorphisms $Aut(K^*)$ of K^* act on the set of these functors. Hence, item (3.9) requires that multiplicative group automorphisms $Aut(K^*)$ and field automorphisms $Aut(K/\mathbb{Q})$ have the same orbits on the sequences $\xi_n, (\xi_{mn})^m = \xi_n, m, n > 0$ of roots of unity.

Kummer theory arises in a similar way if we consider endpoints of liftings of paths joining 1 and arbitrary elements $a_1, ..., a_n$.

3.4.2. Elliptic curves and Abelian varieties. Kummer theory and Serre's open image theorem for elliptic curves.— Kummer theory for elliptic curves and Abelian varieties arises in the same way if we consider π_1-like functors on the category $\langle A \rangle$ generated by an Abelian variety.

Similarly, our conjectures about π_1-like functors on $\langle A \rangle_K$ require that the action of $Aut_{EndA\text{-mod}}(A(K))$ and $\mathrm{Gal}(\bar{\mathbb{Q}}/k)$ on the torsion points do not differ much. This is true for elliptic curves but fails for Abelian varieties of $\dim A > 1$, hence the extra assumption in (4) on the family of π-like functors.

3.4.3. Arbitrary variety. Etale topology and an analogue of Lefshetz theorem for the fundamental group. — To prove Statement 3.13, we need several facts about étale topology. Most of these facts are well-known for smooth varieties; what we use is that they hold "up to finite index" for arbitrary (not necessarily smooth or normal) subvarieties of a smooth projective variety.

Consider the inverse limit $\varprojlim \tilde{V}(\mathbb{C})$ of finite étale covers $\tilde{V}(\mathbb{C}) \longrightarrow V(\mathbb{C})$ of a complex algebraic variety V. The universal analytic covering map $U \longrightarrow V(\mathbb{C})$ gives rise to covering maps $U \longrightarrow \tilde{V}(\mathbb{C})$ and hence a map $U \longrightarrow \varprojlim \tilde{V}(\mathbb{C})$. Zariski topology on the étale covers makes $\varprojlim \tilde{V}(\mathbb{C})$ into a topological space. Hence there are two topologies on U – the complex analytic topology and the "more algebraic" topology on U induced from the map $U \longrightarrow \varprojlim V(\mathbb{C})$. Call the latter *étale* topology on U.

Definition 3.15. — The étale topology on the universal covering space of the topological space of complex points of an algebraic variety is defined as the topology induced from the map to the inverse limit of the spaces of complex points of finite étale covers of the variety equipped with the Zariski topology. The étale topology on a covering space of the topological space of complex points of an algebraic variety is defined similarly.

To prove Statement3.13 we use that these two topologies are similar and nicely related. In particular,

Lemma 3.16. — *Assume V satisfies the assumptions of Statement 3.13.*

- *Closed irreducible sets in étale topology are closed irreducible in complex analytic topology (by definition).*

- *For a set closed in étale topology, its irreducible components in complex analytic topology are also closed in étale topology [GavrDPhil,III.1.4.1(4,5)].*
- *The image of an étale closed irreducible subset of $U \times \ldots \times U$ under a coordinate projection is étale closed [GavrDPhil, III.2.2.1].*

Note that this is easy to see that *connected* components of a set closed in étale topology are also closed in étale topology, and hence that the properties above holds for smooth or normal closed sets.

Let $f : W \longrightarrow V$ be a morphism of varieties, and let $f_* : U_W \longrightarrow U_V$ be the map of the universal covering spaces of $W(\mathbb{C})$ and $V(\mathbb{C})$. We may assume that V is smooth projective but it is essential that W is arbitrary. In applications, W is an arbitrary closed subvariety of a Cartesian power of a fixed variety V.

Lemma 3.17. — *Under the assumptions above, if $f : W \longrightarrow V$ is proper, then the image $f(U_W)$ is closed in U_V in étale topology.*

This is related to the following well-known geometric fact:

Lemma 3.18. —
- *If $f : W(\mathbb{C}) \longrightarrow V(\mathbb{C})$ is a morphism of smooth normal algebraic varieties, g a generic point of $V(\mathbb{C})$ and $W_g = f^{-1}(g)$ then*
$$\pi_1(W_g, w, w) \longrightarrow \pi_1(W, w, w) \longrightarrow \pi_1(V, g, g)$$
is exact up to finite index.
- *Moreover, if $f(W(\mathbb{C}))$ is dense in $V(\mathbb{C})$, then $\pi_1(W, w, w) \longrightarrow \pi_1(V, g, g)$ is surjective.*

In fact we need to use a generalisation of this, namely that it holds up to finite index for arbitrary varieties if one considers the image of the fundamental group in the ambient smooth projective variety, see [GavrDPhil, V.3.3.6, V.3.4.1] for details.

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