A NAIVE DIAGRAM-CHASING APPROACH TO FORMALISATION OF TAME TOPOLOGY

by

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in memoriam: evgenii shurygin

... due to internal constraints on possible architectures of unknown to us functional "mental structures".
Misha Gromov. Structures, Learning and Ergosystems: Chapters 1-4, 6.

Abstract. — We rewrite excerpts of [Bourbaki, General Topology] using the category-theoretic notation of arrows and are thereby led to concise reformulations of classical topological definitions in terms of simplicial categories and orthogonality of morphisms, which we hope might be of use in the formalisation of topology and in developing the tame topology of Grothendieck.

Namely, we observe that topological and uniform spaces are simplicial objects in the same category, a category of filters or, equivalently, the category of pointed topological spaces with maps continuous at the point, and that a number of elementary properties can be obtained by repeatedly passing to the left or right orthogonal (in the sense of Quillen model categories) starting from a simple class of morphisms, often a single typical (counter)example appearing implicitly in the definition.

Examples include the notions of: compact, discrete, connected, and totally disconnected spaces, dense image, induced topology, and separation axioms, and, outside of topology, finite groups being nilpotent, solvable, torsion-free, p-groups, and prime-to-p groups; injective and projective modules; injective and surjective (homo)morphisms.

Contents

A draft of a research proposal. Comments welcome at mishap@sdf.org. I thank Martin Bays, Sergei Ivanov and Vladimir Sosnilo for discussions. http://mishap.sdf.org/mints/ updated: 10.18,11.11.18,21.11.18.

Die Mathematiker sind eine Art Franzosen: Redet man zu ihnen, so übersetzen sie es in ihre Sprache, und dann ist es alsobald ganz etwas anderes. [(1)]
—— Johann Wolfgang von Goethe. Maximen und Reflexionen. Nr. 1005.

if a man bred to the seafaring life ... should take it into his head to philosophize concerning the faculties of the mind, it cannot be doubted, but he would draw his notions from the fabric of his ship, and would find in the mind, sail, masts, rudder, and compass.
—— Thomas Reid. An Inquiry into the Human Mind on the Principles of Common Sense. 1764.

As a scholar, meantime, he was trivial and incapable of labour.
—— Thomas de Quincey. Cicero. Blackwood's magazine. v.52, 1842.

1. Introduction.

1.1. Main ideas.— In this note we rewrite several classical definitions and constructions in topology in terms of category theory and diagram chasing. We do so by by first "transcribing" excerpts of [Bourbaki, General Topology] and [Engelking, Topology] by means of notation extensively using arrows, and then recognizing familiar patterns of standard category-theoretic constructions and diagram chasing arguments.[(2)]

Arguably, *we transcribe the ideas of Bourbaki into a language of category theory appropriate to these ideas*, and our analysis of the text of Bourbaki shows these ideas (but not notation) are implicit in Bourbaki and reflect their logic (or perhaps their ergologic in the sense of [Gromov. Ergobrain; Memorandum Ergo]).

Doing so, we observe that a number of elementary textbook properties are obtained by taking the orthogonal (in the sense of Quillen lifting property) to the simplest morphism-counterexample, and this leads to a concise syntax expressing these properties in two or three bytes in which e.g. denseness, separation property Kolmogoroff/T_0, compactness is expressed as

(dense image)	(Kolmogoroff/T_0)	(compact)
$(\{c\} \longrightarrow \{o \to c\})^l$	$(\{x \leftrightarrow y\} \longrightarrow \{x = y\})^r$	$((\{\{o\} \longrightarrow \{o \to c\}\}^r)_{<5})^{lr}$

this shows their Kolmogoroff complexity is very low (byte or two).

We also observe that the categories of topological spaces, uniform spaces, and simplicial sets are all, in a natural way, full subcategories of the same larger category, namely the simplicial category of filters; coarse spaces of large scale

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(1) Mathematicians are like Frenchmen: whatever you say to them they translate into their own language, and forthwith it is something entirely different. In: Johann Wolfgang von Goethe. Aphorismen und Aufzeichnungen. Nach den Handschriften des Goethe- und Schiller-Archivs hg. von Max Hecker, Verlag der Goethe-Gesellschaft, Weimar 1907. Aus dem Nachlass, Nr. 1005, Uber Natur und Naturwissenschaft. Maximen und Reflexionen.

(2) Arrows and other category-theoretic notations are conspicuously absent from [Bourbaki, General Topology] and little used in his other books. [Corry, Nicolas Bourbaki and the Concept of Mathematical Structure], also [Dieudonné, The work of Bourbaki during the last thirty years] might suggest that Bourbaki conciously avoided category-theory notation.

metric geometry are also simplicial objects of a category of filters with different morphisms. This is, moreover, implicit in the definitions of a topological, uniform, and coarse space.

The exposition is in form of a story where we pretend to "read off" category-theoretic constructions from the text of excerpts of [Bourbaki] and [Engelking] in a straightforward, unsophisticated, almost mechanical manner. We hope word "mechanical" can be taken literally: we pretend to search for correlations between the structure of allowed category-theoretic diagram-chasing constructions and the text of arguments in topology, and hope this search can be done by a short program.

No attempt is made to develop a theory or prove a theorem: our goal is to explain the process of transcribing by working out a few examples in detail. In fact, we think that understanding and formalising this process is a very interesting question.

This note is a research proposal suitable for a polymaths project: transcribing topological arguments into category theory involves rather independent tasks: finding topological arguments worth transcribing and working out the precise meaning of category theoretic reformulations are best suited for general topologists; spotting category theoretic patterns is best suited for category theorists; working out formal syntax is best suited for logicians.

We hope our way of translating might of use in the formalisation of topology and suggests an approach to the tame topology of Grothendieck.

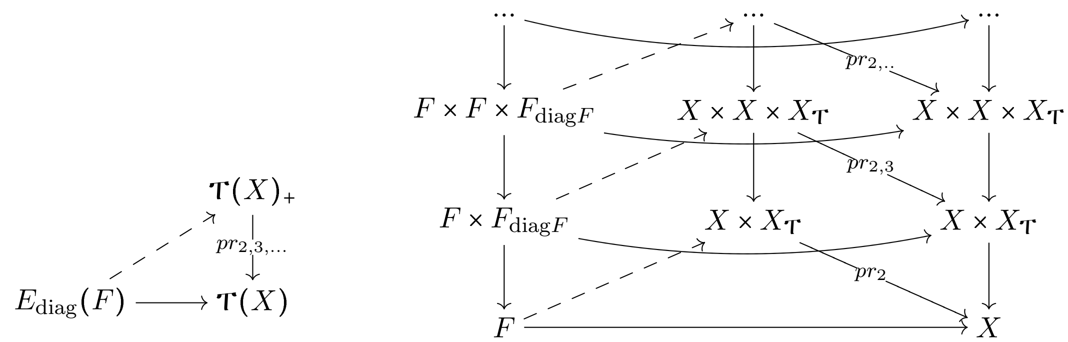
1.2. Contents.— In §1.4, as a warm-up and an example of our translation, we discuss the definition of surjection; in §1.5, we suggest the intuition that orthogonality is category-theoretic *negation*. Appendix §5.1.1 gives a verbose exposition of the same ideas aimed at a student.

In §2.1 we start with a detailed translation of the definitions by Bourbaki of a dense subspace and a separation axiom of being Kolmogoroff/T_0 and show these definitions implicitly describe the simplest counterexamples involving spaces consisting of one or two points, and in fact require orthogonality to these counterexamples. Appendix §5.2.1 and §5.3.2 gives more examples of properties defined by iterated orthogonals. Examples include the notions of: compact, discrete, connected, and totally disconnected spaces, dense image, induced topology, and separation axioms. Appendix §5.3.1 introduces a formal syntax and semantics which expresses these properties in several bytes in both human- and a computer- readable form. Outside of topology, examples in §5.3.2 include finite groups being nilpotent, solvable, torsion-free, p-groups, and prime-to-p groups; injective and projective modules; injective, surjective.

Compactness is discussed in §2.2 we reformulate the Bourbaki's definition in terms of convergence of ultrafilters as an iterated orthogonal of the simplest counterexample. With help of this, we show in §2.2.5 that there is a factorisation system corresponding to Stone-Čech compactification, and thus it is somewhat analogous to Axiom M2 $(cw)(f)$- and $(c)(wf)$- decompositions required in Quillen model categories.

In §2.3 we reformulate the axioms of a topology in a form almost ready to be implemented in a theorem prover based on diagram chasing with finite preorders. In §2.3.2 we observe that the axiom of topology saying that an arbitrary union of open subsets is open can be expressed as a formula of form $\forall\exists\varphi \implies \exists\forall\varphi$ and state a speculation Remark 2.2 that topology is really about permuting quantifiers, a language to talk about dependencies.

In §3, we "transcribe" the informal considerations in [Bourbaki, Introduction]. We "read off" from there in §3.1.1 and §3.1.6 that topological and uniform spaces are 2-dimensional simplicial objects in the same category, the category of filters. The discussion in §3.2 of the notion of a limit of a filter $\mathfrak{F}$ on a topological space X leads to a reformulation in terms of a lifting property wrt shift (décalage) simplicial maps forgetting first face and degeneracy maps $(X \times X_{\mathbf{T}}, X \times X \times X_{\mathbf{T}}, ...) \longrightarrow (X_{\mathbf{T}}, X \times X_{\mathbf{T}}, X \times X \times X_{\mathbf{T}}, ...)$.



where $F \times ... \times F$ is equipped with the finest filter such that the face maps (diagonal embeddings) $F \longrightarrow F \times ... \times F,\ x \mapsto (x, ..., x)$ are continuous, i.e. a subset of $F \times ... \times F$ is diag$F$-big iff it contains the image of a F-big subset of F under the face map $F \longrightarrow F \times ... \times F$.

We end the section with a discussion in §3.3 of path spaces and cylinder objects in the category of topological spaces; this also leads to constructions reminiscent of the shift (décalage). Note that the décalage of a simplicial set is a model for the path space object of a topological space, somewhat smaller than the usual model we discuss.(3)

In §4, we formulate a number of open questions. Unfortunately, interesting open questions are rather vague and concern the expressive power and formalisation of the new category theoretic, diagram-chasing way to talk about topology; to what extent the new language helps to avoid irrelevant set-theoretic

(3) See [nlab:décalage] for a detailed discussion.

details and counterexamples. An important precise open question in this spirit is to define a model structure on the simplicial category of filters compatible with a model structure on the full subcategory of topological spaces. An application which would be of interest in geometry is to formulate a clean version of Arzela-Ascoli theorem (cf. Question 4.7), i.e. a compactness principle for function spaces of maps between topological, metric and/or measurable spaces.

1.3. Speculations.— Does your brain (or your kitten's) have the lifting property (orthogonality), simplicial objects or diagram chasing built-in? §2 suggests a broader and more flexible context making contemplating an experiment possible. Namely, some standard arguments in point-set topology are computations with category-theoretic (not always) commutative diagrams of finite categories (which happen to be preorders, or, equivalently, finite topological spaces) in the same way that lifting properties define injection and surjection. In that approach, the lifting property is viewed as a rule to add a new arrow, a computational recipe to modify diagrams.

Can one find an experiment to check whether humans *subconsciously* use diagram chasing to reason about topology?

Does it appear implicitly in old original papers and books on point-set topology?

Is diagram chasing with preorders too complex to have evolved? Perhaps; but note the self-similarity: preorders are categories as well, with the property that there is at most one arrow between any two objects; in fact sometimes these categories are thought of as 0-categories. So essentially your computations are in the category of (finite 0-) categories.

Is it universal enough? Diagram chasing and point-set topology, arguably a formalisation of "nearness", is used as a matter of course in many arguments in mathematics.

Finally, isn't it all a bit too obvious? Curiously, in my experience it's a party topic people often get stuck on. If asked, few if any can define a surjective or an injective map without words, by a diagram, or as a lifting property, even if given the opening sentence of §5.1 as a hint. No textbooks seem to bother to mention these reformulations (why?). An early version of [Gavrilovich, Hasson] states $(*)_{\rightthreetimes}$ and $(**)_{\rightthreetimes}$ of §1.4 and §5.1 as the simplest examples of lifting properties we were able to think up; these examples were removed while preparing for publication.

1.4. Surjection: an example. — Let us now explain what we mean by translation. A map $f : X \longrightarrow Y$ is *surjective* iff it is left-orthogonal to the simplest non-surjective map $\varnothing \longrightarrow \{\bullet\}$, i.e.

$$(*)_{\rightthreetimes} \qquad \varnothing \longrightarrow \{\bullet\} \rightthreetimes X \xrightarrow{f} Y$$

Recall that for morphisms $f : A \longrightarrow B$, $g : X \longrightarrow Y$ in a category, *a morphism f has the left lifting property wrt a morphism g, f is (left) orthogonal to g*, and we write $f \rightthreetimes g$ or $A \xrightarrow{f} B \rightthreetimes X \xrightarrow{g} Y$, iff for each $i : A \longrightarrow X$, $j : B \longrightarrow Y$ such that $ig = fj$ ("the square commutes"), there is $j' : B \longrightarrow X$ such that $fj' = i$ and $j'g = j$ ("there is a diagonal making the diagram commute").

With this definition, $(*)_{\rightthreetimes}$ reads as

$(*)_{\text{words}}$ *for each map $\{\bullet\} \xrightarrow{y} Y$, i.e. a point $y \in Y$, there is a map $\{\bullet\} \xrightarrow{x} X$, i.e. a point $x \in X$, such that $f \circ x = y$, i.e. $f(x) = y$.*

This is the *text* of the usual definition of surjectivity of a function found in an elementary textbook. Conversely, we can read off $(*)_{\rightthreetimes}$ from the text of the definition of surjectively, by drawing the commutative diagram as we read $(*)_{\text{words}}$.

It is this kind of direct, almost syntactic, relationship between the usual text and its category theoretic reformulation we are looking for in this paper. This is what we mean by saying *the reformulation $(*)_{\rightthreetimes}$ is implicit in the text $(*)_{words}$*.

For a property (class) C of arrows (morphisms) in a category, define its *left* and *right orthogonals*, which we also call its *left* and *right negation*

$$C^l := \{f : \text{ for each } g \in C \ f \rightthreetimes g\}$$
$$C^r := \{g : \text{ for each } f \in C \ f \rightthreetimes g\}$$
$$C^{lr} := (C^l)^r, C^{ll} := (C^l)^l, \ldots$$

Take $C = \{\varnothing \longrightarrow \{*\}\}$ in Top. A calculation shows that a few of its iterated negations are meaningful: in Top, C^r is the class of surjections (as we saw earlier), C^{rr} is the class of subsets, C^{rl} is the class of maps of form $A \longrightarrow A \sqcup D$, D is discrete; $\{\bullet\} \longrightarrow A$ is in C^{rll} iff A is connected; Y is totally disconnected iff $\{\bullet\} \xrightarrow{y} Y$ is in C^{rllr} for each map $\{\bullet\} \xrightarrow{y} Y$ (or, in other words, each point $y \in Y$). C^l is the class of maps $A \longrightarrow B$ such that either $A \neq \varnothing$ or $A = B = \varnothing$. C^{ll} is the class of isomorphisms. C^{lr} is the class of maps $\varnothing \longrightarrow B$, B is arbitrary C^{lrl} is the class of maps which admit a section. $C^{lll} = C^{llr} = ..$ is the class of all maps.

Thus we see that already in this simplest case, taking iterated orthogonals (negation) produces several notions from a textbook, namely surjective, subset, discrete, connected, non-empty, and totally disconnected.

1.5. Intuition/Yoga of orthogonality.— We suggest the following intuition/yoga is helpful.(4)

(4)We were unable to find literature which explicitly describes this intuition, and will be thankful for any references which either discuss this intuition or list potential (counter)examples.

– taking iterated orthogonals (negation) is a cheap way to automatically "generate" interesting notions; a number of standard textbook notions are obtained in this way. We saw that taking iterated negations of the simplest map of topological spaces, $\{\} \longrightarrow \{\bullet\}$, generates 5 classes worthy of being defined in a first year course of topology (surjective, subset, discrete, connected, non-empty and totally disconnected).
– it helps to think of *orthogonality as a category-theoretic (substitute for) negation*; taking orthogonal is perhaps the simplest way to define a class of morphisms without a property in a manner useful for a diagram chasing calculation.
– often a morphism-counterexample can be "read off" from the text of the definition of an elementary textbook property, and the property can be concisely reformulated as the orthogonal of the class consisting of that counterexample.

1.6. Intuition/Yoga of transcription.— We suggest the following intuition/yoga is helpful.

– "transcribing" the usual text of mathematical definitions and arguments by means of notation extensively using arrows sometimes makes it possible to recognise familiar patterns of standard category-theoretic constructions and diagram chasing arguments.
– orthogonality of morphisms often appears in this way, and so do simplicial objects
– from the text of the definition of a topological property sometimes it is possible to "read off" a definition of a topology or a filter or a continuous function; it is worthwhile to try to interpret "for each open subsets there exists ..." as a requirement that some function is continuous

2. Examples of translation. Orthogonality as negation.

2.1. Dense subspaces and Kolmogoroff T_0 spaces.— We shall now transcribe the definitions of *dense* and *Kolmogoroff T_0* spaces.

2.1.1. "A is a dense subset of X."— By definition [Bourbaki, I§1.6, Def.12],

Definition 12. ***A subset* A *of a topological space* X *is said to be dense in* X (or simply *dense*, if there is no ambiguity about X) *if* $\overline{\mathrm{A}} = \mathrm{X}$, *i.e. if every non-empty open set* U *of* X *meets* A.**

Let us transcribe this by means of the language of arrows.

A subset A of a topological space X is an arrow $A \longrightarrow X$. (Note we are making a choice here: there is an alternative translation analogous to the one used in the next sentence). An open subset U of X is an arrow $X \longrightarrow \{U \to U'\}$; here $\{U \to U'\}$ denotes the topological space consisting of one open point U

and one closed point U'; by the arrow $\to$ we mean that that $U' \in cl(U)$. Non-empty: a subset U of X is *empty* iff the arrow $X \longrightarrow \{U \to U'\}$ factors as $X \longrightarrow \{U'\} \longrightarrow \{U \to U'\}$; here the map $\{U'\} \longrightarrow \{U \to U'\}$ is the obvious map sending U' to U'. set U of X meets A: $U \cap A = \varnothing$ iff the arrow $A \longrightarrow X \longrightarrow \{U \to U'\}$ factors as $A \longrightarrow \{U'\} \longrightarrow \{U \to U'\}$.

Collecting above (Figure 1c), we see that a map $A \xrightarrow{f} X$ has dense image iff

$$A \xrightarrow{f} X \rightthreetimes \{U'\} \longrightarrow \{U \to U'\}$$

Note a little miracle: $\{U'\} \longrightarrow \{U \to U'\}$ is the simplest map whose image isn't dense. We'll see it happen again.

2.1.2. Kolmogoroff spaces, axiom T_0.— By definition [Bourbaki,I§1, Ex.2b; p.117/122],

> **b) A topological space is said to be a *Kolmogoroff space* if it satisfies the following condition : given any two distinct points *x*, *x'* of X, there is a neighbourhood of one of these points which does not contain the other. Show that an ordered set with the right topology is a Kolmogoroff space.**

Let us transcribe this. given any two ... points x, x' of X: given a map $\{x, x'\} \xrightarrow{f} X$. two *distinct* points: the map $\{x, x'\} \xrightarrow{f} X$ does not factor through a single point, i.e. $\{x, x'\} \longrightarrow X$ does not factor as $\{x, x'\} \longrightarrow \{x = x'\} \longrightarrow X$. The negation of the sentence there is a neighbourhood which does not contain the other defines a topology on the set $\{x, x'\}$: indeed, the antidiscrete topology on the set $\{x, x'\}$ is the only topology with the property that there is [no] neighbourhood of one of these points which does not contain the other. Let us denote by $\{x \leftrightarrow x'\}$ the antidiscrete space consisting of x and x'. Now we note that the text implicitly defines the space $\{x \leftrightarrow x'\}$, and the only way to use it is to consider a map $\{x \leftrightarrow x'\} \xrightarrow{f} X$ instead of the map $\{x, x'\} \xrightarrow{f} X$.

Collecting above (see Figure 1d), we see that *a topological space X is said to be a* Kolmogoroff *space iff any map $\{x \leftrightarrow x'\} \xrightarrow{f} X$ factors as $\{x \leftrightarrow x'\} \longrightarrow \{x = x'\} \longrightarrow X$.*

Note another little miracle: it also reduces to orthogonality of morphisms

$$\{x \leftrightarrow x'\} \longrightarrow \{x = x'\} \rightthreetimes X \longrightarrow \{x = x'\}$$

and $\{x \leftrightarrow x'\}$ is the simplest non-Kolmogoroff space.

2.1.3. Finite topological spaces as categories.— Our notation $\{U'\} \longrightarrow \{U \to U'\}$ and $\{x \leftrightarrow x'\} \longrightarrow \{x = x'\}$ suggests that *we reformulated the two topological properties of being dense and Kolmogoroff in terms of diagram chasing in (finite) categories.* And indeed, we may think of finite topological spaces as categories and of continuous maps between them as *functors*, as follows; see

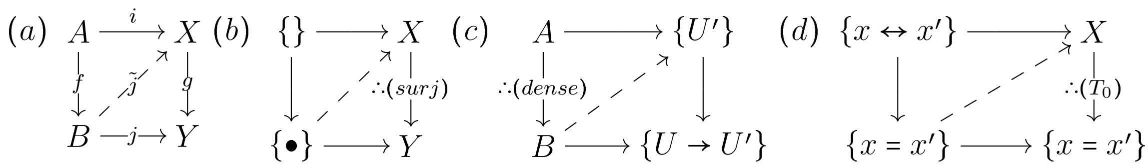


FIGURE 1. Lifting properties. Dots ∴ indicate free variables and what property of these variables is being defined; in a diagram chasing calculation, "$\therefore(dense)$" reads as: given a (valid) diagram, add label $(dense)$ to the corresponding arrow.
(a) The definition of a lifting property $f \rightthreetimes g$: for each $i : A \longrightarrow X$ and $j : B \longrightarrow Y$ making the square commutative, i.e. $f \circ j = i \circ g$, there is a diagonal arrow $\tilde{j} : B \longrightarrow X$ making the total diagram $A \xrightarrow{f} B \xrightarrow{\tilde{j}} X \xrightarrow{g} Y, A \xrightarrow{i} X, B \xrightarrow{j} Y$ commutative, i.e. $f \circ \tilde{j} = i$ and $\tilde{j} \circ g = j$. (b) $X \longrightarrow Y$ is surjective (c) the image of $A \longrightarrow B$ is dense in B (d) X is Kolmogoroff/T_0

Appendix 5.3.1 for details and a definition of our notation for finite topological spaces and maps between them.

A *topological space* comes with a *specialisation preorder* on its points: for points $x, y \in X$, $x \leq y$ iff $y \in clx$ (y is in the *topological closure* of x). The resulting *preordered set* may be regarded as a *category* whose *objects* are the points of X and where there is a unique *morphism* $x{\to}y$ iff $y \in clx$.

For a *finite topological space* X, the specialisation preorder or equivalently the corresponding category uniquely determines the space: a *subset* of X is *closed* iff it is *downward closed*, or equivalently, it is a subcategory such that there are no morphisms going outside the subcategory.

The monotone maps (i.e. *functors*) are the *continuous maps* for this topology.

We denote a finite topological space by a list of the arrows (morphisms) in the corresponding category; '↔' denotes an *isomorphism* and '=' denotes the *identity morphism*. An arrow between two such lists denotes a *continuous map* (a functor) which sends each point to the correspondingly labelled point, but possibly turning some morphisms into identity morphisms, thus gluing some points.

2.2. Compactness via ultrafilters.— We try to interpret the definition of compactness in [Bourbaki,I§9.1, Def.1(C′)] in terms of arrows, or rather we try to rewrite it using the arrow notation as much as possible. Doing so we shall see that this definition, in appropriate notation, condenses to *a Hausdorff space K is quasi-compact iff $K \longrightarrow \{\bullet\}$ is in*

$$((\{\{o\} \longrightarrow \{o \to c\}\}^r)_{<5})^{lr},$$

and that the latter expression almost appears in [Bourbaki, I§10.2,Thm.1d] as a characterisation of the class of proper maps.

2.2.1. Reading the definition of quasi-compactness.— We read the definition of quasi-compactness [Bourbaki,I§9.1, Def.I]; we do not know how to read (C) and therefore we read its reformulation (C′).

Definition 1. ***A topological space* X *is said to be quasi-compact if it satisfies the following axiom:***

(C) ***Every filter on* X *has at least one cluster point.***

A topological space is said to be compact if it is quasi-compact and Hausdorff.

It follows immediately from this axiom that if f is a mapping of a set Z into a quasi-compact space X, and $\mathfrak{F}$ is any filter on Z, then f has at least one cluster point with respect to $\mathfrak{F}$. In particular, every sequence of points of a quasi-compact space has at least one cluster point; but this condition is not equivalent to (C) (Exercise 11).

We give three axioms each of which is *equivalent to axiom* (C):

(C′) ***Every ultrafilter on* X *is convergent.***

A space K is *quasi-compact* iff each ultrafilter $\mathfrak{U}$ on the set of points of K converges, i.e. for each ultrafilter $\mathfrak{U}$ on the set of points of K there is a point $x \in K$ such that each open neighbourhood of x is $\mathfrak{U}$-big. This contains a quantification over open subsets; this suggests to us that we should try to extract a definition of topology from the text and to interpret the requirement as continuity of a certain map. each open neighbourhood of x is $\mathfrak{U}$-big suggest we define a topology such that an open subset is an $\mathfrak{U}$-big open neighbourhood of some $x \in K$. This defines a topology on $K \sqcup \{\text{“x”}\}$:

$$\{U : U \subset K \text{ is open}\} \cup \{U \cup \{\text{“x”}\} : U \subset K \text{ is open and } \mathfrak{U}\text{-big}\}$$

Denote the set equipped with this topology by $K \sqcup_{\mathfrak{U}} \{\text{“x”}\}$. (Note [Bourbaki, I§6.5, Definition 5, Example] define this space.)

Thus, in terms of arrows the definition becomes (see Figure 2a): K is quasi-compact iff the identity map $K \xrightarrow{\text{id}} K$ factors as

$$K \longrightarrow K \sqcup_{\mathfrak{U}} \{\text{“x”}\} \longrightarrow K$$

for each ultrafilter $\mathfrak{U}$ on the set of points of K.

Now note that Figure 2a is a particular case of orthogonality $K \longrightarrow K \sqcup_{\mathfrak{U}} \{\text{“x”}\} \rightthreetimes K \longrightarrow \{\bullet\}$, see Figure 2b where the map $K \longrightarrow K$ is arbitrary. Using orthogonals (negation), we express this by saying that $K \longrightarrow K \sqcup_{\mathfrak{U}} \{\text{“x”}\} \in \{K \longrightarrow \{\bullet\}\}^l$. As usual, we are tempted to define compactness as an orthogonal (negation) of a class (property) of morphisms, and therefore we check that all maps of form $A \longrightarrow A \sqcup_{\mathfrak{U}} \{\bullet\}$ lie in this orthogonal $\{K \longrightarrow \{\bullet\}\}^l$. Conversely, this also means that $K \longrightarrow \{\bullet\}$, for K quasi-compact, lies in the right orthogonal (negation) $\{\, A \longrightarrow A \sqcup_{\mathfrak{U}} \{\bullet\} : \mathfrak{U}$ is an ultrafilter on a space $A\,\}^r$.

Summing up, we read Definition I as

> **Definition I.** ***A topological space X is said to be quasi-compact if it satisfies the following axiom:***
>
> $(\mathrm{C}')_{\rightthreetimes}$ $A \longrightarrow A \sqcup_{\mathfrak{U}} \{\bullet\} \rightthreetimes X \longrightarrow \{\bullet\}$ for each ultrafilter $\mathfrak{U}$ on each space A

Note that there is another, more direct, way to read off this lifting property $(\mathrm{C}')_{\rightthreetimes}$ from a remark in the proof of $(C) \implies (C')$:

(C) ⟹ (C′): For if an ultrafilter has a cluster point then it converges to this point (§ 7, no. 2, Corollary to Proposition 4).

> **If *f* is a mapping of a set Z into a quasi-compact space X, and 𝔘 is an ultrafilter on Z, *f* has at least one limit point with respect to 𝔘 (§ 6, no. 6, Proposition 10).**

In terms of arrows, this reformulation is *precisely* the lifting property

$$Z \longrightarrow Z \sqcup_{\mathfrak{U}} \{\text{“x”}\} \rightthreetimes X \longrightarrow \{\bullet\}$$

We'd like to view the fact that Bourbaki chooses to formulate explicitly *precisely* a lifting property immediately following a key definition as evidence that Bourbaki is implicitly doing category theoretic reasoning.

2.2.2. Proper maps.— If we were to think that [Bourbaki, General Topology] does implicitly uses category theoretic reasoning and orthogonality, we'd hope to find there the definition of the class

$$\{\, A \longrightarrow A \sqcup_{\mathfrak{U}} \{\bullet\} : \mathfrak{U} \text{ is an ultrafilter on a space } A\,\}^r$$

And indeed, this is how Bourbaki characterises the class of proper maps in [Bourbaki, General Topology, I§10.2,Th.1(d)] (cf. Figure 2d), almost exactly. We see this as evidence that Bourbaki does indeed use category theoretic reasoning, or perhaps as an explanation of what do we mean by saying so.

Note we might have started our translation with this characterisation of proper maps in terms of ultrafilters [Bourbaki, General Topology, I§10.2,Th.1(d)], and we'd then arrive at Figure 2d directly.

Theorem 1. *Let $f: X \to Y$ be a continuous mapping. Then the following four statements are equivalent:*

a) *f is proper.*

b) *f is closed and $\overset{-1}{f}(y)$ is quasi-compact for each $y \in Y$.*

c) *If $\mathfrak{F}$ is a filter on X and if $y \in Y$ is a cluster point of $f(\mathfrak{F})$ then there is a cluster point x of $\mathfrak{F}$ such that $f(x) = y$.*

d) *If $\mathfrak{U}$ is an ultrafilter on X and if $y \in Y$ is a limit point of the ultrafilter base $f(\mathfrak{U})$, then there is a limit point x of $\mathfrak{U}$ such that $f(x) = y$.*

However, this reformulation is unsatisfactory for us: it uses non-elementary, infinitary, set-theoretic notion of ultrafilters which we do not know how to manipulate category-theoretically..

We'd like to have a definition which relies on maps between finite spaces.

An argument similar to a linear algebra about dual vector spaces gives the following. For any class C of maps we have that $C^l = C^{lrl}$ and $C^r = C^{rlr}$ and $C_1 \subset C_2$ implies $C_1^l \supset C_2^l$ and $C_1^r \supset C_2^r$. This implies $P^{lr} \subset C^{rlr} = C^r$ whenever $P \subset C^r$.

Take P to be some class of proper maps between finite spaces. By above we see that P^{lr} is a subclass of the class of proper maps. We want to take P to be large enough so that P^{lr} is the whole class of proper maps. And indeed, we find that a classical theorem in general topology tells us we can do so, at least if we only care about spaces satisfying separation axioms. Moreover, we will see it is enough to take P to consist of the following maps between spaces of size at most 3:

$$\{\boldsymbol{B_1} \leftarrow O \rightarrow \boldsymbol{B_2}\} \longrightarrow \{\bullet\} \qquad \{U\} \longrightarrow \{U \rightarrow U'\}$$
$$\{x \leftrightarrow y\} \longrightarrow \{x = y\} \qquad \{o \rightarrow c\} \longrightarrow \{o = c\}$$

2.2.3. Reducing to finite spaces.— Now we are back translating; we ignore the considerations of the previous subsubsection which give us a rather good idea of what we would get as the result of translation.

Reduction to finite spaces is provided by Smirnov-Vulikh-Taimanov theorem in the form by [Engelking, 3.2.1,p.136] ("compact" below stands for "compact Hausdorff"):

3.2.1. THEOREM. *Let A be a dense subspace of a topological space X and f a continuous mapping of A to a compact space Y. The mapping f has a continuous extension over X if and only if for every pair B_1, B_2 of disjoint closed subsets of Y the inverse images $f^{-1}(B_1)$ and $f^{-1}(B_2)$ have disjoint closures in the space X.*

Let us transcribe this. We are given a dense subspace $A \xrightarrow{i} X$ of a topological space X and a continuous mapping $A \xrightarrow{f} Y$ of A to a [Hausdorff] compact space Y. The mapping f has a continuous extension over X means that the arrow

$A \xrightarrow{f} Y$ factors via $A \xrightarrow{i} X$ (cf. Figure 2f). A pair $\boldsymbol{B_1}$, $\boldsymbol{B_2}$ of disjoint closed subsets of Y is an arrow $Y \longrightarrow \{\boldsymbol{B_1} \leftarrow O \rightarrow \boldsymbol{B_2}\}$ where $\{\boldsymbol{B_1} \leftarrow O \rightarrow \boldsymbol{B_2}\}$ is the space with one open point denoted by O and two closed points denoted by $\boldsymbol{B_1}$ and $\boldsymbol{B_2}$. To say the inverse images $f^{-1}(\boldsymbol{B_1})$ and $f^{-1}(\boldsymbol{B_2})$ have disjoint closures in the space X is to say that the composition $A \xrightarrow{f} Y \longrightarrow \{\boldsymbol{B_1} \leftarrow O \rightarrow \boldsymbol{B_2}\}$ factors as $A \xrightarrow{i} X \longrightarrow \{\boldsymbol{B_1} \leftarrow O \rightarrow \boldsymbol{B_2}\}$ (cf. Figure 2g).

Now we need to define the class of dense subspaces. A dense subspace is an injective map with dense image such that the topology on the domain is induced from the target. This suggests we try to define this class by taking left negations (orthogonals) of the simplest archetypal examples of maps with non-dense image, a non-injective map, and a map such that the topology on the domain is not induced from the target.

3.2.1. THEOREM. Let Y be Hausdorff compact and let $A \xrightarrow{i} X$ satisfy (cf. Figure 2(ijk))

(i) (dense) $A \xrightarrow{i} X \rightthreetimes \{U\} \longrightarrow \{U \rightarrow U'\}$

(ii) (injective) $A \xrightarrow{i} X \rightthreetimes \{x \leftrightarrow y\} \longrightarrow \{x = y\}$

(iii) (induced topology) $A \xrightarrow{i} X \rightthreetimes \{o \rightarrow c\} \longrightarrow \{o = c\}$

Then the properties of $A \xrightarrow{f} Y$ defined by Figure 2(f) and Figure 2(g) are equivalent.

This implies that, for Hausdorff compact Y, items 3.2.1(i-iii) and $A \xrightarrow{i} X \rightthreetimes \{\boldsymbol{B_1} \leftarrow O \rightarrow \boldsymbol{B_2}\} \longrightarrow \{\boldsymbol{B_1} = O = \boldsymbol{B_2}\}$ imply that $A \xrightarrow{i} X \rightthreetimes Y \longrightarrow \{\bullet\}$.

Further, note that if $X = A \sqcup \{\text{“x”}\}$ is obtained from A by adjoining a single closed non-open point, then

$$A \xrightarrow{i} X \rightthreetimes \{\boldsymbol{B_1} \leftarrow O \rightarrow \boldsymbol{B_2}\} \longrightarrow \{\boldsymbol{B_1} = O = \boldsymbol{B_2}\}$$

iff there exists an ultrafilter $\mathfrak{U}$ such that $A \xrightarrow{i} X$ is of form $A \longrightarrow A \sqcup_{\mathfrak{U}} \{\text{“x”}\}$.

This implies that maps of form $A \longrightarrow A \sqcup_{\mathfrak{U}} \{\text{“x”}\}$ are in P^l and, finally, that a Hausdorff space K is quasi-compact iff $K \longrightarrow \{\bullet\}$ is in P^{lr} where P consists of

$$\begin{array}{ll} \{\boldsymbol{B_1} \leftarrow O \rightarrow \boldsymbol{B_2}\} \longrightarrow \{\bullet\} & \{U\} \longrightarrow \{U \rightarrow U'\} \\ \{x \leftrightarrow y\} \longrightarrow \{x = y\} & \{o \rightarrow c\} \longrightarrow \{o = c\} \end{array}$$

2.2.4. The simplest counterexample negated three times.— Note that all maps between finite spaces mentioned in the preceeding subsubsection are closed, hence proper by [Bourbaki, I§10.2,Thm.1b].

A verification shows that, for Y and Z finite, the map $Y \xrightarrow{g} Z$ is closed iff

$$\{o\} \longrightarrow \{o \rightarrow c\} \rightthreetimes Y \xrightarrow{g} Z$$

Denote by $(\{\{o\} \longrightarrow \{o \to c\}\}^r)_{<5}$ the subclass of $\{\{o\} \longrightarrow \{o \to c\}\}^r$ consisting of maps between spaces of size at most 4.

Considerations above could be summarized by:

Claim. — *In the category of topological spaces,*

– *a Hausdorff space K is quasi-compact iff*
$K \longrightarrow \{\bullet\}$ *is in* $((\{\{o\} \longrightarrow \{o \to c\}\}^r)_{<5})^{lr}$.

– *every map in* $((\{\{o\} \longrightarrow \{o \to c\}\}^r)_{<5})^{lr}$ *is proper*

And we conjecture that the latter is in fact the class of all proper maps.

Conjecture. — *In the category of topological spaces, the following orthogonal defines the class of proper maps:*

$$((\{\{o\} \longrightarrow \{o \to c\}\}^r)_{<5})^{lr}$$

2.2.5. Compactification as factorisation system/M2-decomposition. — By a simple diagram chasing argument,[(5)] [(6)] each morphism $X \longrightarrow Y$ decomposes as either $X \xrightarrow{(P)^{rl}} \cdot \xrightarrow{(P)^r} Y$ and $X \xrightarrow{(P)^l} \cdot \xrightarrow{(P)^{lr}}$ whenever P is a class of morphisms and the underlying category has enough limits and colimits.

We shall now see that Stone-Čech compactification is an example of such a decomposition when P is the class of proper maps and is thus somewhat analogous to the $(cw)(f)$- and $(c)(wf)$-decomposition required by Axiom M2 of Quillen model categories.

Almost this observation is mentioned explicitly in [Bousfield, Constructions of factorization systems in categories][(7)]:

5.1 Example. In the category Top of topological spaces, let E_1 be the class of all maps $X \to Y$ inducing a bijection $\mathbf{Top}(Y, I) \approx \mathbf{Top}(X, I)$ where I is the closed unit interval. Then $(E_1, \mathcal{M}(E_1))$ is a factorization system in Top by 3.4. One can show that the $(E_1, \mathcal{M}(E_1))$-localization (2.5) on Top is just the Stone–Cech compactification (cf. [7, p. 127]).

In fact, this observation can be found by transcribing [Engelking, Theorem 3.6.1, p.173] by means of diagram chasing (see Figure 3 for the statement and

(5) See Thm. 3.1 of [Bousfield, Constructions of factorization systems in categories] for details of such an argument and assumptions which are enough to make it work. However, note that his definitions are somewhat different from ours: unlike us, he considers the *unique* lifting property, cf.§2 [ibid.].

(6) See [Holgate,PhD,2.1(Perfect Maps)] and references therein for examples of factorisation systems related to Stone-Ćech decomposition and proper maps. Note [Holgate] says "perfect" instead of "proper", as is common in topology.

(7) In our notation $\mathcal{M}(\mathbf{E}_1)$ is almost $(\mathbf{E}_1)^r$ but not quite: $\mathcal{M}(\mathbf{E}_1)$ is the right orthogonal ($\rightthreetimes$-negation) with respect to the *unique* lifting property; [7] is [S. MacLane, Categories for the Working Mathematician (Springer-Verlag, New York, 1971)].

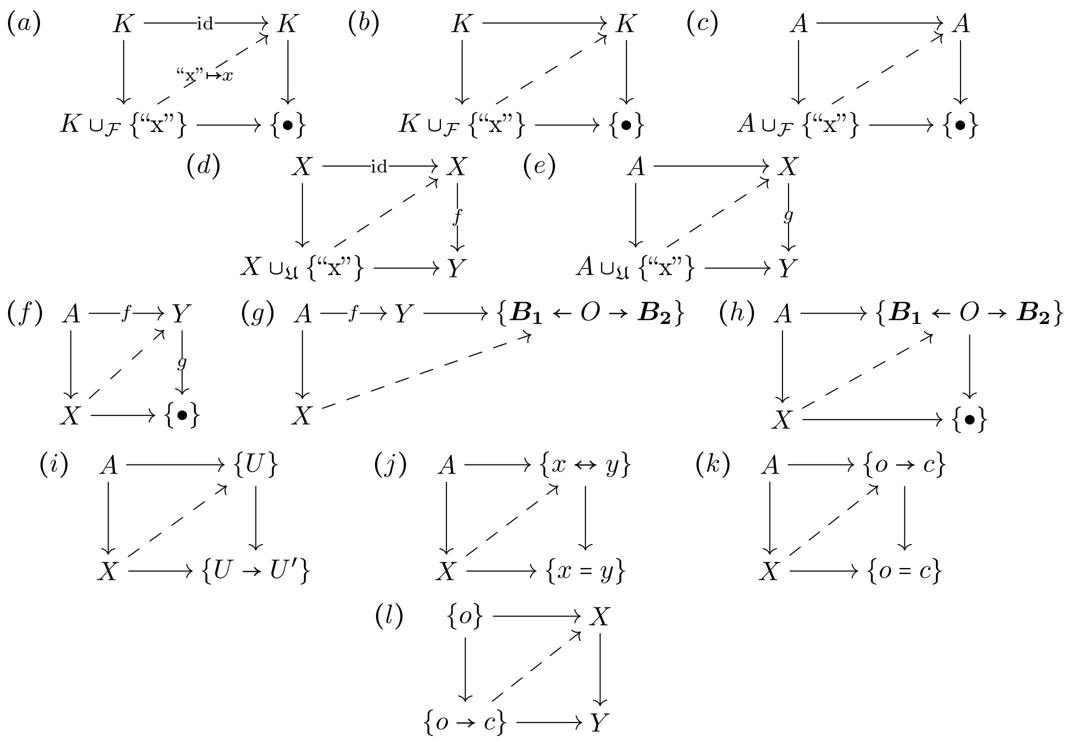


FIGURE 2. These are equivalent reformulations of quasi-compactness of spaces and its generalisation to maps, that of properness of maps. (a) the identity map $K \xrightarrow{\text{id}} K$ factors as $K \longrightarrow K \cup_{\mathcal{F}} \{\text{“x”}\} \longrightarrow K$ (b) this is also equivalent to K being quasi-compact (we no longer require the arrow $K \longrightarrow K$ to be identity) (c) and in fact quasi-compact spaces are orthogonal to maps associated with ultrafilters (d) $X \xrightarrow{f} Y$ is proper, i.e. d) If $\mathfrak{U}$ is an ultrafilter on X and if $y \in Y$ is a limit point of the ultrafilter base $f(U)$, then there is a limit point x of $\mathfrak{U}$ such that $f(x) = y$. [Bourbaki, General Topology, I§10.2,Th.1(d)] (e) this is also equivalent to $X \xrightarrow{f} Y$ is proper, i.e. this holds for each ultrafilter $\mathfrak{U}$ on each space A (f) The mapping f has a continuous extension over X (h) for every pair B_1, B_2 of disjoint closed subsets of Y the inverse images $f^{-1}(B_1)$ and $f^{-1}(B_2)$ have disjoint closures in the space X (i) the image of A is dense in B (j) the map $A \longrightarrow B$ is injective (k) the topology on A is induced from B (l) for X and Y finite, this means that the map $X \longrightarrow Y$ is closed, or, equivalently, proper

related lifting diagrams). In fact, corollaries [Engelking, 3.6.2-3.6.9] could also be seen in a diagram chasing way; below we only reformulate Corollary 3.6.3.

Let us now transcribe Theorem 3.6.1 [Engelking] by means of the notation of arrows. We will deliberately ignore the separability assumptions that X is Tychonoff and βX and Z are assumed to be Hausdorff.

Let P be the class of proper maps. Figure 3ab represent the statement of the theorem [Engelking, Theorem 3.6.1]. Figure 3a suggests that the compactification map $X \longrightarrow \beta X$ is in the class $(P)^l$; Figure 3b suggests that there is a unique decomposition $X \xrightarrow{(P^l)} X' \xrightarrow{(P)^{lr}} \{\bullet\}$.

And indeed, this is implied by a simple diagram chasing argument. Uniqueness follows from orthogonality of $(P)^l$ and $(P)^r$. The decomposition is constructed by an argument which looks roughly as follows:[(8)] consider all the decompositions of form $X \xrightarrow{(P^l)} X' \longrightarrow Y$ and take the pushout $X \xrightarrow{(P)^l} X_l$ of all the maps $X \xrightarrow{(P^l)} X'$ appearing in the decompositions of this form. The map belongs to $(P)^l$ because left orthogonals are closed under pushouts, By the universality property of pushouts you obtain a decomposition $X \xrightarrow{(P)^l} X_l \longrightarrow Y$ and a diagram chasing argument based on the definition of pushout and orthogonality properties of $(P)^l$ and $(P)^{lr}$ shows the map $X_l \longrightarrow Y$ is right orthogonal to $(P)^l$, i.e. belongs to $(P)^{lr}$ as required. An argument of this kind is known as Quillen small object argument and originally was used to prove Axiom M2 $(cw)(f)$- and $(c)(wf)$-decomposition of model categories.

The argument shows that under suitable assumptions that a category has enough limits and colimits, any morphism $X \longrightarrow Z$ decomposes as $X \xrightarrow{(P^l)} Y \xrightarrow{(P)^{lr}} Z$, for any class (P) of morphisms. Here we take (P) to be the class of proper morphisms.

We end our discussion of compactness with the following rather vague considerations; we hope they might suggest the reader something about the arrow notation (calculus) appropriate for topology. We admit that what we say below is very vague.

2.2.6. Compactness as being uniform. $\forall\exists \implies \exists\forall$. — Often an application of compactness is as follows. We know that certain choices can be made for each value of parameters; if we also know that the parameters vary over a compact domain, then we may assume that these choices are uniform, i.e. that they do not depend on the value of the parameters. Put another way, compactness

(8) For details see footnote (5).

Let us recall that the largest element in the family $C(X)$ of all compactifications of a Tychonoff space X is called the Čech-Stone compactification of X and is denoted by βX.

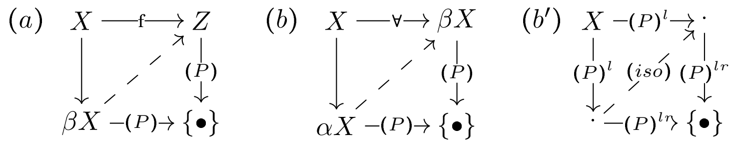
3.6.1. THEOREM. *Every continuous mapping $f: X \to Z$ of a Tychonoff space X to a compact space Z is extendable to a continuous mapping $F: \beta X \to Z$.*

If every continuous mapping of a Tychonoff space X to a compact space is continuously extendable over a compactification αX of X, then αX is equivalent to the Čech-Stone compactification of X.

From Theorem 3.6.1 a series of important corollaries follows.

3.6.3. COROLLARY. *Every continuous function $f: X \to I$ from a Tychonoff space X to the closed interval I is extendable to a continuous function $F: \beta X \to I$.*

If every continuous function from a Tychonoff space X to the closed interval I is continuously extendable over a compactification αX of X, then αX is equivalent to the Čech-Stone compactification of X. ∎



$$(c) \quad X \xrightarrow{(P^l)} \beta X \xrightarrow{(P)} \{\bullet\} \qquad (d) \quad X \overset{(P^l)}{\dashrightarrow} Y \overset{(P)^{lr}}{\dashrightarrow} Z$$

$$(e) \quad X \xrightarrow{(P^l)} \beta X \rightthreetimes [0,1] \longrightarrow \{\bullet\}$$

$$(f) \qquad X \longrightarrow \alpha X \xrightarrow{(P)} \{\bullet\} \text{ and } X \xrightarrow{(P^l)} \alpha X \rightthreetimes [0,1] \longrightarrow \{\bullet\} \text{ implies } \alpha X = \beta X$$

FIGURE 3. A diagram chasing reformulation of [Engelking, Theorem 3.6.1, p.173]. (a) Every continuous mapping $f : X \longrightarrow Z$ of a Tychonoff space X to a compact space Z is extendable to a continuous mapping $F : \beta X \longrightarrow Z$. (b) If every continuous mapping of a Tychonoff space X to a compact space is continuously extendable over a compactification αX of X, then αX is equivalent to the Cech-Stone compactification of X. This is reformulated as follows: if diagram (b) holds, then the diagonal map $\alpha X \longrightarrow \beta X$ can be chosen to be an isomorphism. (b′) this is an analogue of (b) formulated in terms of category theory as uniqueness of $\cdot \xrightarrow{(P)^l} \cdot \xrightarrow{(P)^{lr}} \cdot$ decomposition; the diagonal arrow exists because $(P)^l \rightthreetimes (P)^{lr}$ and thus we require it to be an isomorphism. (c) Both diagrams above can summarized as: there exists a unique decomposition of this form. (d) Further, this is implied by an analogue of Axiom M2 $(cw)(f)$- and $(c)(wf)$-decomposition of model categories: each morphism $X \longrightarrow Z$ decomposes as $X \overset{(P^l)}{\dashrightarrow} Y \overset{(P)^{lr}}{\dashrightarrow} Z$ (e) Every continuous function $f : X \longrightarrow X$ from a Tychonoff space X to the closed interval I is extendable to a continuous function $F : \beta X \longrightarrow I$. (f) If every continuous function from a Tychonoff space X to the closed interval I is continuously extendable over a compactification αX of X, then αX is equivalent to the Čech-Stone compactification of X. Note the conclusion $\alpha X = \beta X$ is stated somewhat imprecisely; we rather need to say that morphisms $X \longrightarrow \alpha X$ and $X \longrightarrow \beta X$ are the same.

allows to change the order of quantifiers $\forall\exists \implies \exists\forall$ in certain formulas. See Appendix 5.5 for a list of examples.[9]

The subsection of [Stacks Project, I.5§15, tag 005M] dealing with the Bourbaki characterisation of proper maps starts with a lemma of this kind:

***Lemma 1* (Tube lemma).** — *Let X and Y be topological spaces. Let $A \subset X$ and $B \subset Y$ be quasi-compact subsets. Let $A \times B \subset W \subset X \times Y$ with W open in $X \times Y$. Then there exists opens $A \subset U \subset X$ and $B \subset V \subset Y$ such that $U \times V \subset W$.*

In a somewhat more old-fashioned way, this lemma can be reformulated as follows:

***Lemma 2* (Tube lemma).** — *Let X and Y be topological spaces. Let $A \subset X$ and $B \subset Y$ be quasi-compact subsets. Let $A \times B \subset W \subset X \times Y$.*

If for each pair of points $a \in A$ and $b \in B$ we can pick neighbourhoods $U = U(a,b) \ni a$ and $V = V(a,b) \ni b$ such that $(a,b) \in U \times V \subset W$, then we can do so uniformly in $a \in A$ and $b \in B$, i.e. such that $U = U(a,b)$ and $V = V(a,b)$ do not depend on a and b.

As a formula, this could be expressed as change of order of quantifiers:

$$\frac{\forall a \in A \forall b \in B\, \exists U \subset X\, \exists V \subset Y\, (U \times V \subset W \text{ and } a \in U \text{ is open and } b \in V \text{ is open})}{\exists U \subset X\, \exists V \subset Y\, \forall a \in A \forall b \in B\, (U \times V \subset W \text{ and } a \in U \text{ is open and } b \in V \text{ is open})}$$

The following example of change or order of quantifiers is simpler but perhaps more telling.

For a connected topological space X, the following are equivalent:

- Each real-valued function on X is bounded
- $\forall x \in K \exists M (f(x) < M) \implies \exists M \forall x \in K (f(x) < M)$
- $\emptyset \longrightarrow K \rightthreetimes \sqcup_{n\in\mathbb{N}}(-n,n) \longrightarrow \mathbb{R}$
  here $\cup_n(-n,n) \longrightarrow \mathbb{R}$ denotes the map to the real line from the disjoint union of intervals $(-n,n)$ which cover it. Note this is a standard example of an open covering of $\mathbb{R}$ which shows it is not compact.

The following is even more vague.

2.2.7. "An open covering has a finite subcovering". — Mathematically, this reformulation is based on the following observation:

> a space K is compact iff for each open covering U of K, the subset K is closed in $K \cup \{\infty\}$ in the topology generated elements of U as *closed* subsets.

This lets us express being *finite* with the help of the notion of the topology generated by a family of sets.

[9] For a discussion see Remark 8 of [Gavrilovich, Lifting Property]

[Hausdorff, Set theory] denotes by $U(x)$ a neighbourhood of a point x, which suggests viewing $U(x)$ as a (possibly multivalued) function of a point x ; We'd like to develop "arrow" notation where this would be expressed as

$$\{x\} \longrightarrow K \xrightarrow{(U(x))} \{x \to y\} \qquad (*)$$

here it is implicit that x maps to x by the composition of the two arrows; "(x)" in "$U(x)$" signifies that $U(x)$ depends on x.

Changing a single symbol "$\to$" into "$\leftarrow$" leads us to consider elements of U as closed subsets of K:

$$\{x\} \longrightarrow K \xrightarrow{(U(x))} \{x \leftarrow y\} \qquad (**)$$

We'd like to assume (or require) that $(**)$ inherits some properties of $(*)$, in the arrow calculus we'd like to define; this would be what corresponds to considering *the topology generated by.*

2.2.8. Summary.— These three examples suggest that orthogonality, or $\rightthreetimes$-negation, has a surprising generative power as a means of defining natural elementary mathematical concepts. In Appendix 5.2.1 and Appendix 5.3.2 we give a number of examples in various categories, in particular showing that many standard elementary notions of abstract topology can be defined by applying the lifting property to simple morphisms of finite topological spaces. Examples in topology include the notions of: compact, discrete, connected, and totally disconnected spaces, dense image, induced topology, and separation axioms. Examples in algebra include: finite groups being nilpotent, solvable, torsion-free, p-groups, and prime-to-p groups; injective and projective modules; injective, surjective, and split homomorphisms.

2.3. Hausdorff axioms of topology as diagram chasing computations with finite categories . — We shall now reformulate the axioms of a topology in a form almost ready to be implemented in a theorem prover based on diagram chasing.

Early works talk of topology in terms of *neighbourhood* systems U_x where U_x varies though *neighbourhoods of points* of a topological space. This is how the notion of topology was defined by Hausdorff; in words of [Bourbaki], "We shall say that a set E carries a topological structure whenever we have associated with each element of E, by some means or other, a family of subsets of E which are called neighbourhoods of this element - provided of course that these neighbourhoods satisfy certain conditions (the axioms of topological structures)."

Whenever we are speaking of a neighbourhood U_x of a point $x \in E$, we are speaking of two functions

$$\{x\} \longrightarrow X \xrightarrow{U} \{x \to x'\}$$

We would like to be able to say that a set E carries a topological structure whenever we have associated with each element x of E, by some means or other, a family of arrows, or functions, $\{x\} \longrightarrow E \longrightarrow \{x \to x'\}$ provided of course that these arrows satisfy certain conditions (corresponding to the axioms of topological structures).

This simple observation allows us to show that the axioms of topology formulated in the more modern language of open subsets can be seen as diagram chasing rules for manipulating diagrams involving notation such as

$$\{x\} \longrightarrow X \qquad X \longrightarrow \{x \searrow y\} \qquad X \longrightarrow \{x \leftrightarrow y\}$$

in the following straightforward way.

2.3.1. Axioms of open sets as diagram chasing rules.— As is standard in category theory, identify a point x of a topological space X with the arrow $\{x\} \longrightarrow X$, a subset Z of X with the arrow $X \longrightarrow \{z \leftrightarrow z'\}$, and an open subset U of X with the arrow $X \longrightarrow \{u \searrow u'\}$. With these identifications, the Hausdorff axioms of a topological space become rules for manipulating such arrows, as follows.

Both the empty set and the whole of X are open says that the compositions

$$X \longrightarrow \{c\} \longrightarrow \{o \searrow c\} \text{ and } X \longrightarrow \{o\} \longrightarrow \{o \searrow c\}$$

behave as expected (the preimage of {o} is empty under the first map, and is the whole of X under the second map).

The intersection of two open subsets is open means the arrow

$$X \longrightarrow \{o \searrow c\} \times \{o' \searrow c'\}$$

behaves as expected (the "two open subsets" are the preimages of points $o \in \{o \searrow c\}$ and $o' \in \{o' \searrow c'\}$; "the intersection" is the preimage of (o, o') in $\{o \searrow c\} \times \{o' \searrow c'\}$).

The preimage of an open set is open says the composition

$$X \longrightarrow Y \longrightarrow \{u \searrow u'\} \longrightarrow \{u \leftrightarrow u'\}$$

is well-defined.

We need the following terminology to formulate the next diagram chaning reformulation. We say that *a diagram commutes from vertex A to vertex B* iff the composition of morphisms along any two paths from A to B is the same. We say *a diagram commutes at (to) a vertex A* iff it commutes from A to any vertex B (from any vertex A to B, resp.).

Finally, let us write a diagram chasing rule which corresponds to the fact that in topology we consider subsets which of elements and that functions are defined element-wise. It allows to reduce diagram chasing to finite objects.

– for each arrows $A \xrightarrow{f} B$, $A \xrightarrow{g} B$ it holds
$A \rightrightarrows_{g}^{f} B$ iff for each $\{u\} \longrightarrow A$, the diagram commutes at vertex $\{u\}$: $\{u\} \longrightarrow A \rightrightarrows_{g}^{f} B$

– for each arrows $A \xrightarrow{f} B$, $A \xrightarrow{g} B$ it holds
$A \rightrightarrows_{g}^{f} B$ iff for each $B \longrightarrow \{x \leftrightarrow y\}$, the diagram commutes to vertex $\{x \leftrightarrow y\}$: $A \rightrightarrows_{g}^{f} B \longrightarrow \{x \leftrightarrow y\}$

– for each arrows $A \xrightarrow{f} B$, $A \xrightarrow{g} B$ it holds
$A \rightrightarrows_{g}^{f} B$ iff for each $\{u\} \longrightarrow A$ and $B \longrightarrow \{x \leftrightarrow y\}$, the diagram commutes from $\{u\}$ to $\{x \leftrightarrow y\}$: $\{u\} \longrightarrow A \rightrightarrows_{g}^{f} B \longrightarrow \{x \leftrightarrow y\}$

2.3.2. An arbitrary union of open subsets is open as $\forall\exists \implies \exists\forall$.— Finally, the axiom that an arbitrary union of open subsets is necessarily open can be reformulated in the following ways:

- *A subset U of X is open iff each point u of U has an open neighbourhood inside of U.*
- *If for each point x of a subset U we can pick an open neighbourhood $x \in U_x \subseteq U$ within U, then we can do so in such a way that U_x does not depend on x* (and therefore $U_x = U$ for each $x \in U$).
- As an $\forall\exists \implies \exists\forall$ implication (cf. §2.2.6,§5.5),

$$\frac{\forall x \in U\, \exists V\, (x \in V \text{ and } V \subseteq U \text{ and } V \text{ is open})}{\exists V\, \forall x \in U\, (x \in V \text{ and } V \subseteq U \text{ and } V \text{ is open})}$$

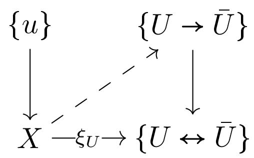
Let us give several reformulations in terms of diagram chasing:

for each arrow $X \xrightarrow{\xi_U} \{U \leftrightarrow \bar{U}\}$ it holds

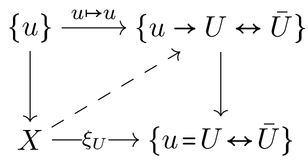
$$\begin{array}{ccc} & & \{U \to \bar{U}\} \\ & \nearrow & \downarrow \\ X & \xrightarrow{\xi_U} & \{U \leftrightarrow \bar{U}\} \end{array} \quad \text{iff for each } \{u\} \longrightarrow X \quad \begin{array}{ccc} \{u\} & \longrightarrow & \{u \to U \leftrightarrow \bar{U}\} \\ \downarrow & \nearrow & \downarrow \\ X & \xrightarrow{\xi_U} & \{u = U \leftrightarrow \bar{U}\} \end{array} \qquad (\star)$$

The following reformulation uses that sets consist of points:
for each arrow $X \xrightarrow{\xi_U} \{U \leftrightarrow \bar{U}\}$ the following are equivalent:

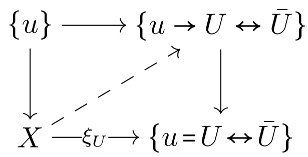
- there is an arrow $X \longrightarrow \{U \longrightarrow \bar{U}\}$
  for each $\{u\} \longrightarrow X$
  the diagram commutes at vertex $\{u\}$



- for each $\{u\} \longrightarrow X$
  making the square commute
  there is an arrow $X \longrightarrow \{u \to U \leftrightarrow \bar{U}\}$
  making the diagram commute



- for each $\{u\} \longrightarrow X$
  for each $\{u\} \longrightarrow \{u \to U \leftrightarrow \bar{U}\}$
  making the square commute
  there is an arrow $X \longrightarrow \{u \to U \leftrightarrow \bar{U}\}$
  making the diagram commute



Question 2.1. — *It is tempting to rewrite the axioms about infinite unions as the following sequence of diagram chasing rules, which seem to be closer to intuitive considerations:*

- *pick a new arrow* $\{u\} \longrightarrow X$
- *construct an arrow* $X \longrightarrow \{u \to U \leftrightarrow \bar{U}\}$ *in some way such that the diagram* $(\star)$ *commutes*
- *remove the dependency of* $X \longrightarrow \{u \to U \leftrightarrow \bar{U}\}$ *on* $\{u\} \longrightarrow X$*, i.e. label the arrow* $X \longrightarrow \{u \to U \leftrightarrow \bar{U}\}$ *as* not dependent *on* $\{u\} \longrightarrow X$

Define a formal syntax and a proof system capturing this kind of derivations.

We cannot stress enough the speculation below.

Remark 2.2. — *We find it extremely intriguing that an axiom of topology admits an* $\forall\exists \implies \exists\forall$ *reformulation. We view it as a sign that topology is really about, so to say, permuting quantifiers, or, in other words, expressly manipulating what variable/term/construction depends on what; so to say, topology reasons about dependency rather than continuity. A technical way to start thinking about this point of view is provided by the* $\forall\exists \implies \exists\forall$ *reformulations in §2.3.2 and* $\forall\exists \implies \exists\forall$ *reformulations of compactness in §2.2.6 and §5.5. See also a discussion in §4.3.3.*

We hope that this reinterpretation may help clarify the nature of the axioms of a topological space, in particular it offers a constructive approach and a diagram chasing formalisation of certain elementary arguments, may clarify to what extent set-theoretic language is necessary, and perhaps help to suggest an approach to ”tame topology” of Grothendieck, i.e. a foundation of topology "without false problems" and "wild phenomena" "at the very beginning".

2.4. Urysohn lemma as a definition of the real line (Unfinished). — We sketch a concise exposition of the proof of Urysohn Lemma and then speculate how it leads to concrete suggestions on how to define the notions of a path and of a map to $\mathbb{R}$ in terms of diagram chasing with finite spaces. Namely, we speculate a map to/from $\mathbb{R}$ or $[0,1]$ determines a collection of maps $X \longrightarrow$ /\/\, resp. a collection of maps $X \xrightarrow{\text{(path)}}$ ... /\/\/\ .... \|/ . which are maximal collections of arrows with certain diagram chasing properties. (Here /\/\ and ... /\/\/\ .... \|/ . denote particular preorders.) These diagram chasing properties should represent the operations of taking finer and finer partitions $0 = t_0 < t_1 < t_2 < ... < t_n < 1$ of $[0,1]$ and that of covering a path by smaller and smaller neighbourhoods.

We'd like to reformulate these properties of collections as diagram chasing rules to manipulate labels or arrows to particular finite preorders, and below we make very preliminary suggestions towards this.

Our considerations are probably a representation of ideas of Cech cohomology, the Freyd definition of the interval and Drinfeld note on geometric realization.

2.4.1. A concise exposition of the proof of Urysohn Lemma. — We hope this exposition may make the argument more transparent to students, and be used to shorten and make more transparent parts of formalization of the proof of Urysohn lemma, e.g. see the proof in Mizar. Note that a minor modification delivers the proof (and statement) of Tietze extension theorem. First we give the standard exposition of the proof of Urysohn lemma as written in the notes by Terrence Tao, and then rewrite it in our notation.

> ***Lemma 1* (Urysohn's lemma).** — *Let X be a topological space. Then the following are equivalent:*
>
> - (i) *Every pair of disjoint closed sets K, L in X can be separated by disjoint open neighbourhoods $U \supset K$, $V \supset L$.*
> - (ii) *For every closed set K in X and every open neighbourhood U of K, there exists an open set V and a closed set L such that $K \subset V \subset L \subset U$.*
> - (iii) *For every pair of disjoint closed sets K, L in X, there exists a continuous function $f : X \to [0,1]$ which equals 1 on K and 0 on L.*
> - (iv) *For every closed set K in X and every open neighbourhood U of K, there exists a continuous function $f : X \to [0,1]$ such that $1_K(x) \leq f(x) \leq 1_U(x)$ for all $x \in X$.*
>
> A topological space which obeys any (and hence all) of (i-iv) is known as a normal space; definition (i) is traditionally taken to be the standard

definition of normality. We will give some examples of normal spaces shortly.

Proof: The equivalence of (iii) and (iv) is clear, as the complement of a closed set is an open set and vice versa. The equivalence of (i) and (ii) follows similarly.

To deduce (i) from (iii), let K, L be disjoint closed sets, let f be as in (iii), and let U, V be the open sets $U := \{x \in X : f(x) > 2/3\}$ and $V := \{x \in X : f(x) < 1/3\}$.

The only remaining task is to deduce (iv) from (ii). Suppose we have a closed set $K = K_1$ and an open set $U = U_0$ with $K_1 \subset U_0$. Applying (ii), we can find an open set $U_{1/2}$ and a closed set $K_{1/2}$ such that

$$K_1 \subset U_{1/2} \subset K_{1/2} \subset U_0. \qquad (U_{1/2})$$

Applying (ii) two more times, we can find more open sets $U_{1/4}, U_{3/4}$ and closed sets $K_{1/4}, K_{3/4}$ such that

$$K_1 \subset U_{3/4} \subset K_{3/4} \subset U_{1/2} \subset K_{1/2} \subset U_{1/4} \subset K_{1/4} \subset U_0. \qquad (U_{1/4})$$

Iterating this process, we can construct open sets U_q and closed sets K_q for every dyadic rational $q = a/2^n$ in $(0,1)$ such that $U_q \subset K_q$ for all $0 < q < 1$, and $K_{q'} \subset U_q$ for any $0 \leq q < q' \leq 1$.

If we now define $f(x) := \sup\{q : x \in U_q\} = \inf\{q : x \notin K_q\}$, where q ranges over dyadic rationals between 0 and 1, and with the convention that the empty set has sup 0 and inf 1, one easily verifies that the sets $\{f(x) > \alpha\} = \bigcup_{q>\alpha} U_q$ and $\{f(x) < \alpha\} = \bigcup_{q<\alpha} X\backslash K_q$ are open for every real number α, and so f is continuous as required. □

Rewrite (i) and (iii) as lifting properties as follows:

$$(i)_\rightthreetimes \qquad \varnothing \longrightarrow X \rightthreetimes \{K \swarrow^{U\backslash K} \searrow x \swarrow^{V\backslash L} \searrow L\} \longrightarrow \{K \swarrow^{U\backslash K = x = V\backslash L} \searrow L\}$$

$$(iii)_\rightthreetimes \qquad \varnothing \longrightarrow X \rightthreetimes \{0'\} \cup [0,1] \cup \{1'\} \longrightarrow \{0 = 0' \searrow x \swarrow 1 = 1'\}$$

where $\{0'\} \cup [0,1] \cup \{1'\}$ is the interval with two closed points glued to the endpoints, $cl(0) = \{0, 0'\}, cl(1) = \{1, 1'\}$[(10)]

When (i) and (iii) are rewritten in this form, it is tempting to prove $(iii)_\rightthreetimes$ by iterating $(i)_\rightthreetimes$ infinitely many times.

And indeed, we may do so by doing pullbacks and liftings along the 3 maps /\/\ ⟶ /\ (we renamed points to simplify notation):

$$\{t_1 \swarrow^{o_1} \searrow t_2 \swarrow^{o_2} \searrow t_3\} \longrightarrow \{t_1 = o_1 = t_2 \swarrow^{o_2} \searrow t_3\}$$

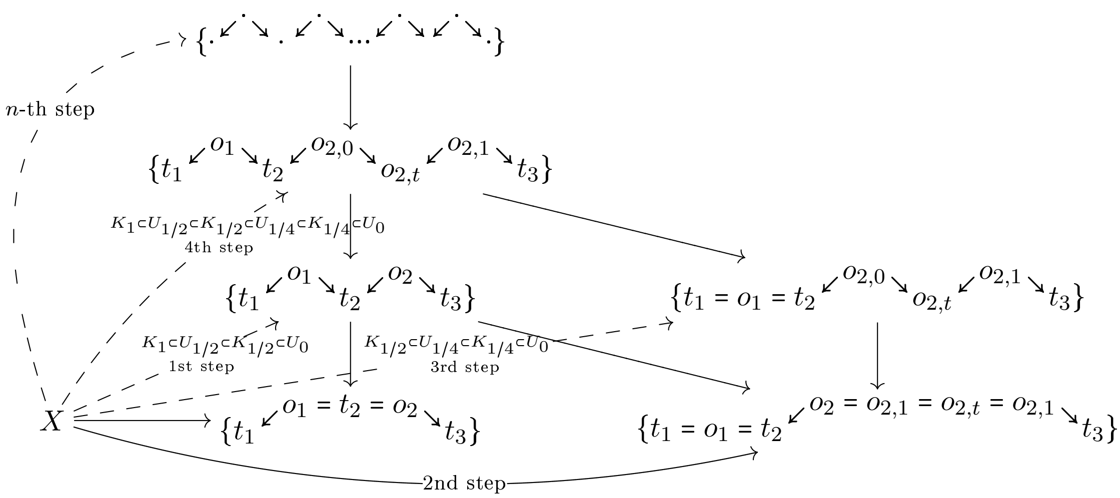
$$\{t_1 \swarrow^{o_1} \searrow t_2 \swarrow^{o_2} \searrow t_3\} \longrightarrow \{t_1 \swarrow^{o_1 = t_2 = o_2} \searrow t_3\}$$

---

(10) In more detail, the topology on $\{0'\} \cup [0,1] \cup \{1'\}$ is defined by $\{U : U \subset [0,1] \text{ open}\} \cup \{0' \cup U : 0 \in U \subset [0,1] \text{ open}\} \cup \{1' \cup U : 1 \in U \subset [0,1] \text{ open}\}$ in particular $0'$ and $1'$ are closed, and $cl(0) = \{0, 0'\}, cl(1) = \{1, 1'\}$.

$$\{t_1 \swarrow^{o_1} \searrow t_2 \swarrow^{o_2} \searrow t_3\} \longrightarrow \{t_1 \swarrow^{o_1} \searrow t_2 = o_2 = t_3\}$$

We draw the first two steps; applying $(ii)_\curlywedge$ we get at the 1st step $X \longrightarrow /\backslash/\backslash$, in Tao's notation $K_1 \subset U_{1/2} \subset K_{1/2} \subset U_0$. Applying $(ii)_\curlywedge$ once more, at the 4th step we find $X \longrightarrow /\backslash/\backslash/\backslash$. Tao applies (ii) two more times and finds decomposition $K_1 \subset U_{3/4} \subset K_{3/4} \subset U_{1/2} \subset K_{1/2} \subset U_{1/4} \subset K_{1/4} \subset U_0$, which in our notation would correspond to $X \longrightarrow /\backslash/\backslash/\backslash/\backslash$.



Iterating this builds maps $X \longrightarrow \{\cdot \swarrow^{\cdot} \searrow \cdot \swarrow^{\cdot} \searrow \dots \swarrow^{\cdot} \searrow \swarrow^{\cdot} \searrow \cdot\}$. Now an argument using the definition of real numbers finishes the argument, for example, as follows.

A sequence of real numbers $0 = t_0 < t_1 < t_2 < \dots < t_n = 1$, $t_i \in \mathbb{R}$ determines a decomposition of the interval $[0,1] \subset \mathbb{R}$ as a union of points and open intervals; and this decomposition can be viewed as a map to a finite space which contracts the open subintervals (t_i, t_{i+1}) to open points o_i: $[0,1] \longrightarrow \{t_0 \swarrow^{o_0} \searrow t_1 \swarrow^{o_1} \searrow t_2 \swarrow^{o_2} \searrow \dots \swarrow^{o_n} \searrow t_n\}$. Taking finer and finer decompositions gives rise to a system of maps from $[0,1]$ to these finite spaces; on the other hand, as we saw above, iterating (i) also gives rise to a system of maps to these finite spaces. A verification now shows such a system of maps determine a map to $[0,1]$.

Tietze extension theorem. — *Let X be a normal topological space, let $[a,b]$ be a bounded interval, let K be a closed subset of X, and let $f : X \longrightarrow [a,b]$ be a continuous function. Then there exists a continuous function $\tilde{f} : X \longrightarrow [a,b]$ which extends f, i.e. $\tilde{f}(x) = f(x)$ for all $x \in K$.*

Rewrite the conclusion as a lifting property (here $\{\bullet\}$ denotes the space consisting of a single point):

$$K \longrightarrow X \rightthreetimes [a,b] \longrightarrow \{\bullet\}$$

A diagram chasing argument shows that the assumption implies the lifting properties (the first one is $(i)_\rightthreetimes$ above for $K \longrightarrow X$):

$$K \longrightarrow X \rightthreetimes \{t_1 \swarrow^{o_1} \searrow t_2 \swarrow^{o_2} \searrow t_3\} \longrightarrow \{t_1 \swarrow^{o_1 = t_2 = o_2} \searrow t_3\}$$

$$K \longrightarrow X \rightthreetimes \{t_1 \swarrow^{o_1} \searrow t_2\} \longrightarrow \{t_1 = o_1 = t_2\}$$

Now as in the proof of Urysohn lemma, iterating these lifting property constructs $\tilde f : X \longrightarrow [a,b]$ as required.

2.4.2. Definition of the notion of a map to $\mathbb{R}$. — Analysing the proof leads to the following suggestions towards a diagram-chasing definition of maps to the real line or the interval.

Real line $\mathbb{R}$ comes equipped with the collection of maps $\mathbb{R} \longrightarrow \cdot \swarrow^{\cdot} \searrow \cdot \swarrow^{\cdot} \searrow \cdot$ associated with decompositions, for $s < t \in \mathbb{R}$

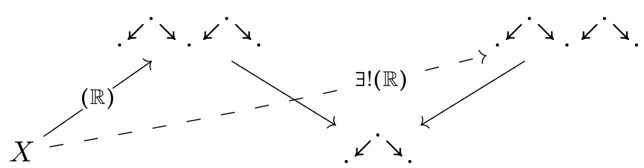
$$\mathbb{R} = (-\infty, s] \cup (s, \frac{s+t}{2}) \cup \{\frac{s+t}{2}\} \cup (\frac{s+2}{2}, +\infty)$$

Hence, a map $X \longrightarrow \mathbb{R}$ determines a collection of maps $X \longrightarrow \cdot \swarrow^{\cdot} \searrow \cdot \swarrow^{\cdot} \searrow \cdot$, and it is tempting to give a diagram-chasing definition of a map $X \longrightarrow \mathbb{R}$ as a maximal collection of maps $X \longrightarrow \cdot \swarrow^{\cdot} \searrow \cdot \swarrow^{\cdot} \searrow \cdot$ compatible in a certain sense.

***Question* (Notion of a map to $\mathbb{R}$).** — *Define a map $f : X \longrightarrow [0,1]$ as a label on maps $X \overset{(\mathbb{R})}{\longrightarrow} \cdot \swarrow^{\cdot} \searrow \cdot \swarrow^{\cdot} \searrow \cdot$ which satisfies some diagram chasing compatibility properties.*

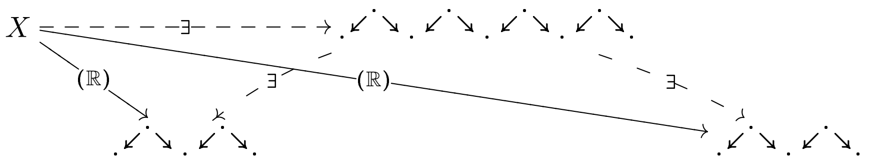
These conditions should probably include:

- (any open interval can be uniquely divided in half) for any two of the three maps $\cdot \swarrow^{\cdot} \searrow \cdot \swarrow^{\cdot} \searrow \cdot \longrightarrow \cdot \swarrow^{\cdot} \searrow \cdot$ corresponding to contracting a subinterval



- in $X \overset{(\mathbb{R})}{\longrightarrow} \cdot \swarrow^{\cdot} \searrow \cdot \swarrow^{\cdot} \searrow \cdot$, the preimage of the closed middle point is a point

– (points are linearly ordered) for any two maps $X \xrightarrow{(\mathbb{R})} .\swarrow\overset{.}{}\searrow.\swarrow\overset{.}{}\searrow.$ there are maps $.\swarrow\overset{.}{}\searrow.\swarrow\overset{.}{}\searrow. \longleftarrow .\swarrow\overset{.}{}\searrow.\swarrow\overset{.}{}\searrow.\swarrow\overset{.}{}\searrow.\swarrow\overset{.}{}\searrow. \longrightarrow .\swarrow\overset{.}{}\searrow.\swarrow\overset{.}{}\searrow.$ preserving order such that



2.4.3. Definition of the notion of a path. — It is tempting to define the notion of a path in a dual way. Below are very preliminary considerations.

An essential property of a (smooth enough) path is that it can be covered by a sequence of small open neighbourhoods. Any collection of open subsets of a space determines a map to a finite space, and for an non-self-intersecting path such a sequence determines a map to the following finite space

$$X \longrightarrow t_0 \swarrow^{o_0}\searrow t_1 \swarrow^{o_1}\searrow t_2 \swarrow^{o_2}\searrow \ldots \swarrow^{o_n}\searrow t_n$$
$$\searrow * \swarrow \quad \searrow * \swarrow \quad \searrow * \swarrow \quad \searrow * \swarrow$$
.

or equivalently a map to a partial order $\begin{matrix} . \; . \; ... \; . \; . \\ /\backslash/\backslash/ \; \backslash/\backslash/\backslash \\ . \; . \; ..... \; . \; . \\ \backslash\backslash\backslash // / \\ . \end{matrix}$ We hope that a (possibly self-intersecting) path determines a collection of such maps which is a maximal collection satisfying certain diagram chasing properties. Moreover, we hope these diagram chasing properties correspond to lifting properties representing the operation of splitting one small neighbourhood into two, and saying these open neighbourhoods do form a sequence.

Note the partial orders $/\backslash/\backslash ... /\backslash$ occuring in the definition of a map to $\mathbb{R}$ are suborders of the partial orders occuring in the definition of a path, and we hope this leads to a tautological diagram chasing corstruction corresponding to the identity map $\mathbb{R} \longrightarrow \mathbb{R}$.

Question **(Notion of a path).** — *Define a the notion of path $\gamma : \mathbb{R} \longrightarrow X$ or $\gamma : [0,1] \longrightarrow X$ as a label on maps* $X \xrightarrow{(path)} \begin{matrix} . \; . \; . \\ /\backslash/\backslash/\backslash \\ . \; . \; . \; . \\ \backslash\backslash // \\ . \end{matrix}$ *which satisfies some diagram chasing compatibility properties, and is maximal such.*

3. Topological spaces as simplicial filters.

We shall now introduce terminology which we feel allows us to more directly give precise meaning to phrases such as "*such and such a property holds for all points sufficiently near a*". We will do so by "reading it off" the informal considerations of [Bourbaki, Introduction] of the intuitive notions of limit, continuity and neighbourhood.

This section may be read independently of the rest of the paper. Unlike the previous section, we do not introduce a concise formal syntax to describe the categorical structures that arise.

3.1. Reading the definition of topology.— Now we pretend to directly transcribe the following explanations of Bourbaki of the intuition of topology and analysis [Bourbaki, Introduction, p.13]

> **To formulate the idea of neighbourhood we started from the vague concept of an element "sufficiently near" another element. Conversely, a topological structure now enables us to give precise meaning to the phrase "such and such a property holds for all points *sufficiently near a*": by definition this means that the set of points which have this property is a neighbourhood of *a* for the topological structure in question.**
>
> **As we have already said, a topological structure on a set enables one to give an exact meaning to the phrase "whenever *x* is sufficiently near *a*, *x* has the property P{*x*}". But, apart from the situation in which a "distance" has been defined, it is not clear what meaning ought to be given to the phrase "every pair of points *x*, *y* which are sufficiently near each other has the property P{*x*, *y*}", since *a priori* we have no means of comparing the neighbourhoods of two different points. Now the notion of a pair of points near to each other arises frequently in classical analysis (for example, in propositions which involve uniform continuity). It is therefore important that we should be able to give a precise meaning to this notion in full generality, and we are thus led to define structures which are richer than topological structures, namely *uniform structures*. They are the subject of Chapter II.**

Call a subset $P \subseteq X \times X$ *topoic* iff $(a,x) \in P$ whenever x is sufficiently near a, i.e. for each $a \in X$ there is a neighbourhood $a \in U_a \subseteq X$ of a such that $\{a\} \times U_a \subseteq P$. This terminology enables us to give precise meaning to the phrase "for each point a such and such a property [$P(a,x)$] holds for all points *sufficiently near* a": by definition this means that the set of pairs of points which have this property is topoic for the topological structure in question.

Call a subset $P \subset X \times X \times X$ *topoic* iff we can ensure $(a,x,y) \in P$ by first picking x sufficiently near to a, and then picking y sufficiently near x, i.e. for each $a \in X$ there is a neighbourhood $a \in U_a \subseteq X$ of a such that for each $x \in U_a$ there is a neighbourhood $x \in U_{a,x} \subseteq X$ such that $(a,x,y) \in P$ for each $y \in U_{a,x}$. Similarly we define topoic subsets of X^n for $n > 2$, see §3.1.1 for details.

Topoic subsets of X^n form a filter where by a *filter* we mean a topological space such that a subset containing a non-empty open set is necessarily open. Axioms of topology say that the Cartesian powers of X with these filters form a 2-dimensional simplicial object $\mathcal{T}(X)$ in the category of filters.[11] Moreover

$$\mathcal{T} : \mathrm{Top} \longrightarrow Func(\mathrm{Ord}^{op}_{<\omega}\,,\ \mathcal{F}ilt)$$

is a fully faithful embedding of the category of topological spaces into the category of simplicial objects (in fact, of dimension 2) in the category of filters.

Here $\mathcal{F}ilt \subset \mathrm{Top}$ is the full subcategory of the category of topological spaces formed by filters (for us a filter is a topological space such that a set containing a non-empty open subset is necessarily open, hence by a morphism of filters we mean a map such that the preimage of a big set is big). or, equivalently, it is the category of pointed topological spaces and morphisms are maps continuous at the point and preserving the point. $\mathrm{Ord}_{<\omega}$ denotes the category of categories corresponding to finite linear orders

$$\bullet_1 \longrightarrow .. \longrightarrow \bullet_n,\ \ 0 < n < \omega.$$

In particular, the axiom a neighbourhood of a point x is also a neighbourhood of all points sufficiently near to x, says that $pr_{1,3} : X \times X \times X \longrightarrow X \times X,\ (x_1, x_2, x_3) \mapsto (x_1, x_3)$ is continuous wrt the filters defined above, or, equivalently, wrt the filter on $X \times X \times X$ defined as pullback along projections $X \times X \xleftarrow{pr_{1,2}} X \times X \times X \xrightarrow{pr_{2,3}} X \times X$.

Continuity of a function $f : X \longrightarrow X'$ "given any point $x_0 \in X$ and any neighbourhood V' of $f(x_0)$ in X', there is a neighbourhood V of x_0 in X such that the relation $x \in V$ implies $f(x) \in V'$" is expressed by saying that the preimage $f^{-1}(P) \subset X \times X$ of any topoic subset $P \subset X' \times X'$ is topoic, or, by saying that the obvious natural transformation $\mathcal{T}(X) \xrightarrow{\mathcal{T}(f)} \mathcal{T}(Y)$ is well-defined.

As we have already said, talking about topoic subsets wrt a topological structure on a set enables one to give an exact meaning to the phrase "whenever x is sufficiently near a, x has the property $P\{x\}$". But, apart from the situation in which a "distance" has been defined, it is not clear what meaning ought to

[11] We find it convenient to allow filters where the empty set is big, i.e. we allow the filter of all subsets of a set.
The category $\mathcal{F}ilt$ of filters can be thought in three equivalent ways: (i) it is a full subcategory of the category of topological spaces whose objects are spaces such that a subset containing a non-empty open subset is necessarily open (ii) its objects are pointed topological spaces with morphisms being maps continuous at the point (iii) its objects are sets equipped with a finitely additive measure taking only two values 0 and 1 and such that a subset of a measure 0 set has necessarily measure 0; morphisms are measurable maps preserving the measure (iv) its objects are sets equipped with a collection of subsets called *big* such that the intersection of two big subsets is big and a subset containing a big subset is necessarily big as well; morphisms are maps such that the preimage of a big subset is necessarily big.
(ii) and (iii) suggest that one may also consider the category $\mathcal{F}ilt$ of filters localised as follows: two maps $f, g : X \longrightarrow Y$ are considered equal as morphisms iff they are equal locally, resp. almost everywhere, i.e. the subset $\{x : f(x) = g(x)\}$ is big in X.

be given to the phrase "every pair of points x, y which are sufficiently near each other has the property $P\{x, y\}$", since *a priori* topoic subsets wrt a topological structure give no means of comparing the neighbourhoods of two different points. Now the notion of a pair of points near to each other arises frequently in classical analysis (for example, in propositions which involve uniform continuity). It is therefore important that we should be able to give a precise meaning to this notion in full generality, and we are thus led to define the notion of topoic subsets wrt structures which are richer than topological structures, namely *uniform structures*.

For a metric space M, call a subset $P \subseteq M \times M$ *topoic* iff "every pair of points x, y which are sufficiently near each other has the property $P\{x, y\}$", i.e. there is $\varepsilon > 0$ such that $(x, y) \in P$ whenever $dist(x, y) < \varepsilon$. More generally, call a subset $P \subseteq M^n$ *topoic* iff there is $\varepsilon > 0$ such that $(x_1, ..., x_n) \in P$ whenever $dist(x_i, x_j) < \varepsilon$ for each $1 \leq i \leq j \leq n$.

The axioms of uniform structure, cf. [Bourbaki,II§I.1], say that the filters of topoic subsets on M^n, $n \geq 0$, define a 2-dimensional simplicial object

$$\mathfrak{M}(M) : \mathrm{Ord}^{op}_{<\omega} \longrightarrow \mathfrak{F}ilt$$

which factors as

$$\mathrm{Ord}^{op}_{<\omega} \longrightarrow FinSets^{op} \longrightarrow \mathfrak{F}ilt$$

where $FinSets^{op}$ is the category of finite sets.

In particular, to say the map of filters $M \xrightarrow{(x,x)} M \times M$ is continuous and the filter on M is antidicrete is almost to say Axiom $(\mathrm{U_I})$ Every set belonging to the set of entourages $\mathfrak{U}$ contains the diagonal Δ of $M \times M$. Axiom $(\mathrm{U_{II}})$ If $V \in \mathfrak{U}$ then $V^{-1} \in \mathfrak{U}$, where $V^{-1} = \{(y, x) : (x, y) \in V\}$, says the permutation of coordinates $M \times M \xrightarrow{(x,y) \mapsto (y,x)} M \times M$ is continuous.

Axiom $(\mathrm{U_{III}})$ For each entourage $V \in \mathfrak{U}$ there exists entourage $W \in \mathfrak{U}$ such that $W \circ W \subset V$ says that $pr_{1,3} : M \times M \times M \longrightarrow M \times M,\ (x_1, x_2, x_3) \mapsto (x_1, x_3)$ is continuous wrt the filter on $M \times M \times M$ defined as pullback along projections $M \times M \xleftarrow{pr_{1,2}} M \times M \times M \xrightarrow{pr_{2,3}} M \times M$.

Thus Axiom $(\mathrm{U_I})$ and $(\mathrm{U_{III}})$ of uniform structures say the functor $\mathfrak{M}(M) : \mathrm{Ord}^{op}_{<\omega} \longrightarrow \mathfrak{F}ilt$ is well-defined, and Axiom $(\mathrm{U_{II}})$ says it factors via $\mathrm{Ord}^{op}_{<\omega} \longrightarrow FinSets^{op}$.

This terminology gives us means of comparing the neighbourhoods of two different points and give a precise meaning to the notion of a pair of points near to each other which arises frequently in classical analysis (for example, in propositions which involve uniform continuity).

A topological argument often relies on consequently choosing "sufficiently near" points; in this case we expect that it implicitly constructs a topoic subset of $E \times .. \times E$.

Sometimes an argument chooses points not consequently, and we hope that often enough it implicitly constructs a topoic subset of $E \times .. \times E$, albeit in a topoic structure not associated with a topological structure and possibly specific to the argument.

Let us now define the topoic structure on a set E associated with a topological structure on E.

3.1.1. Topoic structure of a topological space.— Let X be a topological space. Call a property (subset) $P \subseteq X \times X$ *topoic* iff $(a, x) \in P$ holds whenever x is sufficiently near a, i.e. for each point $a \in X$ there is a neighbourhood U_a such that $(a, x) \in P$ whenever $x \in U_a$. Call a property $P \subseteq X^n$ *topoic* iff we can ensure that $(x_1, .., x_n) \in P$ provided we pick x_2 sufficiently near x_1, then pick x_3 sufficiently near x_2, then ... then pick x_n sufficiently near x_{n-1}, given any $x_1 \in X$, i.e.
for each point $x_1 \in X$ there is an open neighbourhood $U_{x_1} \ni x_1$ such that
for each point $x_2 \in U_{x_1}$ there is an open neighbourhood $U_{x_1,x_2} \ni x_2$ such that
for each point $x_3 \in U_{x_1,x_2}$ there is an open neighbourhood $U_{x_1,x_2,x_3} \ni x_3$ such that
for each point $x_4 \in U_{x_1,x_2,x_3}$ ....
.....
for each point $x_n \in U_{x_1,x_2,...,x_{n-1}}$ there is a neighbourhood $U_{x_1,x_2,...,x_{n-1}} \ni x_n$ such that $(x_1, ..., x_n) \in P$.
Topoic subsets form a filter (as well as a topology) on X^n: $P' \supset P$, P topoic implies P' is topoic, and the intersection of finitely many topoic sets is topoic.

As noted above, the *filter of topoic subsets* allows us to directly speak about "sufficiently near" points. If a topological argument relies on consequently choosing of "sufficiently near" points, then we expect that it implicitly constructs a topoic subset of $X \times .. \times X$.

3.1.2. Topoic structure of a metric space.— Let M be a metric space. Let us define the topoic structure associated with metric space M: a subset $P \subset M^n$ is topoic iff there is $\varepsilon > 0$ such that $(x_1, .., x_n) \in M$ provided $dist(x_i, x_j) < \varepsilon$ for each $1 \leq i \leq j \leq n$. Thereby we give the phrase "every pair of points x, y which are sufficiently near each other has the property $P\{x, y\}$" the precise meaning that P is topoic with respect to the topoic structure associated with the metric (distance) on M.

3.1.3. Continuity in topological spaces.— Let us now see that the intuitive explanation of continuity of a function by [Bourbaki, Introduction, p.13] transcribes directly to the language of topoic subsets.

> **Once topological structures have been defined, it is easy to make precise the idea of *continuity*. Intuitively, a function is continuous at a point if its value varies as little as we please whenever the argument remains sufficiently near the point in question. Thus continuity will have an exact meaning whenever the space of arguments and the space of values of the function are topological spaces. The precise definition is given in Chapter I, § 2.**

This reads as: given a topoic subset $W \subset Y \times Y$, we can find a topoic subset of $V \subset X \times X$ such that $(f(x_0), f(x)) \in W$ provided $(x_0, x) \in V$. Here "given" corresponds to as we please; "we can find" to sufficiently; the subset W being "topoic" corresponds to its value varies as little; and whenever the argument remains sufficiently *near the point in question* corresponds to finding a topoic subset W of $X \times X$ such that $(f(x_0), f(x)) \in W$ whenever $(x_0, x) \in V$.

Thus continuity of $f : X \longrightarrow Y$ means the map $f \times f : X \times X \longrightarrow Y \times Y$ is continuous wrt the topoic filters, or, equivalently, the obvious map of simplicial objects $\mathrm{T}(X) \xrightarrow{\mathrm{T}(f)} \mathrm{T}(Y)$ is well-defined.

3.1.4. Axioms of topology.— Bourbaki reformulate the axioms of topology as Axioms $(\mathrm{V_I} - \mathrm{V_{IV}})$ stated in terms of neighbourhood filters [Bourbaki,I§1.2], also cf. [ibid, Proposition 2]. Note that the notion of a neighbourhood is all that is need to define topoic subsets, and let us now try to understand these axioms in terms of topoic subsets and coordinate maps between X^n.

Let us denote by $\mathfrak{B}(x)$ the set of all neighbourhoods of x. The sets $\mathfrak{B}(x)$ have the following properties :

$(\mathrm{V_I})$ ***Every subset of X which contains a set belonging to $\mathfrak{B}(x)$ itself belongs to $\mathfrak{B}(x)$.***

$(\mathrm{V_{II}})$ ***Every finite intersection of sets of $\mathfrak{B}(x)$ belongs to $\mathfrak{B}(x)$.***

$(\mathrm{V_{III}})$ ***The element x is in every set of $\mathfrak{B}(x)$.***

Indeed, these three properties are immediate consequences of Definition 4 and axiom $(\mathrm{O_{II}})$.

$(\mathrm{V_{IV}})$ ***If V belongs to $\mathfrak{B}(x)$, then there is a set W belonging to $\mathfrak{B}(x)$ such that, for each $y \in$ W, V belongs to $\mathfrak{B}(y)$.***

By Proposition 1, we may take W to be any open set which contains x and is contained in V.

> **This property may be expressed in the form that *a neighbourhood of x is also a neighbourhood of all points sufficiently near to x.***

Axioms $(\mathrm{V_I})$ and $(\mathrm{V_{II}})$ say that topoic subsets (as defined above) do indeed form a filter.

$(\mathrm{V_{III}})$ ***The element x is in every set of $\mathfrak{B}(x)$.***

The filter on X is antidiscrete, and thus $(\mathrm{V_{III}})$ implies that the diagonal embedding

$$X \longrightarrow X \times X, \quad x \mapsto (x, x)$$

is continuous wrt the topoic filters. In fact, continuity of the diagonal embedding means that either for each x element x is in every set of $\mathfrak{B}(x)$ or for each x element x is in no set of $\mathfrak{B}(x)$.

(V_{IV}) *If* V *belongs to* $\mathfrak{B}(x)$, *then there is a set* W *belonging to* $\mathfrak{B}(x)$ *such that, for each* $y \in$ W, V *belongs to* $\mathfrak{B}(y)$.

By Proposition 1, we may take W to be any open set which contains *x* and is contained in V.

> **This property may be expressed in the form that *a neighbourhood of x is also a neighbourhood of all points sufficiently near to x*.**

This means that the map

$$X \times X \times X \longrightarrow X \times X, \quad (x_1, x_2, x_3) \mapsto (x_1, x_3)$$

is continuous; to see this, consider the preimage of $\{x\} \times V \subset X \times X$, or rather a topoic subset $\{x\} \times V \cup ((X \setminus \{x\}) \times X) \subset X \times X$. It has to be topoic, and by definition that means that for each point $x \in X$, there is a neighbourhood W of x such that, for each $y \in W$, V is a neighbourhood of y. (We may take W to be any open set which contains x_1 and is contained in V.)

A further verification shows that the maps

$$X^n \longrightarrow X^m, \; (x_1, ..., x_n) \mapsto (x_{i_1}, ..., x_{i_m}), \; 1 \leq i_1 \leq ... \leq i_m \leq n$$

have the property that the preimage of a topoic set is topoic.

3.1.5. Summing up: a simplicial object of a topological space.— The coordinate maps between Cartesian powers remind us of the simplicial object of Cartesian powers, and we are tempted to understand the topological structure as a construction of a simplicial object. And indeed, considerations above show that we obtain a functor

$$\mathrm{T}(X) : \mathrm{Ord}_{<\omega}^{op} \longrightarrow \mathcal{F}ilt$$

where $\mathcal{F}ilt$ is the category of filters, and $\mathrm{Ord}_{<\omega}$ denotes the category of categories corresponding to finite linear orders

$$\bullet_1 \longrightarrow .. \longrightarrow \bullet_n, \; 0 \leq n < \omega.$$

For a space X, the simplicial object $\mathrm{T}(X)$ is the object

$$(X_{\mathrm{T}}, X \times X_{\mathrm{T}}, X \times X \times X_{\mathrm{T}}, ...)$$

consisting of Cartesian powers of the set of points of X equipped with the filter of topoic subsets corresponding to the topological structure on X defined in §3.1.1.

Continuous maps $f : X \longrightarrow X'$ are in one-to-one correspondence with natural transformations $\mathrm{T}(X) \implies \mathrm{T}(X')$, and in fact there is a fully faithful embedding of the category of topological spaces in the category of simplicial filters

$$Top \subset Func(\mathrm{Ord}_{<\omega}^{op}, \mathcal{F}ilt)$$

3.1.6. Metric spaces.— Consider the topoic structure associated with a metric space M. A straightforward verification shows that permutations of coordinates $M^n \longrightarrow M^m, (x_1, ..., x_n) \mapsto (x_{i_1}, ..., x_{i_m})$, $1 \leq i_1, ..., \leq i_m \leq n$ have the property that the preimage of a topoic set is topoic, and hence we obtain a functor

$$\mathcal{M}(M) : \mathrm{Ord}^{op}_{<\omega} \longrightarrow \mathcal{F}ilt$$

which factors as

$$\mathrm{Ord}^{op}_{<\omega} \longrightarrow FinSets^{op} \longrightarrow \mathcal{F}ilt$$

where $FinSets^{op}$ is the category of finite sets. This functor sends n to the set M^n equipped with the filter of topoic subsets, i.e. the filter of subsets containing an ε-neighbourhood of the diagonal, for some $\varepsilon > 0$.

Given a mapping $f : M \longrightarrow M'$ of sets of points, the condition that the preimage of a topoic subset of $M \times M$ is necessarily a topoic subset of $M' \times M'$, says that for each $\delta > 0$ there is $\varepsilon > 0$ such that $\mathrm{dist}(f(x), f(y)) < \delta$ whenever $\mathrm{dist}(x, y) < \varepsilon$, i.e. the mapping f is uniformly continuous.

In fact, as is easy to see, this construction also works for uniform spaces, and we obtain a fully faithful embedding of the category of uniform spaces in the category of simplicial filters(12)

$$\mathcal{M} : UniformSpaces \subset Func(\mathrm{Ord}^{op}_{<\omega}, \mathcal{F}ilt)$$

For a filter $\mathfrak{F}$, let $\mathcal{L}(\mathfrak{F}) = Hom(-, \mathfrak{F})$ denote the simplicial filter $(\mathfrak{F}, \mathfrak{F} \times \mathfrak{F}, \mathfrak{F} \times \mathfrak{F} \times \mathfrak{F}, ...)$ consisting of Cartesian powers of $\mathcal{F}$ and coordinate maps. Let $\mathfrak{i}(\mathfrak{F})$ denote the simplicial filter $(\mathfrak{F}, \mathfrak{F}, \mathfrak{F}, ...)$ consisting of $\mathcal{F}$ itself and identity maps.

A *Cauchy filter* $\mathfrak{F}$ on a metric space M (cf. [Bourbaki,II§3.1,Def.2]) is a filter on the set of points of M such that the obvious map $\mathcal{L}(\mathfrak{F}) \longrightarrow \mathcal{M}(M)$ is well-defined.

A *Cauchy sequence* in M is a map $\mathcal{L}(\mathbb{N}_{cofinite}) \longrightarrow \mathcal{M}(M)$ where $\mathbb{N}_{cofinite}$ is the set of natural numbers equipped with cofinite topology (i.e. a subset is closed iff it is finite).

This allows to define various notions of equicontinuity of sequences of functions.

Let X be a topological space, let M be a metric space, and let $(f_i)_{i \in \mathbb{N}}$ be a family of functions $f_i : X \longrightarrow M$.

The family f_i is *equicontinuous* if either of the following equivalent conditions holds:

- for every $x \in X$ and $\varepsilon > 0$, there exists a neighbourhood U of x such that $d_Y(f_i(x'), f_i(x)) \leq \varepsilon$ for all $i \in \mathbb{N}$ and $x' \in U$
- the map $\mathcal{T}(X) \times \mathfrak{i}(\{\mathbb{N}\}) \longrightarrow \mathcal{M}(M)$, $(x, i) \longmapsto f_i(x)$ is well-defined
- the map $\mathcal{T}(X) \times \mathfrak{i}(\mathbb{N}_{cofinite}) \longrightarrow \mathcal{M}(M)$, $(x, i) \longmapsto f_i(x)$ is well-defined

(12) For more details see [Gavrilovich, Simplicial Filters], in particular Claim 2 which characterises the category of functors corresponding to uniform spaces.

If $X = (X, d_X)$ is also a metric space, we say that the family f_i is *uniformly equicontinuous* iff either of the following equivalent conditions holds:

– for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $d_Y(f_i(x'), f_i(x)) \le \varepsilon$ for all $i \in \mathbb{N}$ and $x', x \in x$ with $d_X(x, x') \le \delta$
– the map $\mathcal{M}(X) \times \mathcal{L}(\{\mathbb{N}\}) \longrightarrow \mathcal{M}(M),\ (x,i) \longmapsto f_i(x)$ is well-defined
– the map $\mathcal{M}(X) \times \mathcal{L}(\mathbb{N}_{cofinite}) \longrightarrow \mathcal{M}(M),\ (x,i) \longmapsto f_i(x)$ is well-defined

The family is *uniformly Cauchy* iff either of the following equivalent conditions holds:

– for every $\varepsilon > 0$ there exists a $\delta > 0$ and $N > 0$ such that $d_Y(f_i(x'), f_j(x)) \le \varepsilon$ for all $i, j > N$ and $x', x \in x$ with $d_X(x, x') \le \delta$.
– the map $\mathcal{M}(X) \times \mathcal{L}(\mathbb{N}_{cofinite}) \longrightarrow \mathcal{M}(M),\ (x,i) \longmapsto f_i(x)$ is well-defined

Here $\{\mathbb{N}\}$ denotes the trivial filter on $\mathbb{N}$ with a unique big subset $\mathbb{N}$ itself, and $\mathbb{N}_{cofinite}$ denotes the filter of cofinite subsets of $\mathbb{N}$.

This suggest we might reformulate Arzela-Ascoli theorem as something about inner Hom in $s\mathcal{F}ilt$, see Question 4.7.

3.1.7. Measure spaces.— [13] We can associate a topoic structure on a set with a measure on the set as follows.

Let X be a set and μ be a measure on X. We say a subset U of X^n is *$\mathcal{M}_{<\omega}$-big* iff there exists finitely many subsets $A_1, .., A_N$ such that $\mu(X \setminus \cup_{1\le i\le N} A_i) = 0$ and $\cup_{1\le i \le N} A_i^n \subset U$.

We say a subset U of X^n is *$\mathcal{M}$-big* iff there exists countably many subsets $A_1, A_2, ..$ such that $\mu(X \setminus \cup_{1\le i<\omega} A_i) = 0$ and $\cup_{1\le i<\omega} A_i^n \subset U$.

Similarly to above, these filters on the Cartesian powers of X and the coordinate maps define simplicial objects in the category of filters

$$(X_{\mathcal{M}_{<\omega}}, (X\times X)_{\mathcal{M}_{<\omega}}, (X\times X\times X)_{\mathcal{M}_{<\omega}}, ...)$$

$$(X_{\mathcal{M}}, (X\times X)_{\mathcal{M}}, (X\times X\times X)_{\mathcal{M}}, ...)$$

which we denote by $\mathcal{M}_{<\omega}(X,\mu)$ and $\mathcal{M}(X,\mu)$, resp. In fact these functors factor through the category of finite sets:

$$\mathcal{M}_{<\omega}(X,\mu) : \mathrm{Ord}^{op}_{<\omega} \longrightarrow FinSets^{op} \longrightarrow \mathcal{F}ilt$$

$$\mathcal{M}(X,\mu) : \mathrm{Ord}^{op}_{<\omega} \longrightarrow FinSets^{op} \longrightarrow \mathcal{F}ilt$$

Let Y be a set and ν be a measure on Y. A measurable map $f : X \longrightarrow Y$ such that $\mu(f^{-1}(A)) = 0$ whenever $\nu(A) = 0$ induces morphisms of functors $\mathcal{M}_{<\omega}(X,\mu) \xrightarrow{\mathcal{M}_{<\omega}(f)} \mathcal{M}_{<\omega}(Y,\nu)$ and $\mathcal{M}(X,\mu) \xrightarrow{\mathcal{M}(f)} \mathcal{M}(Y,\nu)$, and conversely, each morphism in $s\mathcal{F}ilt$ between these objects is of this form.

[13] This section is not finished, in a very preliminary state and may contain mistakes. I will appreciate any corrections and suggestions sent by readers.

3.1.8. Maps of metric spaces preserving geodesics.— [14]

For a metric space M, call a subset $P \subset M^n$ *topoic wrt geodesic structure* iff $(x_1, ..., x_n) \in P$ whenever

(∗) there is a geodesic in M first passing through x_1, then passing through x_2, ..., then passing through x_n.

The condition (∗) is preserved by coordinate maps $M^n \longrightarrow M^m$ preserving the order of coordinates, hence this does define a simplicial filter based on $(M, M^2, M^3, ...)$.

A map $f : M \longrightarrow M'$ induces a map of these topoic structures on M and M' if a geodesic in M maps into a geodesic in M'. The converse holds for M and M' nice enough, e.g. Riemannian manifolds where geodesics are locally unique.

Note that the topoic structure on M^2 is trivial if each pair of points on M can be connected by a geodesic.

Note that this definition can be modified in some obvious ways, e.g. call a subset $P \subset M^n$ *topoic wrt* $+\varepsilon$*-geodesic structure*, resp. $\cdot\varepsilon$*-geodesic structure*, iff there is $\varepsilon > 0$ such that $(x_1, ..., x_n) \in P$ whenever $(*)_{+\varepsilon}$, resp. $(*)_{\cdot\varepsilon}$, holds:

$(*)_{+\varepsilon}$ for any $1 \le i < j < k \le n$ $\quad \mathrm{dist}(x_i, x_j) + \mathrm{dist}(x_j, x_k) < \mathrm{dist}(x_i, x_k) + \varepsilon$

$(*)_{\cdot\varepsilon}$ for any $1 \le i < j < k \le n$ $\quad \mathrm{dist}(x_i, x_j) + \mathrm{dist}(x_j, x_k) < \mathrm{dist}(x_i, x_k)(1 + \varepsilon)$

3.2. Limits as maps to shifted (décalage) topological spaces.— We now try to transcribe the explanation of the notion of filter in [Bourbaki,I,Introduction].

As with continuity, the idea of a *limit* involves two sets, each endowed with suitable structures, and a mapping of one set into the other. For example, the limit of a sequence of real numbers a_n involves the set N of natural numbers, the set R of real numbers, and a mapping of the former set into the latter. A real number a is then said to be a limit of the sequence if, whatever neighbourhood V of a we take, this neighbourhood contains all the a_n except for a finite number of values of n; that is, if the set of natural numbers n for which a_n belongs to V is a subset of N whose complement is finite. Note that R is assumed to carry a topological structure, since we are speaking of neigbourhoods; as to the set N, we have made a certain family of subsets play a particular

[14] This section is not finished, in a very preliminary state and may contain mistakes. I will appreciate any corrections and suggestions sent by readers.

part, namely those subsets whose complement is finite. This is a general fact: whenever we speak of limit, we are considering a mapping f of a set E into a topological space F, and we say that f has a point a of F as a limit if the set of elements x of E whose image $f(x)$ belongs to a neighbourhood V of a [this set is just the "inverse image" $\overset{-1}{f}(V)$] belongs, whatever the neighbourhood V, to a certain family $\mathfrak{F}$ of subsets of E, given beforehand. For the notion of limit to have the essential properties ordinarily attributed to it, the family $\mathfrak{F}$ must satisfy certain axioms, which are stated in Chapter I, § 6. Such a family $\mathfrak{F}$ of subsets of E is called a *filter* on E. The notion of a filter, which is thus inseparable from that of a limit, appears also in other contexts in topology; for example, the neighbourhoods of a point in a topological space form a filter.

As with continuity, the idea of a limit involves two sets, each endowed with suitable structures, and a mapping of one set into the other. For example, the limit of a sequence of real numbers a_n involves the set $\mathbb{N}$ of natural numbers, the set $\mathbb{R}$ of real numbers, and a mapping of the former set into the latter. A real number a is then said to be a limit of the sequence if whatever neighbourhood V of a we take, this neigh- bourhood contains all the a_n except for a finite number of values of n; that is, if the set of natural numbers n for which a_n belongs to V is a subset of $\mathbb{N}$ whose complement is finite. Note that $\mathbb{R}$ is assumed to carry a topological structure, since we are speaking of neigbourhoods; as to the set $\mathbb{N}$, we have made a certain family of subsets play a particular part, namely those subsets whose complement is finite play the part of open subsets, for we require preimages of certain subsets to belong to this family. Using terminology of topology, we reformulate the definition as follows: *a real number a is said to be a limit of the sequence if the sequence determines a continuous map from $\mathbb{N}$ with filter (topology) formed by complement of finite subsets, to $\mathbb{R}$ with filter (topology) formed by neighbourhoods of a.* We'd like to think of $\mathbb{R}$ with this filter as a fibre above a of some total space, and we do so as follows. Define $\mathbb{R}\times\mathbb{R}_{\mathcal{T}}$ to be the finest filter on $\mathbb{R}\times\mathbb{R}$ such that the topology on fibre $\{a\}\times\mathbb{R}\subset\mathbb{R}\times\mathbb{R}$ is the filter formed by neighbourhoods of a, i.e. we define a subset $U\subset\mathbb{R}\times\mathbb{R}$ to be big (open) iff for each $a\in\mathbb{R}$, its fibre $U_a = U\cap\{a\}\times\mathbb{R} = \{(a,x) : (a,x)\in U\}$ above a contains a neighbourhood of a. (This is the filter of topoic subsets of $\mathbb{R}\times\mathbb{R}$ defined earlier.) With this, we further reformulate the definition of limit as follows: a is the limit of sequence $a_\bullet : \mathbb{N}\longrightarrow\mathbb{R}$ iff $(a, a_\bullet) : \mathbb{N}\longrightarrow\mathbb{R}\times\mathbb{R}_{\mathcal{T}},\quad n\longmapsto(a,a_n)$ *is continuous where $\mathbb{R}\times\mathbb{R}_{\mathcal{T}}$ is $\mathbb{R}\times\mathbb{R}$ equipped with the filter of topoic subsets in the topological structure on $\mathbb{R}$.* In other words, taking the limit of the sequence of real numbers a_n is taking a factorisation such that the diagonal map is constant on the first coordinate:

$$\begin{array}{ccc} & & \mathbb{R}\times\mathbb{R}_{\mathcal{T}} \\ & \overset{n\mapsto(a,a_n)}{\nearrow} & \downarrow pr_2 \\ \mathbb{N} & \xrightarrow[a\mapsto a_n]{} & \mathbb{R}_{\text{antidiscrete}} \end{array}$$

Now let us rewrite this diagram in simplicial terms. First we notice that there seem to be a simplicial object implicitly present in the diagram: the maps $\mathbb{R}\times\mathbb{R}_{\mathcal{T}} \underset{pr_2}{\overset{pr_1}{\rightrightarrows}} \mathbb{R}_{\text{antidiscrete}}$ look like a beginning of a simplicial object. Therefore we consider their 2-coskeleton $cosk_2(\mathbb{R}_{\text{antidiscrete}} \underset{pr_2}{\overset{pr_1}{\leftrightarrows}} \mathbb{R}\times\mathbb{R}_{\mathcal{T}})$, which is the object of Cartesian powers of $\mathbb{R}$ and coordinate maps

$$\mathbb{R}_{antidiscrete} \leftrightarrows \mathbb{R}\times\mathbb{R}_{\mathcal{T}} \leftrightarrows \mathbb{R}\times\mathbb{R}\times\mathbb{R}_{\mathcal{T}}...$$

and the filter $\mathbb{R}\times...\times\mathbb{R}_{\mathcal{T}}$ is the coarsest filter such that each degeneracy map $\mathbb{R}\times...\times\mathbb{R}_{\mathcal{T}} \longrightarrow \mathbb{R}\times\mathbb{R}_{\mathcal{T}}$, $(x_1,...,x_n) \longmapsto (x_i,x_j)$, $0 < i < j < n$, is continuous. Call this object $\mathcal{T}(\mathbb{R})$.

Replace in the diagram $\mathbb{R}_{\text{antidiscrete}}$ by $\mathcal{T}(\mathbb{R})$, and $\mathbb{R}\times\mathbb{R}_{\mathcal{T}}$ by "shifted" $\mathcal{T}(\mathbb{R})$

$$\mathbb{R}\times\mathbb{R}_{\mathcal{T}} \underset{(pr_1,pr_3)}{\overset{(pr_1,pr_2)}{\leftrightarrows}} \mathbb{R}\times\mathbb{R}\times\mathbb{R}_{\mathcal{T}} \leftrightarrows ...$$

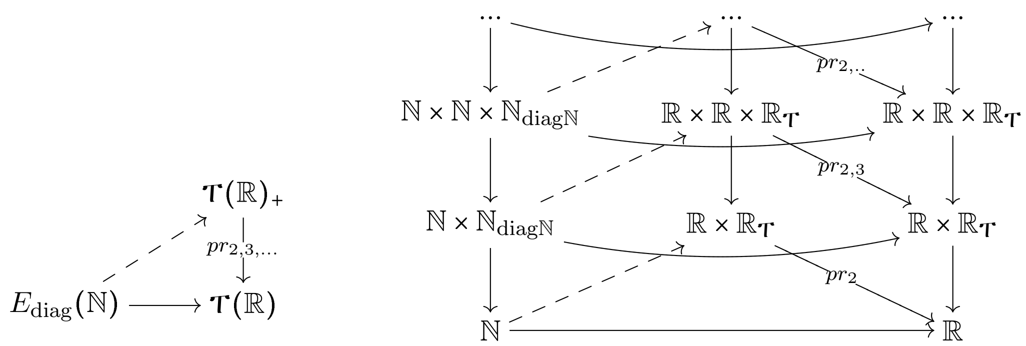
which forgets the first face and degeneracy maps. We call it $\mathcal{T}(\mathbb{R})_+$ and it is the composition $\mathcal{T}(\mathbb{R})\circ[+1]$ where $[+1] : \mathrm{Ord}_{<\omega} \longrightarrow \mathrm{Ord}_{<\omega}$, $n \longmapsto n+1$, is the endomorphism adding a new minimal element -1 to each linear order (and this new minimal element is kept minimal by the image of any morphism, i.e. $[+1](f)(-1) = -1$ for any $f \in Mor\mathrm{Ord}_{<\omega}$). Finally, the map $\mathbb{R}\times\mathbb{R}_{\mathcal{T}} \xrightarrow{pr_2} \mathbb{R}_{antidiscrete}$ then corresponds to the shift/decalage map $\mathcal{T}(\mathbb{R})_+ \longrightarrow \mathcal{T}(\mathbb{R})$ forgetting the first face and degeneracy maps.

In summary, in the diagram we replace $\mathbb{R}_{antidiscrete}$ by $\mathcal{T}(\mathbb{R})$, and $\mathbb{R}\times\mathbb{R}_{\mathcal{T}}$ by $\mathcal{T}(\mathbb{R})_+$, and the map $\mathbb{R}\times\mathbb{R}_{\mathcal{T}} \xrightarrow{pr_2} \mathbb{R}_{antidiscrete}$ by $\mathcal{T}(\mathbb{R})_+ \longrightarrow \mathcal{T}(\mathbb{R})$; we choose $\mathcal{T}(\mathbb{R})$ because it is a simplicial object already implicitly present in the diagram.

The underlying sets of $\mathcal{T}(\mathbb{R})_+$ form a disconnected union $\sqcup_{a\in\mathbb{R}}\{a\}\times E(\mathbb{R})$ of simplicial objects, and therefore any map from a connected simplicial object is necessarily constant on the first coordinate.

This suggests we replace $\mathbb{N}$ by a connected simplicial object, and the object of Cartesian powers is a natural choice.

This suggests that we rewrite the diagram above as the following lifting diagram of simplicial objects:

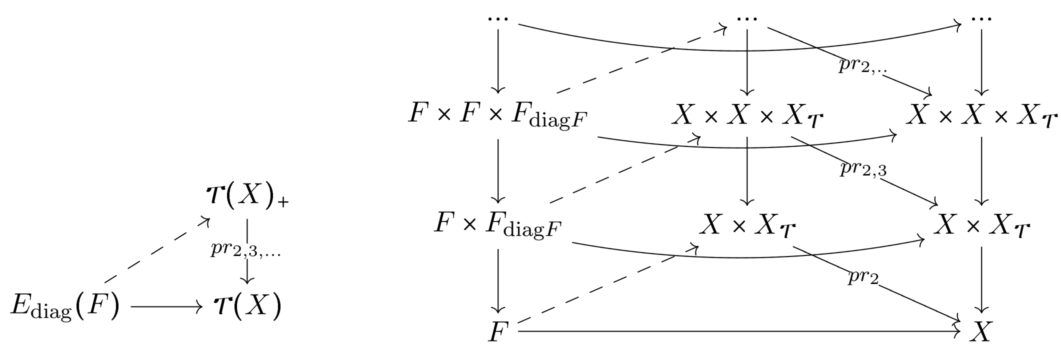


The filters on Cartesian powers of $\mathbb{N}$ are defined to be the finest possible so that continuity does not place unnecessary restrictions, i.e. the filter diag($\mathbb{N}$) on $\mathbb{N}\times\ldots\times\mathbb{N}$ is the finest possible such that the diagonal embedding $\mathbb{N} \longrightarrow \mathbb{N}\times\ldots\times\mathbb{N}$ is continuous, i.e. a subset of $\mathbb{N}^n$ is big iff it contains $\{(n, n, ..., n) : n > N\}$ for some $N > 0$.

Note that $\mathbb{R}$ is assumed to carry a topological structure which we represent as a simplicial object in $s\mathcal{F}ilt$; as to the set $\mathbb{N}$, we have made a certain family of subsets play a particular part, namely those subsets whose complement is finite, and we also represent this structure on $\mathbb{N}$ as a simplicial object in $s\mathcal{F}ilt$. Hence, both $\mathbb{R}$ and $\mathbb{N}$, with respective structures considered, live in the same category, and the notion of limit is expressed as a lifting property in $s\mathcal{F}ilt$.

This is a general fact:

***Reformulation 3.1* (limit).** — whenever we speak of limit, we are considering *a lifting diagram*

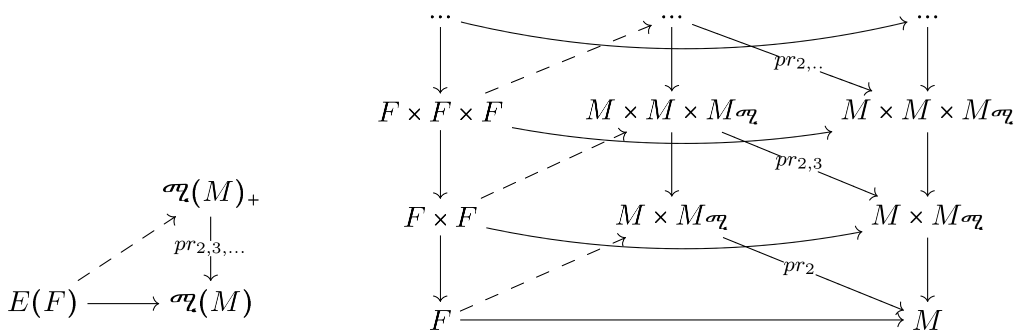


where F is a filter and X is a topological space, and the map $\mathcal{T}(X)_+ \longrightarrow \mathcal{T}(X)$ is the simplicial map forgetting the first face and degeneracy maps, and E_{diag} : $\mathcal{F}ilt \longrightarrow s\mathcal{F}ilt$ as defined above.

The notion of a limit of a filter on a *metric* space is similar, but its reformulation can be made self-contained. We do so now.

Recall that, for a metric space M, we denote by $M \times M \times ... \times M_{\mathcal{M}}$ the filter of ε-neighbourhoods of the diagonal, i.e. $U \subset M^n$ is $\mathcal{M}$-big (open) iff there is $\varepsilon > 0$ such that $(x_1, ..., x_n) \in M^n$ whenever $\operatorname{dist}(x_i, x_j) < \varepsilon$ for all $0 < i \leq j \leq n$. Recall that $E : \mathcal{F}ilt \longrightarrow s\mathcal{F}ilt$ is the fully faithful embedding sending a filter F into the simplicial object $(F, F \times F, F \times F \times F, ...)$ consisting of Cartesian powers of Fand coordinate maps.

***Reformulation 3.2* (limit).** — whenever we speak of limit *of a Cauchy filter on a metric space,* we are considering *a lifting diagram*



where F is a filter and M is a metric space, and the map $\mathcal{M}(M)_+ \longrightarrow \mathcal{M}(M)$ is the simplicial map forgetting the first face and degeneracy maps, and $E : \mathcal{F}ilt \longrightarrow s\mathcal{F}ilt$ as defined above.

This allows us to formulate several properties of spaces as lifting properties (here $\bot$ and $\top$ denote the initial and terminal object of $s\mathcal{F}ilt$):

- a topological space K is quasi-compact iff each ultrafilter $\mathfrak{U}$ on K converges, i.e.,

$$\bot \longrightarrow E_{\text{diag}}(\mathfrak{U}) \rightthreetimes \mathbf{T}(K)_+ \longrightarrow \mathbf{T}(K)$$

- a metric space M is quasi-compact iff each ultrafilter $\mathfrak{U}$ on M converges

$$\bot \longrightarrow E_{\text{diag}}(\mathfrak{U}) \rightthreetimes \mathcal{M}(M)_+ \longrightarrow \mathcal{M}(M)$$

- a metric space M is complete iff each Cauchy filter $\mathfrak{F}$ on M converges

$$\bot \longrightarrow E(\mathfrak{F}) \rightthreetimes \mathcal{M}(M)_+ \longrightarrow \mathcal{M}(M)$$

Also note

- a metric space M is pre-compact iff each ultrafilter $\mathfrak{U}$ on M is Cauchy

$$\operatorname{const}\mathfrak{U} \longrightarrow E(\mathfrak{U}) \rightthreetimes \mathcal{M}(M) \longrightarrow \top$$

3.2.1. Arzela-Ascoli theorems as lifting properties. — The reformulations above allow us express equicontinuity and Arzela-Ascoli theorems, and we attempt to do so in this subsection. This subsection is preliminary, work in progress and likely to contain misprints and inaccuracies.

Below we formulate various equicontinuity and convergence notions for sequences of functions, and then reformulate them as lifting properties is $s\mathcal{F}ilt$.

Recall that $E : \mathcal{F}ilt \longrightarrow s\mathcal{F}ilt$ is the fully faithful embedding sending a filter $\mathfrak{F}$ into the simplicial object $(\mathfrak{F}, \mathfrak{F}\times\mathfrak{F}, \mathfrak{F}\times\mathfrak{F}\times\mathfrak{F}, ...)$ consisting of Cartesian powers of $\mathcal{F}$ and coordinate maps. Let const $(\mathfrak{F})$ denote the simplicial filter $(\mathfrak{F}, \mathfrak{F}, \mathfrak{F}, ...)$ consisting of $\mathcal{F}$ itself and identity maps.

A *Cauchy filter* $\mathfrak{F}$ on a metric space M (cf. [Bourbaki,II§3.1,Def.2]) is a filter on the set of points of M such that the obvious map $E(\mathfrak{F}) \longrightarrow \mathcal{M}(M)$ is well-defined.

A *Cauchy sequence* in M is a map $E(\mathbb{N}_{cofinite}) \longrightarrow \mathcal{M}(M)$ where $\mathbb{N}_{cofinite}$ is the set of natural numbers equipped with cofinite topology (i.e. a subset is closed iff it is finite).

This allows to define various notions of equicontinuity of sequences of functions.

Let X be a topological space, let M be a metric space, and let $(f_i)_{i\in\mathbb{N}}$ be a family of functions $f_i : X \longrightarrow M$.

The family f_i is *equicontinuous* if either of the following equivalent conditions holds:

- for every $x \in X$ and $\varepsilon > 0$, there exists a neighbourhood U of x such that $d_Y(f_i(x'), f_i(x)) \leq \varepsilon$ for all $i \in \mathbb{N}$ and $x' \in U$
- the map $\mathbb{T}(X) \times \mathrm{const}\,(\{\mathbb{N}\}) \longrightarrow \mathcal{M}(M),\ (x, i) \longmapsto f_i(x)$ is well-defined
- the map $\mathbb{T}(X) \times \mathrm{const}\,(\mathbb{N}_{cofinite}) \longrightarrow \mathcal{M}(M),\ (x, i) \longmapsto f_i(x)$ is well-defined
- the map $\mathbb{T}(X) \times E_{\mathrm{diag}}(\{\mathbb{N}\}) \longrightarrow \mathcal{M}(M),\ (x, i) \longmapsto f_i(x)$ is well-defined
- the map $\mathbb{T}(X) \times E_{\mathrm{diag}}(\mathbb{N}_{cofinite}) \longrightarrow \mathcal{M}(M),\ (x, i) \longmapsto f_i(x)$ is well-defined

If $X = (X, d_X)$ is also a metric space, we say that the family f_i is *uniformly equicontinuous* iff either of the following equivalent conditions holds:

- for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $d_Y(f_i(x'), f_i(x)) \leq \varepsilon$ for all $i \in \mathbb{N}$ and $x', x \in X$ with $d_X(x, x') \leq \delta$
- the map $\mathcal{M}(X) \times \mathrm{const}\,(\{\mathbb{N}\}) \longrightarrow \mathcal{M}(M),\ (x, i) \longmapsto f_i(x)$ is well-defined
- the map $\mathcal{M}(X) \times \mathrm{const}\,(\mathbb{N}_{cofinite}) \longrightarrow \mathcal{M}(M),\ (x, i) \longmapsto f_i(x)$ is well-defined
- the map $\mathcal{M}(X) \times E_{\mathrm{diag}}(\{\mathbb{N}\}) \longrightarrow \mathcal{M}(M),\ (x, i) \longmapsto f_i(x)$ is well-defined
- the map $\mathcal{M}(X) \times E_{\mathrm{diag}}(\mathbb{N}_{cofinite}) \longrightarrow \mathcal{M}(M),\ (x, i) \longmapsto f_i(x)$ is well-defined

The family f_i is *uniformly Cauchy* iff either of the following equivalent conditions holds:

- for every $\varepsilon > 0$ there exists a $\delta > 0$ and $N > 0$ such that $d_Y(f_i(x'), f_j(x)) \leq \varepsilon$ for all $i, j > N$ and $x', x \in X$ with $d_X(x, x') \leq \delta$.
- the map $\mathfrak{M}(X) \times E(\mathbb{N}_{cofinite}) \longrightarrow \mathfrak{M}(M),\ (x,i) \longmapsto f_i(x)$ is well-defined

An equicontinuous family f_i *uniformly converges* to a function f iff either of the following equivalent conditions holds:

- for every $\varepsilon > 0$ there exists $N > 0$ such that $d_Y(f(x), f_i(x)) \leq \varepsilon$ for all $i > N$ and $x \in X$.
- the map $\mathfrak{T}(X) \times E_{\text{diag}}(\mathbb{N}_{cofinite}) \longrightarrow \mathfrak{M}(M)_+ \longrightarrow \mathfrak{M}(M),\ (x,i) \longmapsto (f, f_i(x))$ is well-defined
- the map $\mathfrak{T}(X) \times E(\mathbb{N}_{cofinite}) \longrightarrow \mathfrak{M}(M)_+ \longrightarrow \mathfrak{M}(M),\ (x,i) \longmapsto (f, f_i(x))$ is well-defined

An equicontinuous family f_i has a subsequence which uniformly converges to a function f iff for some ultrafilter $\mathfrak{U}$ extending the filter $\mathbb{N}_{cofinite}$ of cofinite subsets either of the following equivalent conditions holds:

- the map $\mathfrak{T}(X) \times E_{\text{diag}}(\mathfrak{U}) \longrightarrow \mathfrak{M}(M)_+ \longrightarrow \mathfrak{M}(M),\ (x,i) \longmapsto (f, f_i(x))$ is well-defined

Here $\{\mathbb{N}\}$ denotes the trivial filter on $\mathbb{N}$ with a unique big subset $\mathbb{N}$ itself, and $\mathbb{N}_{cofinite}$ denotes the filter of cofinite subsets of $\mathbb{N}$.

This suggest we might reformulate Arzela-Ascoli theorem as something about inner Hom in $s\mathcal{F}ilt$, see Question 4.7.

This allows to formulate a couple of versions of Arzela-Ascoli theorems as implications between lifting properties:

Reformulation 3.3 **(Arzela-Ascoli)**. — *For K a quasi-compact topological space and M a quasi-compact metric space, each equicontinuous sequence $f_i : K \longrightarrow M$, $i \in \mathbb{N}$ has a subsequence which uniformly converges to a continuous function:*

$\bot \longrightarrow E_{\text{diag}}(\mathfrak{U}) \rightthreetimes \mathfrak{T}(K)_+ \longrightarrow \mathfrak{T}(K)$

$\bot \longrightarrow E_{\text{diag}}(\mathfrak{U}) \rightthreetimes \mathfrak{M}(M)_+ \longrightarrow \mathfrak{M}(M)$ *for each ultrafilter* $\mathfrak{U}$

implies

$\bot \longrightarrow \mathfrak{T}(K) \times E_{\text{diag}}(\mathfrak{U}) \rightthreetimes \mathfrak{M}(M)_+ \longrightarrow \mathfrak{M}(M)$ *for each ultrafilter* $\mathfrak{U}$

For K a quasi-compact topological space and M a pre-compact metric space, each equicontinuous sequence $f_i : K \longrightarrow M$, $i \in \mathbb{N}$, has a uniformly convergent subsequence:

$\bot \longrightarrow E_{\text{diag}}(\mathfrak{U}) \rightthreetimes \mathfrak{T}(K)_+ \longrightarrow \mathfrak{T}(K)$

$\text{const}\,\mathfrak{U} \longrightarrow E(\mathfrak{U}) \rightthreetimes \mathfrak{M}(M) \longrightarrow \top$ *for each ultrafilter* $\mathfrak{U}$

implies

$\mathfrak{T}(K) \times const(\mathfrak{U}) \longrightarrow \mathfrak{T}(K) \times E_{\text{diag}}(\mathfrak{U}) \rightthreetimes \mathfrak{M}(M) \longrightarrow \top$ *for ultrafilter* $\mathfrak{U}$ *on* $\mathbb{N}$

Remark 1. — Note that one may identify the real interval $[0,1]$ with the set of endofunctors $\mathrm{Ord}_{<\omega} \xrightarrow{\tau} \mathrm{Ord}_{<\omega}$ such that $\tau(n) = n+1$ for any $n \in Ob\mathrm{Ord}_{<\omega}$. To see this, note that such an endofunctor is given by inserting a new element in each finite linear order in a compatible manner. This temps us to think of the space of paths.

3.3. Path and cylinder spaces and Axiom M2(cw)(f) and M2(c)(wf) of Quillen model categories. — In the category of topological spaces, there is a simple but very useful way to turn an arbitrary map into either a fibration or cofibration. It it captured by Axiom M2 of model categories which requires that each map decomposes as a composition of a cofibration and a fibration, and any one of them may also be required to be a weak equivalence.

In notation,

$$\begin{array}{ccc} & A \times [0,1] \sqcup_A B & \\ {\scriptstyle (c)}\nearrow & & \searrow{\scriptstyle (wf)} \\ A & \longrightarrow & B \end{array} \qquad \begin{array}{ccc} & A \times_B B^I & \\ {\scriptstyle (cw)}\nearrow & & \searrow{\scriptstyle (f)} \\ A & \longrightarrow & B \end{array}$$

Figure 3 gives drawings representing these decompositions in the category of topological spaces.

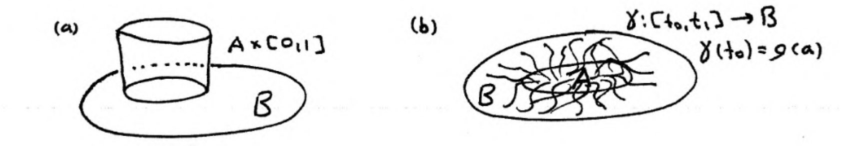


Fig. 3. M2-decompositions. Cones, cylinders and paths. (a) A c-wf-decomposition in Top using a cone object. (b) A wc-f-decomposition in Top using a cocone object of paths.

7.16. *Axiom M2. Downward Lowenheim-Skolem theorem as an instance of M2(c-wf).* Downward

3.3.1. (cw)(f)-decomposition.— Let us analyse Figure 3b (cw-f decomposition).

Recall that, to translate, we care both about intuition and algebraic manipulations.

The construction uses the following algebraic manipulations: it considers pairs $(x, \gamma^{const}_{f(x)})$ and $(x, \gamma(t_1))$. We ignore paths because these are complicated "infinitary" notions we are unable to express in our language, and hence all we are left with are pairs $(x, f(x))$ where $x \in A$ and (x,y) where $x \in A, y \in B$. This suggests we look at the following decomposition:

$$
\begin{array}{ccccccc}
X & \longleftrightarrow & X\times X & \longleftrightarrow & X\times X\times X & \longleftrightarrow & \\
\downarrow{\scriptstyle (x,f(x))} & & \downarrow{\scriptstyle (x_1,f(x_1),x_2,f(x_2))} & & \downarrow{\scriptstyle (x_1,f(x_1),x_2,f(x_2),x_3,f(x_3))} & & \\
X\times Y & \longleftrightarrow & X\times Y\times X\times Y & \longleftrightarrow & X\times Y\times X\times Y\times X\times Y & \longleftrightarrow & \\
\downarrow{\scriptstyle y} & & \downarrow{\scriptstyle (y_1,y_2)} & & \downarrow{\scriptstyle (y_1,y_2,y_3)} & & \\
Y & \longleftrightarrow & Y\times Y & \longleftrightarrow & Y\times Y\times Y & \longleftrightarrow &
\end{array}
$$

What is the filter $\mathfrak{F}_{\text{cw-f}}$ on the elements of the middle row $X\times Y\times X\times Y\times \ldots$? The diagram suggests that we start with the filter corresponding to the product topology on $X\times Y$.

Let us use the intuition. In model categories, weak equivalences are thought of as equivalences and therefore X and $(X\times Y)_{\mathfrak{F}_{\text{cw-f}}}$ should be very similar for purposes we care about. Geometric intuition suggests that we only care about an infinitesimal neighbourhood of $\{(x, id_x) : x\in X\}$ and would prefer our paths to be infinitesimally short.

This motivates us to modify the topoic filter of the product topology on $X\times Y$ by adding as topoic "infinitesimal neighbourhoods of X"

$$\bigcup_{x\in X,\, f(x)\in U_{f(x)} \text{ a neighbourhood}} \{x\}\times U_{f(x)} \subset X\times Y$$

That is, we define a new filter $\mathfrak{F}^n_{\text{cw-f}}$ on $(X\times Y)^n$ generated by subsets $U^n\cap W$ where U is of the form above and W is a topoic subset of $(X\times Y)^n$ with respect to the product topology on $X\times Y$.

Remark 2. — Note that when $X = a$ is a point, this decomposition gives us $\{a\} \xrightarrow{(cw)} Y\circ[0]_a \xrightarrow{(f)} Y$. This suggests that we think of $Y\circ[0]_a$ as the space of infinitesimally short paths starting at a; this would correspond to the intuition that an infinitesimally short path is roughly the same as its endpoint.

3.3.2. (c)(wf)-decomposition.— Dually, the *(c)(wf)*-decomposition of $X\xrightarrow{f} Y$ leads us to consider

$$
\begin{array}{ccccccc}
X & \longleftrightarrow & X\times X & \longleftrightarrow & X\times X\times X & \longleftrightarrow & \\
\downarrow{\scriptstyle (x,f(x))} & & \downarrow{\scriptstyle (x_1,x_2)} & & \downarrow{\scriptstyle (x_1,x_2,,x_3))} & & \\
(X\sqcup Y) & \longleftrightarrow & (X\sqcup Y)\times(X\sqcup Y) & \longleftrightarrow & (X\sqcup Y)\times(X\sqcup Y)\times(X\sqcup Y) & \longleftrightarrow & \\
\downarrow{\scriptstyle y} & & \downarrow{\scriptstyle (y_1,y_2)} & & \downarrow{\scriptstyle (y_1,y_2,y_3)} & & \\
Y & \longleftrightarrow & Y\times Y & \longleftrightarrow & Y\times Y\times Y & \longleftrightarrow &
\end{array}
$$

Intuitively, a (infinitesimal) neighbourhood of a point y contains points $(x,1)$ whenever $f(x) = y$.

This motivates us to modify the topoic filter of the disjoint union topology on $X \sqcup Y$ by requiring its topoic subsets to satisfy also the following property: $x \in P$ whenever $f(x) \in P$, $x \in X$.

That is, we define a new filter $\mathfrak{F}^n_{\text{c-wf}}$ on $(X \times Y)^n$ which consists of the subsets topoic wrt the disjoint union topology which also satisfy the property that $(z_1,..,x_i,...,z_n) \in P$ whenever $x_i \in X$ and $(z_1,..,f(x_i),...,z_n) \in P$.

Arguably, one might find an intuition according to which $(X \sqcup Y)_{\mathfrak{F}_{\text{c-wf}}}$ is similar to $Y_{\mathbf{T}}$.

4. Open questions and directions for research

4.1. Research directions. — Our observations suggest the following broad questions and directions for research.

4.1.1. Category theory implicit in elementary topology. — We'd like to think of our observations as *translations of ideas of Bourbaki on general topology into a language of category theory appropriate to these ideas*, and that these ideas (but not notation) are implicit in Bourbaki and reflect their logic (or perhaps the ergologic in the sense of [Gromov]).

***Question 4.1* (Category theory and topological ideas and intuition)**

- *Translate more of Bourbaki and some intuitive topological arguments into the language of category theory and diagram chasing.*
- *Understand how this translation works and in what way it is a* translation *rather than something new. Formulate what does it mean to say that these category theoretic constructions are implicit in Bourbaki and find evidence that indeed they are implicitly there.*
- *More speculatively, find evidence that these category theoretic diagram chasing arguments are implicitly present in the topological intuition of a student, say by finding correlations between errors of intuition and errors of calculation.*

But is this so and what does it actually mean?

The goal of our analysis is somewhat reminiscent of the goal of [Hodges. Ibn Sina on analysis: 1. Proof search. Or: Abstract State Machines as a tool for history of logic] where he "extract[s] from [the text of Ibn Sina's commentary on a couple of paragraphs of Aristotle's Prior Analytics] all the essential ingredients of an Abstract State Machine for [a proof search] algorithm". We'd like to think that we extract from the text of a couple of paragraphs of Bourbaki all the essential ingredients of certain category theoretic constructions.

4.1.2. Formalisation of topology.— Our translation is unsophisticated and is largely based on textual coincidences and correlations between the text and allowed category theoretic manipulations. Can these coincidences—and the translation—be found by a machine learning algorithm? A hope is that category theoretic manipulations are restrictive enough so that a brute force search for correlations between (long enough sequences of) allowed category theoretic manipulations and the text of Bourbaki may produce meaningful results.

Designing such an algorithm would involve designing a derivation system for the category theoretic constructions used.

Our reformulations of certain notions of topology in terms of orthogonality (negation) are so concise (several bytes) that they can be found, or rather listed, by a brute force search. This might be a starting point in designing such an algorithm: first design an algorithm which can work with the reformulations in terms of iterated orthogonals (negations) of maps between finite spaces, find correlations between these orthogonals and text of Bourbaki, and single out the interesting notions obtained by iterated negation (orthogonals) from very simple morphism. These notions should include quasi-compactness, denseness, connected etc.

Question 4.2 **(Category theory and topological ideas and intuition)**

- *Write a short program which extracts diagram chasing derivations from texts on elementary topology, in the spirit of the ideology of ergosystems/ergostructures. The texts might include [Bourbaki, General Topology] as well as some informal explanations.*

  In particular, it should be able to convert verbal definitions of properties defined by orthogonals into the corresponding orthogonals.
- *Develop a formalisation of topology based on this translation.*

4.1.3. Tame topology and foundations of topology. — Does our point of view shed light on the tame topology of Grothendieck and allows to develop a foundation of topology "without false problems" and "wild phenomena" "at the very beginning"? [Esquisse d'un Programme, translation,§5,p.33]

Our approach does seem to avoid certain set-theoretic issues and constructions. For example, ultrafilters do appear in our reformulation of compactness, but do so only in a combinatorial disguise. Remark 1 in §3.2 suggests a way to think about pathspaces without real numbers.

Question 4.3 **(Tame topology and foundations of topology.)**

- *Develop elementary topology in terms of finite categories (viewed as finite topological spaces) and labelled commutative diagrams, with an emphasis on labels (properties) of morphisms defined by iterated orthogonals ($\rightthreetimes$-negation).*
- *Develop topology in terms of finite categories, labelled commutative diagrams, and simplicial filters. Develop a syntax to describe simplicial filters as concise as the syntax of iterated $\rightthreetimes$-negation of maps between finite spaces.*

> *Does this lead to tame topology of Grothendieck, i.e. a foundation of topology "without false problems" and "wild phenomena" "at the very beginning" ?*

Grothendieck suggests that the following needs to be done first:

> Among the first theorems one expects in a framework of tame topology as I perceive it, aside from the comparison theorems, are the statements which establish, in a suitable sense, the existence and uniqueness of "the" tubular neighbourhood of closed tame subspace in a tame space (say compact to make things simpler), together with concrete ways of building it (starting for instance from any tame map $X \longrightarrow \mathbb{R}^+$ having Y as its zero set), the description of its "boundary" (although generally it is in no way a manifold with boundary!) ∂T, which has in T a neighbourhood which is isomorphic to the product of T with a segment, etc. Granted some suitable equisingularity hypotheses, one expects that T will be endowed, in an essentially unique way, with the structure of a locally trivial fibration over Y, with ∂T as a subfibration.

Question 4.4. — *Write a first year course introducing elementary topology and category theory ideas at the same time, based on the observations above and the calculus to be developed. Compactness would be explained with help of all the definitions above; Tychonoff theorem follows immediately by a diagram chasing argument from the fact that compactness is given by $\rightthreetimes$-negation (orthogonal); $\forall\exists \longrightarrow \exists\forall$ definitions would give students some intuition.*

As a first step, write an exposition aimed at students of the separation axioms and Urysohn Lemma in terms of the lifting properties.[15]

Note that the standard proof of Urysohn lemma can be represented as follows: iterate the lifting property defining normal (T_4) spaces

$$\varnothing \longrightarrow X \rightthreetimes \{x \swarrow x' \searrow X \swarrow y' \searrow y\} \longrightarrow \{x \swarrow x' = X = y' \searrow y\}$$

to prove

$$\varnothing \longrightarrow X \rightthreetimes \{x \swarrow x_1 \searrow \ldots \swarrow x_n \searrow y\} \longrightarrow \{x \swarrow x_1 = \ldots = x_n \searrow y\}$$

Then pass to the infinite limit to construct a map $X \longrightarrow \mathbb{R}$.

---

(15) See `https://ncatlab.org/nlab/show/separation+axioms+in+terms+of+lifting+properties` for a list of reformulations of the separation axioms in terms of orthogonals.

4.1.4. Homotopy and model category structure.— Let $\tilde{\mathcal{F}}ilt$ be the category $\mathcal{F}ilt$ of filters localised as follows: we consider two morphisms equal iff they coincide on a big subset of the domain, i.e. $f, g : X \longrightarrow Y$ are considered equal as morphisms in $\tilde{\mathcal{F}}ilt$ iff the subset $\{x : f(x) = g(x)\}$ is big in X.

Question 4.5 **(Homotopy theory and model category structure on** $s\mathcal{F}ilt$ **or** $s\tilde{\mathcal{F}}ilt$.**)**

Is there an interesting model category structure on $s\mathcal{F}ilt$ *or* $s\tilde{\mathcal{F}}ilt$*? Does it lead to interesting homotopy theory of uniform spaces?*

In §3.3 we suggest examples of (cw)(f)- and (c)(wf)-decompositions. Do the corresponding classes of acyclic cofibrations and fibrations generate a model structure on $s\mathcal{F}ilt$ *or* $s\tilde{\mathcal{F}}ilt$*?*

Does either category have interesting objects corresponding to quotients of topological spaces by a group action?

4.2. Metric spaces, uniform spaces and coarse spaces. —

4.2.1. Uniform structures. —

Question 4.6. — *Rewrite the theory of uniform structures and metric spaces in terms of the category* $s\mathcal{F}ilt$ *of simplicial filters. In particular,*

- *reformulate the Lebesgue's number lemma, partition of unity, and the characterisation of paracompactness by A.Stone mentioned by [Alexandroff] (cf. §5.5).*

Question 4.7 **(Arzela-Ascoli).** — 1. *Reformulate various notions of equicontinuity and convergence of a family of functions* $f_i : X \longrightarrow M$ *in terms of maps in* $s\mathcal{F}ilt$ *using e.g.* const $(\mathbb{N}_{cofinite})$, $E(\mathbb{N}_{cofinite})$, const $(\mathbb{N}_{cofinite} \cup_{\mathbb{N}_{cofinite}} \{\infty\})$, $\mathcal{T}(\mathbb{N}_{cofinite} \cup_{\mathbb{N}_{cofinite}} \{\infty\})$, $E(\mathbb{N}_{cofinite} \cup_{\mathbb{N}_{cofinite}} \{\infty\})$, $\mathcal{T}(\mathbb{N}_{cofinite})$, $\mathcal{T}(X)$, $\mathcal{U}(X)$, *and* $\mathcal{U}(M)$.

2. *Reformulate and prove Arzela-Ascoli theorem in terms something like inner Hom in* $s\mathcal{F}ilt$ *and the lifting properties defining precompactness, compactness etc.*
3. *Define various function spaces in terms of something like inner Hom in* $s\mathcal{F}ilt$.

4.2.2. Large scale geometry.— The category of quasigeodesic metric spaces and large scale Lipschitz maps embeds into another category $s\mathcal{F}^{+}ilt$ of simplicial filters, with maps of filters defined differently: a $\mathcal{F}^{+}ilt$-morphism of filters maps a small subset into a small subset.

Let X be a metric space. Call a subset U of X^n *small* iff the diameters of tuples in U are uniformly bounded, i.e. there is a $d = d(U)$ such that for each $(u_1, ..., u_n) \in U$, $dist(u_i, u_j) \leq d$ for each $1 \leq i, j \leq n$; this defines a filter on X^n. Note that coordinate maps $X^n \longrightarrow X^m$ have the property that the image of a small subset is necessarily small. Hence this construction defines a functor $\mathcal{X} : FinSets^{op} \longrightarrow \mathcal{F}ilt$. A natural transformation $\mathcal{X} \longrightarrow \mathcal{Y}$ of functors associated with metric spaces X and Y, resp., corresponds to a map of metric spaces $f : X \longrightarrow Y$ such that for each $d > 0$ there is $D > 0$ such that

$$dist(f(x'), f(x'')) < D \text{ whenever } dist(x', x'') < d, x', x'' \in X.$$

For X quasi-geodesic, this is the class of large scale Lipschitz maps.

***Question 4.8* (Large scale geometry).** — *Rewrite in terms of the category $s\mathcal{F}ilt$ of simplicial filters the theory of metric spaces and uniformly bounded maps and the theory of coarse structures (cf. [Bunke, Engel]).*

4.2.3. Group theory. — In the category of groups, properties defined by orthogonals (cf. §5.3.2) include groups being nilpotent, solvable, torsion-free, p-groups, and prime-to-p groups.

This suggests it is worthwhile to try to rewrite group theoretic arguments in diagram chasing manner, say the proof that nilpotent groups are solvable, and try to find a semantics for our notation of finite topological spaces in the category of groups.

***Question 4.9* (Group theory).** —

- *Calculate iterated $\rightthreetimes$-negation (orthogonals) of interesting morphisms in the category of groups and find interesting properties defined this way.*
- *Find a diagram chasing reformulation of the Sylow theorems.*
- *Find a semantics in the category of groups for the notation introduced in §5.3.1.*

To reformulate the Sylow theorem, the following characterisation of inner automorphisms may be of help: an automorphism $f : G \longrightarrow G$ is inner iff either of the following equivalent conditions hold (cf. [Schupp,Inn]):

- $f : G \longrightarrow G$ extends to an automorphism of $f' : H \longrightarrow H$, for any $h : G \longrightarrow H$, i.e. $f \circ h = h \circ f'$.
- $f : G \longrightarrow G$ extends to an automorphism of $f' : H \longrightarrow H$, for any $h : H \longrightarrow G$, i.e. $f \circ h = h \circ f'$

To find a semantics, it would help to find a category which contains both groups and finite preorders. One candidate is the category $Cats$ of categories where a group G is identified with a category with a single object O such that $Aut(O) = G$. Intuitively, one may think of the category of groups as analogous to the (sub)category of Hausdorff spaces in the following way: the interesting example are groups (Hausdorff spaces), yet the big ambient category contains useful objects (finite topological spaces) which are very unlike the interesting examples we care about, but are useful to talk about these examples.

4.3. Open problems.— Now we would like to formulate several suggestions with specific details.

4.3.1. Topology.—

Question 4.10. — *Develop a syntax and a derivation calculus based on $\rightthreetimes$-negation, arrows, labelled arrows and diagrams, finite topological spaces, and simplicial filters. Develop an intuition for the calculus as well.*

1. *Standard arguments and definitions in elementary topology should be represented by short formal calculations which are both human readable and computer verifiable.*
2. *In particular, the calculus should express concisely all the three definitions of compactness, and prove their equivalence by short formal calculations.*

Question 4.11. — *Does topological intuition (as developed by a first year student) relate to the formal calculus we'd like to develop? Note that this might be testable by an experiment, namely it might be possible to test whether mistakes of intuition correspond to mistakes of calculation. This might even be used to develop the calculus.*

Question 4.12. — *Write* a very short *program which would "invent" (generate) the (very) basic theory of topology, possibly using unstructured input such as the text of [Bourbaki, General Topology]. Our examples suggest that iterating right and left $\rightthreetimes$-negation up to 5 times and restricting size to 3 or 4 is enough to generate, but not single out, the notions of compactness, connectedness, a subset, a closed subset, separation axioms, and some implications between them.*

What is the length of a shortest such program? To what extent have the axioms of topology to be hardcoded rather than generated?

Let us comment on how such a program may look like.

We observed that there is a simple rule which leads to several notions in topology interesting enough to be introduced in an elementary course. Can this rule be extended to a very short program which learns elementary topology?

We suggest the following naive approach is worth thinking about.

The program maintains a collection of directed labelled graphs and certain distinguished subgraphs. Directed graphs represent parts of a category; distinguished subgraphs represent commutative diagrams. Labels represent properties of morphisms. Further, the program maintains a collection of rules to manipulate these data, e.g. to add or remove arrows and labels.

The program interacts with a flow of signals, say the text of [Bourbaki, General Topology, Ch.1], and seeks correlations between the diagram chasing rules and the flow of signals. It finds a "correlation" iff certain strings occur nearby in the signal flow iff they occur nearby in a diagram chasing rule. To find "what's interesting", by brute force it searches for a valid derivation which exhibits such correlations. To guide the search and exhibit missing correlations in a derivation under consideration, it may ask questions: are these two strings related? Once it finds such a derivation, the program "uses it for building its own structure". Labels correspond to properties of morphisms. Labels defined by the lifting property play an important role, often used to exclude counterexamples making a diagram chasing argument fail. In [DeMorgan] we analysed the text of the definitions of surjective and injective maps showing what such a correlation may look like in a "baby" case.

A related but easier task is to write a theorem prover doing diagram chasing in a model category. The axioms of a (closed or not) model category as stated in [Quillen,I.1.1] can be interpreted as rules to manipulate labelled commutative diagrams in a labelled category. It appears straightforward how to formulate a logic (proof system) based on these rules which would allow to express statements like: Given a labelled commutative diagram, (it is permissible to) add this or that arrow or label. Moreover, it appears not hard to write a theorem prover for this logic doing brute force guided search. What is not clear whether this logic is complete in any sense or whether there are non-trivial inferences of this form to prove.

Writing such a theorem prover is particularly easy when the underlying category of the model category is a partial order [Gavrilovich, Hasson] and [BaysQuilder] wrote some code for doing diagram chasing in such a category. However, the latter problem is particularly severe as well.

The two problems are related; we hope they help to clarify the notion of an ergosystem and that of a topological space.

The following are somewhat more concrete questions.

Question 4.13. —

- *Prove that a compact Hausdorff space is normal by diagram chasing; does it require additional axioms? Note that we know how to express the statement entirely in terms of $\rightthreetimes$-negation and finite topological spaces of small size.*
- *Formalise the argument in [Fox, 1945] which implies the category of topological spaces is not Cartesian closed; does it apply to s$\mathcal{F}$ilt?*

  Namely, Theorem 3 [ibid.] proves that if X is separable metrizable space, R is the real line, then X is locally compact iff there is a topology on $X^{\mathbb{R}}$ such that for any space T, a function $h : X \times T \longrightarrow \mathbb{R}$ is continuous iff the corresponding function $h^* : T \longrightarrow X^{\mathbb{R}}$ is continuous (where $h(x,t) = h^*(t)(x)$)

  Note that here we do not know how to express the statement.

Question 4.14. — *Characterise the interval $[0,1]$, a circle $\mathbb{S}^1$ and, more generally, spheres $\mathbb{S}^n$ using their topological characterisations provided by the Kline* *sphere charterisation theorem and its analogues. An example of such a characterisation is that a topological space X is homomorphic to the circle $\mathbb{S}^1$ iff X is a connected Hausdorff metrizable space such that $X \setminus \{x, y\}$ is not connected for any two points $x \neq y \in X$ ([Hocking, Young. Topology, Thm.2-28,p.55]); another example is that a topological space X is homomorphic to the closed interval $[0,1]$ iff X is a connected Hausdorff metrizable space such that $X \setminus \{x\}$ is not connected for exactly two points $x \neq y \in X$ ([Hocking, Young. Topology, Thm.2-27,p.54]).*

4.3.2. $\rightthreetimes$-negation, or orthogonality.— Call a subcategory $\mathcal{A}$ of $\mathcal{B}$ *$\rightthreetimes^s$-full* iff the value of an s-orthogonal of a class of morhpisms in $\mathcal{A}$ does not depend whether it is calculated in $\mathcal{A}$ or in $\mathcal{B}$. i.e. $(C)^s_{\mathcal{A}} = (C)^s_{\mathcal{B}}$ for any class C of morphisms of $\mathcal{A}$, where $s \in \{l, r\}^n$ is a string.

Question 4.15. — *Calculate left and right $\rightthreetimes$-negations and generalisations, e.g. $(C)^r$, $(C)^l$, $(C)^{rl}$, $(C)^{ll}$, $(C)^{rr}$, $(C)^{llr}$, ... for various simple classes of morphisms in various categories, e.g. morphisms of finite topological spaces or finite groups.*

Develop abstract theory of the lifting property. Find examples of $\rightthreetimes^s$-full subcategories.

4.3.3. Compactness as being uniform.— In §5.5 we observe that a number of consequences of compactness can be expressed as a change of order of quantifiers in a formula, i.e. are of form $\forall\exists\varphi(...) \implies \exists\forall\varphi(...)$ namely that a real-valued function on a compact is necessarily bounded, that a Hausdorff compact is necessarily normal, that the image in X of a closed subset in $X \times K$ is necessarily closed, Lebesgue's number Lemma, and paracompactness. In §2.3.2 we show that an axiom of topology, namely that an infinite union of open sets is open, is also of this form.

Such formulae correspond to inference rules of a special form, and we feel a special syntax should be introduced to state these rules.

For example, consider the statement that "a real-valued function on a compact domain is necessarily bounded". As a first order formula, it is expressed as

$$\forall x \in K \exists M (f(x) \leq M) \implies \exists M \forall x \in K (f(x) \leq M)$$

Another way to express it is:

$$\exists M : K \longrightarrow \mathbb{R} \forall x \in K (f(x) \leq M(x)) \implies \exists M \in \mathbb{R} \forall x \in K (f(x) \leq M)$$

Note that all that happened here is that a function $M : K \longrightarrow \mathbb{R}$ become a constant $M \in \mathbb{R}$, or rather expression "M(x)" of type $K \longrightarrow M$ which used (depended upon) variable "x" become expression "M" which does not use (depend upon) variable "x". We feel there should be a special syntax which would allow to express above as an inference rule *removing dependency of*

"M(x)" on "x", and this syntax should be used to express consequences of compactness in a diagram chasing derivation system for elementary topology.

To summarise, we think that compactness should be formulated with help of inference rules for expressly manipulating which variables are 'new', in what order they 'were' introduced, and what variables terms depend on, e.g. rules replacing a term t(x,y) by term t(x).

Something like the following:

```
... f(x) =< M(x) ...
--------------------
...  f(x) =< M    ...
```

Question 4.16. — *In §5.5 we give several examples where consequences of compactness are expressed as change of order of quantifiers* $\forall\exists \longrightarrow \exists\forall$.

- *Is there a theorem generalising these examples?*
- *Is there a proof system which allows to formulate inference rules corresponding to these reformulations?*

4.4. Open problems.— Here we formulate precise questions one may ask. The choice of these questions is somewhat arbitrary.

***Question 4.17* (Iterated orthogonals in *Top*).** —
- *Are there finitely many different iterated orthogonals of the form* $\{\emptyset \longrightarrow \{\bullet\}\}^s$ *where* $s \in \{l,r\}^{<\omega}$*?*

  More generally, are there finitely many different classes obtained from $\{\emptyset \longrightarrow \{\bullet\}$ *by repeatedly passing to left or right orthogonal* C^l *or* C^r *or the subclass* $C_{<n}$ *of morphisms between spaces of size at most* n*?*

  Is there an algorithm which decides whether two such classes are equal?
- *Find the shortest expressions (Kolmogoroff complexity) of various topological notions.*
- *Is*

$$((\{\{o\} \longrightarrow \{o \to c\}\}^r)_{<5})^{lr}$$

  the class of proper maps?
- *Calculate*[16]

$$((\{c\} \longrightarrow \{o\searrow c\})^r_{<5})^{lr}, \qquad ((\{c\} \longrightarrow \{o\searrow c\})^{lrr}$$

$$(\{a\swarrow U\searrow x\swarrow V\searrow b\} \longrightarrow \{a\swarrow U = x = V\searrow b\})^{lr}$$

- *Characterise the class of covering maps as an iterated negation of a class of maps of finite spaces.*
  - *Calculate* $\rightthreetimes^{lr}$*-orthogonal of the class of maps of finite spaces which have unique path lifting property up to reparametrisation.*

(16) For motivation see Remark 5 of [Gavrilovich, Lifting Property]

- *Calculate* $(\{a,b\} \longrightarrow \{a=b\})^{lr}$ *and* $\{D \longrightarrow \{\bullet\} : D \textit{ is discrete}\}^{lr}$*; these are the iterated orthogonals of the simplest examples of covering spaces.*

Note the orthogonals may depend on the category they are calculated in, which is either Top or $s\mathcal{F}ilt$*.*

A number of elementary topological properties can be defined by, in a sense, combinatorial expressions, by taking iterated orthogonals in Top of a single morphism between finite topological spaces [Gavrilovich, Lifting Property]. Calculate these expressions in $s\mathcal{F}ilt$ using the embedding $\mathcal{T} : Top \longrightarrow s\mathcal{F}ilt$. Note that this would give properties of both topological spaces and metric spaces. Do they define the same properties of topological spaces? Do they provide an interesting analogy between topological spaces and metric spaces, e.g. compactness [Bourbaki, I§10.2, Thm.1(d), p.101] and completeness [Bourbaki, II§3.6, Prop.11]?

The following is an example of a precise conjecture.

Question 4.18 **(Compactness and completeness)**

– *Calculated in Top, is*

$$((\{\{o\} \longrightarrow \{o \to c\}\}^r)_{<5})^{lr}$$

the class of proper maps?

– *Is the following true in n* $s\mathcal{F}ilt$ *or* $s\mathfrak{F}ilt$*?*

1. *A Hausdorff space X is compact iff*

$$\mathcal{T}(X \longrightarrow \{\bullet\}) \in ((\{\mathcal{T}(\{o\} \longrightarrow \{o \to c\})\}^r)_{<5})^{lr}$$

2. *A metric space M is complete iff*

$$\mathcal{M}(M \longrightarrow \{\bullet\}) \in ((\{\mathcal{T}(\{o\} \longrightarrow \{o \to c\})\}^r)_{<5})^{lr}$$

– *Does the value of an orthogonal depend whether it is calculated in Top or* $s\mathcal{F}ilt$*? For example, is it true that for any morphism f of finite topological spaces,*

$$\{f\}^{lr}_{Top} = \{\mathcal{T}(f)\}^{lr}_{s\mathcal{F}ilt} \cap Top \text{ and } \{f\}^{rl}_{Top} = \{\mathcal{T}(f)\}^{rl}_{s\mathcal{F}ilt} \cap Top \ ?$$

Recall $\mathfrak{F}ilt$ is the category $\mathcal{F}ilt$ of filters localised as follows: we consider two morphisms equal iff they coincide on a big subset of the domain, i.e. $f, g : X \longrightarrow Y$ are considered equal as morphisms in $\mathfrak{F}ilt$ iff the subset $\{x : f(x) = g(x)\}$ is big in X.

Question 4.19 **(Is** $s\mathcal{F}ilt$ **or** $s\mathfrak{F}ilt$ **a model category?)**

Let $(cw)_0$*,* $(f)_0$*,* $(c)_0$ *and* $(wf)_0$ *be the classes of maps arising in the examples of M2* $(cw)(f)$*- and* $(c)(wf)$*-decompositions suggested in §3.3.*

Do classes $(cw)_0^{lr}$*,* $(f)_0^{rl}$*,* $(c)_0^{lr}$ *and* $(wf)_0^{rl}$ *define a model structure on* $s\mathcal{F}ilt$ *or on* $s\mathfrak{F}ilt$*?*

Does it induce one of the usual model category structures on the subcategory Top of s𝔭ilt?

***Question 4.20* (Orthogonals in group theory).** — – *Is the class of finite CA-groups or CN-groups defined by a natural lifting property, say as an iterated orthogonal of a single homomorphism? Recall that a group is a CA-group, resp. CN-group, iff the centraliser of a non-identity element is necessarily abelian, resp. nilpotent.*

- *Calculate iterated left and right $\rightthreetimes$-negations and generalisations, e.g. $(C)^r$, $(C)^l$, $(C)^{rl}$, $(C)^{ll}$, $(C)^{rr}$, $(C)^{llr}$, ... for various simple classes of morphisms in various categories, e.g. morphisms of finite topological spaces or finite groups.*
- *Reformulate the Feit-Thompson odd group theorem as inclusion of orthogonals.*(17)

5. Appendix

5.1. Surjection and injection: an example of translation and orthogonals. — This section is part of a note(18) written for *The De Morgan Gazette* to demonstrate that some natural definitions are lifting properties relative to the simplest counterexample, and to suggest a way to "extract" these lifting properties from the text of the usual definitions and proofs. The exposition is in the form of a story and aims to be self-contained and accessible to a first year student who has taken some first lectures in naive set theory, topology, and who has heard a definition of a category. A more sophisticated reader may find it more illuminating to recover our formulations herself from reading either the abstract, or the abstract and the opening sentence of the next two sections. The displayed formulae and Figure 4(a) defining the lifting property provide complete formulations of our theorems to such a reader.

5.1.1. Surjection and injection. — We try to find some "algebraic" notation to (re)write the *text* of the definitions of surjectivity and injectivity of a function, as found in any standard textbook. We want something very straightforward and syntactic—notation for what we (actually) say, for the text we write, and not for its meaning, for who knows what meaning is anyway?

(*)$_{\mathbf{words}}$: "A function f from X to Y is *surjective* iff for every element y of Y there is an element x of X such that $f(x) = y$."

(17)For a partial reformulation see Gavrilovich, Expressing the statement of the Feit-Thompson theorem with diagrams in the category of finite groups].

(18)See [Gavrilovich, DMG]. I thank Vladimir Sosnilo for help with the exposition.

A function from X to Y is an arrow $X \longrightarrow Y$. Grothendieck taught us that a point, say "x of X", is (better viewed as) as $\{\bullet\}$-valued point, that is an arrow

$$\{\bullet\} \longrightarrow X$$

from a (the?) set with a unique element; similarly "y of Y" we denote by an arrow

$$\{\bullet\} \longrightarrow Y.$$

Finally, make dashed the arrows required to "exist". We get the diagram Fig. 1(b) without the upper left corner; there "$\{\}$" denotes the empty set with no elements listed inside of the brackets.

$(**)_{\text{words}}$: "A function f from X to Y is *injective* iff no pair of different points of X is sent to the same point of Y."

"A function f from X to Y" is an arrow $X \longrightarrow Y$. "A pair of points" is a $\{\bullet, \bullet\}$-valued point, that is an arrow

$$\{\bullet, \bullet\} \longrightarrow X$$

from a two element set; we ignore "different" for now. "the same point of Y" is an arrow $\{\bullet\} \longrightarrow Y$. Represent "sent to" by an arrow

$$\{\bullet, \bullet\} \longrightarrow \{\bullet\}.$$

What about "different"? If the points are not "different", then they are "the same" point of X, and thus we need to add an arrow representing a single point of X, that is an arrow

$$\{\bullet\} \longrightarrow X.$$

Now all these arrows combine nicely into diagram Figure 4(c); however, our analysis does not necessarily makes it clear that the diagonal arrow needs to be denoted differently. How do we read it? We want this diagram to have the meaning of the sentence $(**)_{\text{words}}$ above, so we interpret such diagrams as follows:

($\rightthreetimes$) : "for every commutative square (of solid arrows) as shown there is a diagonal (dashed) arrow making the total diagram commutative" (see Fig. 1(a)).

(recall that "commutative" in category theory means that the composition of the arrows along a directed path depends only on the endpoints of the path)

Property ($\rightthreetimes$) has a name and is in fact quite well-known [Qui]. It is called *the lifting property*, or sometimes *orthogonality of morphisms*, and is viewed as the property of the two downward arrows; we denote it by $\rightthreetimes$.

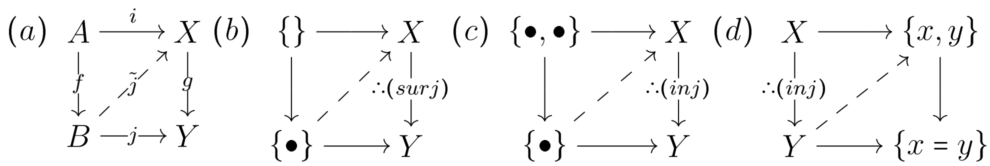


FIGURE 4. Lifting properties. Dots $\therefore$ indicate free variables and what property of these variables is being defined; in a diagram chasing calculation, "$\therefore(surj)$" reads as: given a (valid) diagram, add label $(surj)$ to the corresponding arrow.
(a) The definition of a lifting property $f \rightthreetimes g$: for each $i : A \longrightarrow X$ and $j : B \longrightarrow Y$ making the square commutative, i.e. $f \circ j = i \circ g$, there is a diagonal arrow $\tilde{j} : B \longrightarrow X$ making the total diagram $A \xrightarrow{f} B \xrightarrow{\tilde{j}} X \xrightarrow{g} Y, A \xrightarrow{i} X, B \xrightarrow{j} Y$ commutative, i.e. $f \circ \tilde{j} = i$ and $\tilde{j} \circ g = j$. (b) $X \longrightarrow Y$ is surjective
(c) $X \longrightarrow Y$ is injective; $X \longrightarrow Y$ is an epicmorphism if we forget that $\{\bullet\}$ denotes a singleton (rather than an arbitrary object and thus $\{\bullet,\bullet\} \longrightarrow \{\bullet\}$ denotes an arbitrary morphism $Z \sqcup Z \xrightarrow{(id,id)} Z$)
(d) $X \longrightarrow Y$ is injective, in the category of Sets; $\pi_0(X) \longrightarrow \pi_0(Y)$ is injective, when the diagram is interpreted in the category of topological spaces.

Now we rewrite $(*)_{\text{words}}$ and $(**)_{\text{words}}$ as:

$$(*)_{\rightthreetimes} \qquad \{\} \longrightarrow \{\bullet\} \rightthreetimes X \longrightarrow Y$$

$$(**)_{\rightthreetimes} \qquad \{\bullet,\bullet\} \longrightarrow \{\bullet\} \rightthreetimes X \longrightarrow Y$$

So we rewrote these definitions without any words at all. Our benefits? The usual little miracles happen:

The notation makes apparent a similarity of $(*)_{\text{words}}$ and $(**)_{\text{words}}$: they are obtained, in the same purely formal way, from the two of the simplest arrows (maps, morphisms) in the category of Sets. More is true: it is also apparent that these two arrows are the simplest *counterexamples* to the properties, and this suggests that we think of the lifting property as a category-theoretic (substitute for) negation. Note also that a non-trivial (one which is not an non-isomorphism) morphism never has the lifting property relative to itself, which fits with this interpretation.

Now that we have a formal notation and the little observation above, we start to play around looking at simple arrows in various categories, and also at not-so-simple arrows representing standard counterexamples. You notice a few words from your first course on topology: (i) *connected,* (ii) *the separation axioms* T_0 *and* T_1, (iii) *dense,* (iv) *induced (pullback) topology*, and (v) *Hausdorff* are, respectively,[19]

(i):

$$X \longrightarrow \{\bullet\} \rightthreetimes \{\bullet,\bullet\} \longrightarrow \{\bullet\}$$

(ii):

$$\{\bullet \leftrightarrow \star\} \longrightarrow \{\bullet = \star\} \rightthreetimes X \longrightarrow \{\bullet\}$$

and

$$\{\bullet \rightarrow \star\} \longrightarrow \{\bullet = \star\} \rightthreetimes X \longrightarrow \{\bullet\}$$

(iii):

$$X \longrightarrow Y \rightthreetimes \{\bullet\} \longrightarrow \{\bullet \rightarrow \star\}$$

(iv):

$$X \longrightarrow Y \rightthreetimes \{\bullet \rightarrow \star\} \longrightarrow \{\bullet\}$$

(v):

$$\{\bullet,\bullet'\} \xrightarrow{(inj)} X \rightthreetimes \{\bullet \leftarrow \star \rightarrow \bullet'\} \longrightarrow \{\bullet\}$$

here

$$\{\bullet \rightarrow \star\},\ \{\bullet \leftrightarrow \star\},\ \ldots$$

denote finite preorders, or, equivalently, finite categories with at most one arrow between any two objects, or finite topological spaces on their elements or objects, where a subset is closed iff it is downward closed (that is, together with each element, it contains all the smaller elements). Thus

$$\{\bullet \rightarrow \star\},\ \{\bullet \leftrightarrow \star\}\ \text{ and }\ \{\bullet \leftarrow \star \rightarrow \bullet'\}$$

denote the connected spaces with only one open point $\bullet$, with no open points, and with two open points $\bullet, \bullet'$ and a closed point $\star$. Line (v) is to be interpreted somewhat differently: we consider *all* the injective arrows of form $\{\bullet,\bullet'\} \longrightarrow X$.

We mentioned that the lifting property can be seen as a kind of negation. Confusingly, there are *two* negations, depending on whether the morphism appears on the left or right side of the square, that are quite different: for example, both the pullback topology and the separation axiom T_1 are negations of the same morphism, and the same goes for injectivity and injectivity on π_0 (see Figure 4(c,d)).

Now consider the standard example of something non-compact: the open covering

$$\mathbb{R} = \bigcup_{n\in\mathbb{N}} \{\, x \,:\, -n < x < n \,\}$$

[19] The notation is self-explanatory; for the definition see §5.3.1.

of the real line by infinitely many increasing intervals. A related arrow in the category of topological spaces is

$$\bigsqcup_{n\in\mathbb{N}}\{\,x:-n<x<n\}\longrightarrow\mathbb{R}.$$

Does the lifting property relative to that arrow define compactness? Not quite, but almost:

$$\{\}\longrightarrow X\rightthreetimes\bigsqcup_{n\in\mathbb{N}}\{x:-n<x<n\}\longrightarrow\mathbb{R}$$

reads, for X connected, as "Every continuous real-valued function on X is bounded, i.e. for each continuous $f:X\longrightarrow\mathbb{R}$ there is a natural number $n\in\mathbb{N}$ such that $-n<f(x)<n$ for each $x\in X$", which is an early characterisation of compactness taught in a first course on analysis. Notice that this characterisation mentions explicitly the arrow $X\longrightarrow\mathbb{R}$ and the bounded intervals of the real line, i.e. arrows $\{\,x:-n<x<n\}\xrightarrow{\subseteq}\mathbb{R}$, $n\in\mathbb{N}$ constituting the arrow-counterexample on the right hand side.

In a category of metric spaces with say distance non-increasing maps, a metric space X is *complete*, i.e. each Cauchy sequence $x_n\in X$, $n\in\mathbb{N}$, say $dist(x_n,x_m)\le 1/n$, converges to some point $x_\infty\in X$ such that $dist(x_\infty,x_n)\le 1/n$, iff

$$\{\text{``}x_n\text{''}:\,n\in\mathbb{N}\}\longrightarrow\{\text{``}x_n\text{''}:n\in\mathbb{N}\}\cup\{\text{``}x_\infty\text{''}\}\rightthreetimes X\longrightarrow\{\bullet\}$$

(where $dist(\text{``}x_n\text{''},\text{``}x_m\text{''})=\frac{1}{n}$ for $m>n$, $dist(\text{``}x_\infty\text{''},\text{``}x_n\text{''})=\frac{1}{n}$, as defined above.)

In functional analysis, a (partially defined!) linear operator $f:X\longrightarrow Y$ between Banach spaces X and Y is *closed* iff for every convergent sequence $x_n\in X$, if $f(x_n)\xrightarrow[n\longrightarrow\infty]{}y$ in Y, then there is a $x\in X$ such that $f(x)=y$ and $x_n\xrightarrow[n\longrightarrow\infty]{}x$, i.e.

$$\{\text{``}x_n\text{''}:\,n\in\mathbb{N}\}\longrightarrow\{\text{``}x_n\text{''}:n\in\mathbb{N}\}\cup\{\text{``}x_\infty\text{''}\}\rightthreetimes\mathrm{Domain}(f)\longrightarrow Y$$

A module P over a commutative ring R is *projective* iff for an arbitrary arrow $N\longrightarrow M$ in the category of R-modules it holds

$$0\longrightarrow R\rightthreetimes N\longrightarrow M\implies 0\longrightarrow P\rightthreetimes N\longrightarrow M.$$

Dually, a module I over a ring R is *injective* iff for an arbitrary arrow $N\longrightarrow M$ in the category of R-modules it holds

$$R\longrightarrow 0\rightthreetimes N\longrightarrow M\implies N\longrightarrow M\rightthreetimes I\longrightarrow 0.$$

5.1.2. Finite groups.— There are examples outside of topology; see Appendix 5.2.1. Let us give some examples in group theory. There is no non-trivial homomorphism from a group F to G, write $F \not\to G$, iff

$$0 \longrightarrow F \rightthreetimes 0 \longrightarrow G \text{ or equivalently } F \longrightarrow 0 \rightthreetimes G \longrightarrow 0.$$

A group A is *Abelian* iff

$$\langle a,b\rangle \longrightarrow \langle a,b : ab = ba\rangle \rightthreetimes A \longrightarrow 0$$

where $\langle a,b\rangle \longrightarrow \langle a,b : ab = ba\rangle$ is the abelianisation morphism sending the free group into the Abelian free group on two generators; a group G is *perfect*, $G = [G,G]$, iff $G \not\to A$ for any Abelian group A, i.e.

$$\langle a,b\rangle \longrightarrow \langle a,b : ab = ba\rangle \rightthreetimes A \longrightarrow 0 \implies G \longrightarrow 0 \rightthreetimes A \longrightarrow 0$$

in the category of finite or algebraic groups, a group H is *soluble* iff $G \not\to H$ for each perfect group G, i.e.

$$0 \longrightarrow G \rightthreetimes 0 \longrightarrow H \text{ or equivalently } C \longrightarrow 0 \rightthreetimes H \longrightarrow 0.$$

A prime number p does not divide the number elements of a finite group G iff G has no element of order p, i.e. no element $x \in G$ such that $x^p = 1_G$ yet $x^1 \neq 1_G, ..., x^{p-1} \neq 1_G$, equivalently $\mathbb{Z}/p\mathbb{Z} \not\to G$, i.e.

$$0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \rightthreetimes 0 \longrightarrow G \text{ or equivalently } \mathbb{Z}/p\mathbb{Z} \longrightarrow 0 \rightthreetimes G \longrightarrow 0.$$

A finite group G is a p-group, i.e. the number of its elements is a power of a prime number p, iff in the category of finite groups

$$0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \rightthreetimes 0 \longrightarrow H \implies 0 \longrightarrow H \rightthreetimes 0 \longrightarrow G.$$

5.2. Appendix. Iterated orthogonals: definitions and intuition.— For a property (class) C of arrows (morphisms) in a category, define *its left and right orthogonals*, which we also call *left and right negation*:

$$C^l := \{f : \text{ for each } g \in C \ f \rightthreetimes g\}$$

$$C^r := \{g : \text{ for each } f \in C \ f \rightthreetimes g\}$$

$$C^{lr} := (C^l)^r, ...$$

here $f \rightthreetimes g$ reads "f has the left lifting property wrt g", "f is (left) orthogonal to g", i.e. for $f : A \longrightarrow B$, $g : X \longrightarrow Y$, $f \rightthreetimes g$ iff for each $i : A \longrightarrow X$, $j : B \longrightarrow Y$ such that $ig = fj$ ("the square commutes"), there is $j' : B \longrightarrow X$ such that $fj' = i$ and $j'g = j$ ("there is a diagonal making the diagram commute"), cf. Fig. 5.

The following observation is enough to reconstruct all the examples of iterated orthogonals in this paper, with a bit of search and computation.

Observation.
A number of elementary properties can be obtained by repeatedly passing to the left or right orthogonal $C^l, C^r, C^{lr}, C^{ll}, C^{rl}, C^{rr}, \ldots$ starting from a simple class of morphisms, often a single (counter)example to the property you define.

The counterexample is often implicit in the text of the definition of the property.

A useful intuition is to think that the property of left-lifting against a class C is a kind of negation of the property of being in C, and that right-lifting is another kind of negation. Hence the classes obtained from C by taking orthogonals an odd number of times, such as $C^l, C^r, C^{lrl}, C^{lll}$ etc., represent various kinds of negation of C, so $C^l, C^r, C^{lrl}, C^{lll}$ each consists of morphisms which are far from having property C.

Taking the orthogonal of a class C is a simple way to define a class of morphisms excluding non-isomorphisms from C, in a way which is useful in a diagram chasing computation.

The class C^l is always closed under retracts, pullbacks, (small) products (whenever they exist in the category) and composition of morphisms, and contains all isomorphisms of C. Meanwhile, C^r is closed under retracts, pushouts, (small) coproducts and transfinite composition (filtered colimits) of morphisms (whenever they exist in the category), and also contains all isomorphisms. Under some assumptions on existence of limits and colimits and ignoring set-theoretic difficulties[20], each morphism $X \longrightarrow Y$ decomposes both as $X \xrightarrow{(C)^l} \bullet \xrightarrow{(C)^{lr}} Y$ and $X \xrightarrow{(C)^{rl}} \bullet \xrightarrow{(C)^r} Y$.

For example, the notion of isomorphism can be obtained starting from the class of all morphisms, or any single example of an isomorphism:

$$(Isomorphisms) = (all\ morphisms)^l = (all\ morphisms)^r = (h)^{lr} = (h)^{rl}$$

where h is an arbitrary isomorphism.

5.2.1. Examples of iterated orthogonals.— Here give a list of examples of well-known properties which can be defined by iterated orthogonals starting from a simple class of morphisms.

(i) $(\varnothing \longrightarrow \{*\})^r$, $(0 \longrightarrow R)^r$, and $\{0 \longrightarrow \mathbb{Z}\}^r$ are the classes of surjections in in the categories of Sets, R-modules, and Groups, resp., (where $\{*\}$ is the one-element set, and in the category of (not necessarily abelian) groups, 0 denotes the trivial group)

(20) For an example of a theorem along these lines see [Bousfield, Constructions of factorization systems in categories, 5.1 Ex, 3.1 Thm]. Note that he considers the *unique* lifting property, unlike us.

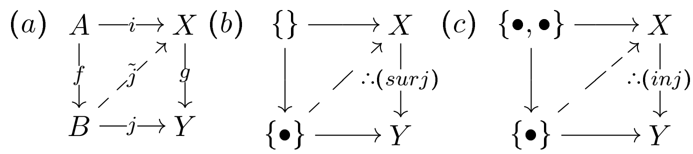


FIGURE 5. Lifting properties. (a) The definition of a lifting property $f \rightthreetimes g$. (b) $X \longrightarrow Y$ is surjective (c) $X \longrightarrow Y$ is injective

(ii) $(\{\star,\bullet\} \longrightarrow \{*\})^l = (\{\star,\bullet\} \longrightarrow \{*\})^r$, $(R \longrightarrow 0)^r$, $\{\mathbb{Z} \longrightarrow 0\}^r$ are the classes of injections in the categories of Sets, R-modules, and Groups, resp

(iii) in the category of R-modules,
a module P is projective iff $0 \longrightarrow P$ is in $(0 \longrightarrow R)^{rl}$
a module I is injective iff $I \longrightarrow 0$ is in $(R \longrightarrow 0)^{rr}$

(iv) in the category of Groups,
a finite group H is nilpotent iff $H \longrightarrow H \times H$ is in $\{0 \longrightarrow G : G \text{ arbitrary}\}^{lr}$
a finite group H is solvable iff $0 \longrightarrow H$ is in $\{\, 0 \longrightarrow A : A \text{ abelian } \}^{lr} = \{\, [G,G] \longrightarrow G : G \text{ arbitrary } \}^{lr}$
a finite group H is of order prime to p iff $H \longrightarrow 0$ is in $\{\mathbb{Z}/p\mathbb{Z} \longrightarrow 0\}^r$
a finite group H is a p-group iff, in the subcategory of finite groups, $H \longrightarrow 0$ is in $\{\mathbb{Z}/p\mathbb{Z} \longrightarrow 0\}^{rr}$
a group F is free iff $0 \longrightarrow F$ is in $\{0 \longrightarrow \mathbb{Z}\}^{rl}$

(v) in the category of metric spaces and uniformly continuous maps,
a metric space X is complete iff $\{1/n\}_n \longrightarrow \{1/n\}_n \cup \{0\} \rightthreetimes X \longrightarrow \{0\}$ where the metric on $\{1/n\}_n$ and $\{1/n\}_n \cup \{0\}$ is induced from the real line
a subset $A \subset X$ is closed iff $\{1/n\}_n \longrightarrow \{1/n\}_n \cup \{0\} \rightthreetimes A \longrightarrow X$

(vi) in the category of topological spaces,
for a connected topological space X, each function on X is bounded iff

$$\varnothing \longrightarrow X \rightthreetimes \cup_n(-n,n) \longrightarrow \mathbb{R}$$

(vii) in the category of topological spaces (see notation defined below),
a space X is path-connected iff $\{0,1\} \longrightarrow [0,1] \rightthreetimes X \longrightarrow \{*\}$
a space X is path-connected iff for each Hausdorff compact space K and each injective map $\{x,y\} \hookrightarrow K$ it holds $\{x,y\} \hookrightarrow K \rightthreetimes X \longrightarrow \{*\}$

Proof. In (iv), we use that a finite group H is nilpotent iff the diagonal $\{(h,h) : h \in H\}$ is subnormal in $H \times H$, cf. [Nilp], and in fact the orthogonal is the class of subnormal subgroups. □

5.3. A concise notation for certain properties in elementary point-set topology. — We introduce a concise, and in a sense intuitive, notation (syntax) able to express a number of properties in elementary point-set topology. It is appropriate for properties defined as iterated orthogonals (negation) starting from maps of finite topological spaces.(21)

For example, surjective, injective, connected, totally disconnected, and dense are expressed as $\{\{\} \longrightarrow \{a\}\}^r$, $\{\{x,y\} \longrightarrow \{x=y\}\}^r$, $\{\{x,y\} \longrightarrow \{x=y\}\}^l$ or $\{\{\} \longrightarrow \{a\}\}^{rll}$, $\{\{\} \longrightarrow \{a\}\}^{rllr}$, $\{\{x\} \longrightarrow \{x \swarrow y\}\}^r$.

5.3.1. Notation for maps between finite topological spaces.— A *topological space* comes with a *specialisation preorder* on its points: for points $x,y \in X$, $x \leq y$ iff $y \in clx$ (y is in the *topological closure* of x). The resulting *preordered set* may be regarded as a *category* whose *objects* are the points of X and where there is a unique *morphism* $x \searrow y$ iff $y \in clx$.

For a *finite topological space* X, the specialisation preorder or equivalently the corresponding category uniquely determines the space: a *subset* of X is *closed* iff it is *downward closed*, or equivalently, is a full subcategory such that there are no morphisms going outside the subcategory.

The monotone maps (i.e. *functors*) are the *continuous maps* for this topology.

We denote a finite topological space by a list of the arrows (morphisms) in the corresponding category; '$\leftrightarrow$' denotes an *isomorphism* and '=' denotes the *identity morphism*. An arrow between two such lists denotes a *continuous map* (a functor) which sends each point to the correspondingly labelled point, but possibly turning some morphisms into identity morphisms, thus gluing some points.

With this notation, we may display continuous functions for instance between the *discrete space* on two points, the *Sierpinski space*, the *antidiscrete space* and the *point space* as follows (where each point is understood to be mapped to the point of the same name in the next space):

$$\begin{array}{ccccccc} \{a,b\} & \longrightarrow & \{a \searrow b\} & \longrightarrow & \{a \leftrightarrow b\} & \longrightarrow & \{a=b\} \\ \text{(discrete space)} & \longrightarrow & \text{(Sierpinski space)} & \longrightarrow & \text{(antidiscrete space)} & \longrightarrow & \text{(single point)} \end{array}$$

In $A \longrightarrow B$, each object and each morphism in A necessarily appears in B as well; we avoid listing the same object or morphism twice. Thus both

$$\{a\} \longrightarrow \{a,b\} \qquad \text{and} \qquad \{a\} \longrightarrow \{b\}$$

denote the same map from a single point to the discrete space with two points. Both

$$\{a \swarrow U \searrow x \swarrow V \searrow b\} \longrightarrow \{a \swarrow U = x = V \searrow b\} \text{ and } \{a \swarrow U \searrow x \swarrow V \searrow b\} \longrightarrow \{U = x = V\}$$

(21)I thank Urs Schreiber for help with the exposition in this subsection.

denote the morphism gluing points U, x, V.

In $\{a \searrow b\}$, the point a is open and point b is closed. We denote points by $a, b, c, .., U, V, ..., 0, 1..$ to make notation reflect the intended meaning, e.g. $X \longrightarrow \{U \searrow U'\}$ reminds us that the preimage of U determines an open subset of X, $\{x, y\} \longrightarrow X$ reminds us that the map determines points $x, y \in X$, and $\{o \searrow c\}$ reminds that o is open and c is closed.

Each continuous map $A \longrightarrow B$ between finite spaces may be represented in this way; in the first list list relations between elements of A, and in the second list put relations between their images. However, note that this notation does not allow to represent *endomorphisms* $A \longrightarrow A$. We think of this limitation as a feature and not a bug: in a diagram chasing computation, endomorphisms under transitive closure lead to infinite cycles, and thus our notation has better chance to define a computable fragment of topology.

5.3.2. Examples of iterated orthogonals obtained from maps between finite topological spaces.— Here give a list of examples of well-known properties which can be defined by iterated orthogonals starting from maps between finite topological spaces, often with less than 5 elements.

In the category of topological spaces (see notation defined below),

- a Hausdorff space K is compact iff $K \longrightarrow \{*\}$ is in $((\{o\} \longrightarrow \{o \searrow c\})^r_{<5})^{lr}$
- a Hausdorff space K is compact iff $K \longrightarrow \{*\}$ is in

$$\{\, \{a \leftrightarrow b\} \longrightarrow \{a = b\}, \{o \searrow c\} \longrightarrow \{o = c\}, \{c\} \longrightarrow \{o \searrow c\}, \{a \swarrow o \searrow b\} \longrightarrow \{a = o = b\} \,\}^{lr}$$

- a space D is discrete iff $\varnothing \longrightarrow D$ is in $(\varnothing \longrightarrow \{*\})^{rl}$
- a space D is antidiscrete iff $D \longrightarrow \{*\}$ is in $(\{a, b\} \longrightarrow \{a = b\})^{rr} = (\{a \leftrightarrow b\} \longrightarrow \{a = b\})^{lr}$
- a space K is connected or empty iff $K \longrightarrow \{*\}$ is in $(\{a, b\} \longrightarrow \{a = b\})^l$
- a space K is totally disconnected and non-empty iff $K \longrightarrow \{*\}$ is in $(\{a, b\} \longrightarrow \{a = b\})^{lr}$
- a space K is connected and non-empty iff for some arrow $\{*\} \longrightarrow K$
  $\{*\} \longrightarrow K$ is in $(\varnothing \longrightarrow \{*\})^{rll} = (\{a\} \longrightarrow \{a, b\})^l$
- a space K is non-empty iff $K \longrightarrow \{*\}$ is in $(\varnothing \longrightarrow \{*\})^l$
- a space K is empty iff $K \longrightarrow \{*\}$ is in $(\varnothing \longrightarrow \{*\})^{ll}$
- a space K is T_0 iff $K \longrightarrow \{*\}$ is in $(\{a \leftrightarrow b\} \longrightarrow \{a = b\})^r$
- a space K is T_1 iff $K \longrightarrow \{*\}$ is in $(\{a \searrow b\} \longrightarrow \{a = b\})^r$
- a space X is Hausdorff iff for each injective map $\{x, y\} \hookrightarrow X$ it holds $\{x, y\} \hookrightarrow X \rightthreetimes \{x \searrow o \swarrow y\} \longrightarrow \{x = o = y\}$
- a non-empty space X is regular (T3) iff for each arrow $\{x\} \longrightarrow X$ it holds $\{x\} \longrightarrow X \rightthreetimes \{x \searrow X \swarrow U \searrow F\} \longrightarrow \{x = X = U \searrow F\}$
- a space X is normal (T4) iff $\varnothing \longrightarrow X \rightthreetimes \{a \swarrow U \searrow x \swarrow V \searrow b\} \longrightarrow \{a \swarrow U = x = V \searrow b\}$

- a space X is completely normal iff $\varnothing \longrightarrow X \rightthreetimes [0,1] \longrightarrow \{0\swarrow x \searrow 1\}$ where the map $[0,1] \longrightarrow \{0\swarrow x \searrow 1\}$ sends 0 to 0, 1 to 1, and the rest $(0,1)$ to x
- a space X is path-connected iff $\{0,1\} \longrightarrow [0,1] \rightthreetimes X \longrightarrow \{*\}$
- a space X is path-connected iff for each Hausdorff compact space K and each injective map $\{x,y\} \hookrightarrow K$ it holds $\{x,y\} \hookrightarrow K \rightthreetimes X \longrightarrow \{*\}$
- a non-empty space X is regular (T3) iff for each arrow $\{x\} \longrightarrow X$ it holds $\{x\} \longrightarrow X \rightthreetimes \{x \searrow X \swarrow U \searrow F\} \longrightarrow \{x = X = U \searrow F\}$
- a space X is normal (T4) iff $\varnothing \longrightarrow X \rightthreetimes \{a \swarrow U \searrow x \swarrow V \searrow b\} \longrightarrow \{a \swarrow U = x = V \searrow b\}$
- a space X is completely normal iff $\varnothing \longrightarrow X \rightthreetimes [0,1] \longrightarrow \{0\swarrow x \searrow 1\}$ where the map $[0,1] \longrightarrow \{0\swarrow x \searrow 1\}$ sends 0 to 0, 1 to 1, and the rest $(0,1)$ to x
- a space X is path-connected iff $\{0,1\} \longrightarrow [0,1] \rightthreetimes X \longrightarrow \{*\}$
- a space X is path-connected iff for each Hausdorff compact space K and each injective map $\{x,y\} \hookrightarrow K$ it holds $\{x,y\} \hookrightarrow K \rightthreetimes X \longrightarrow \{*\}$
- $(\varnothing \longrightarrow \{*\})^r$ is the class of surjections
- $(\varnothing \longrightarrow \{*\})^r$ is the class of maps $A \longrightarrow B$ where $A \neq \varnothing$ or $A = B$
- $(\varnothing \longrightarrow \{*\})^{rr}$ is the class of subsets, i.e. injective maps $A \hookrightarrow B$ where the topology on A is induced from B
- $(\varnothing \longrightarrow \{*\})^{lr}$ is the class of maps $\varnothing \longrightarrow B$, B arbitrary
- $(\varnothing \longrightarrow \{*\})^{lrl}$ is the class of maps $A \longrightarrow B$ which admit a section
- $(\varnothing \longrightarrow \{*\})^{l}$ consists of maps $f : A \longrightarrow B$ such that either $A \neq \varnothing$ or $A = B = \varnothing$
- $(\varnothing \longrightarrow \{*\})^{rl}$ is the class of maps of form $A \longrightarrow A \sqcup D$ where D is discrete
- $\{\bullet\} \longrightarrow A$ is in $(\varnothing \longrightarrow \{*\})^{rll}$ iff A is connected
- Y is totally disconnected iff $\{\bullet\} \xrightarrow{y} Y$ is in $(\varnothing \longrightarrow \{*\})^{rllr}$ for each map $\{\bullet\} \xrightarrow{y} Y$ (or, in other words, each point $y \in Y$).
- $(\{b\} \longrightarrow \{a \searrow b\})^l$ is the class of maps with dense image
- $(\{b\} \longrightarrow \{a \searrow b\})^{lr}$ is the class of closed subsets $A \subset X$, A a closed subset of X
- $(\{a \searrow b\} \longrightarrow \{a = b\})^l$ is the class of injections
- $((\{a\} \longrightarrow \{a \searrow b\})^r_{<5})^{lr}$ is roughly the class of proper maps (see below).

Proof. Items related to compactness and proper maps are discussed in ??. Other items require a simple if tedious verification. □

5.4. Separation axioms as orthogonals.— See `https://ncatlab.org/nlab/show/separation+axioms+in+terms+of+lifting+properties` for a list of reformulations of the separation axioms.

5.5. Appendix. Compactness as being uniform: change of order of quantifiers. — We give several examples where an application of compactness can be reformulated as changing the order of quantifiers in a formula.

5.5.1. Each real-valued function on a compact set is bounded. —

$$\frac{\forall x \in K \exists M (f(x) < M)}{\exists M \forall x \in K (f(x) < M)}$$

Note this is a lifting property, for K connected:

$$\{\} \longrightarrow K \rightthreetimes \sqcup_{n \in \mathbb{N}} (-n, n) \longrightarrow \mathbb{R}$$

here $\sqcup_n (-n, n) \longrightarrow \mathbb{R}$ denotes the map to the real line from the disjoint union of intervals $(-n, n)$ which cover it. Note this is a standard example of an open covering of $\mathbb{R}$ which shows it is not compact.

5.5.2. The image of a closed set is closed. — K is compact iff the following implication holds for each set X and each subset $Z \subset X \times K$:

$$\frac{\forall y \in K \exists U \exists V (U \subset X \text{ open and } V \subset K \text{ open and } a \in U \text{ and } y \in V \text{ and } U \times V \subset Z)}{\exists U \exists V \forall y \in K (U \subset X \text{ open and } V \subset K \text{ open and } a \in U \text{ and } y \in V \text{ and } U \times V \subset Z)}$$

The hypothesis says Z contains a rectangular open neighbourhood of each point of the line $\{a\} \times K$; the conclusion says that Z contains a rectangular open neighbourhood of the whole line $\{a\} \times K$.

5.5.3. A Hausdorff compact is necessarily normal.— The application of compactness in the usual proof of this implication amounts to the following change of order of quantifiers:

$$\frac{\forall a \in A \forall b \in B \exists U \exists V (a \in U \text{ and } b \in V \text{ and } U \cap V = \{\} \text{ and } U \subset K \text{ open and } V \subset K \text{ open})}{\exists U \exists V \forall a \in A \forall b \in B (a \in U \text{ and } b \in V \text{ and } U \cap V = \{\} \text{ and } U \subset K \text{ open and } V \subset K \text{ open})}$$

5.5.4. Lebesgue number Lemma. — Let S be a family of (arbitrary) subsets of a metric space X.

$$\frac{\forall x \in X \exists \delta > 0 \exists U \in S \forall y \in X (dist(x, y) < \delta \implies y \in U)}{\exists \delta > 0 \forall x \in X \exists U \in S \forall y \in X (dist(x, y) < \delta \implies y \in U)}$$

The hypothesis says that $\{Inn\, U \,:\, U \in S\}$ is an open cover of X; the conclusion is as usually stated, that each set of diameter $< \delta$ is covered by a single member of the cover.

Note that this lemma may be expressed in terms of uniform structures.

5.5.5. Paracompactness.— [Alexandroff,§2.3,p.38] writes "as it seems to me, one of the deepest and most interesting properties of paracompacts" is the following theorem of A.Stone: that

> A T_1-space is *paracompact* iff for each open covering α of X there is an open covering β such that for each x in X there is U in A such that $\cup\{V \in \beta : x \in V\} \subset U$

The family of subsets $\cup\{V \in \beta : x \in V\}$ where $x \in X$, $V \in \beta$ forms a covering denoted by β^* by [Alexandroff]. This is somewhat reminiscent of a simplicial construction.

As quantifier exchange, this is:

$$\frac{\text{for each open covering } \alpha \text{ exists open covering } \beta.\ \forall x \in X \forall V \in \beta \exists U \in \alpha (x \in V \implies V \subset U)}{\text{for each open covering } \alpha \text{ exists open covering } \beta.\ \forall x \in X \exists U \in \alpha \forall V \in \beta (x \in V \implies V \subset U)}$$

The hypothesis holds trivially: take $\beta = \alpha, V = U$.

Question 5.1. — *Describe a logic and a class of formulae where such exchange of order quantifiers is permissible. Is there a treatment of compactness in terms of changing order of quantifiers ?*

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This note is elementary, and it was embarrassing and boring, and embarrassingly boring, to think or talk about matters so trivial, but luckily I had no obligations for a time.

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