A FIRST-ORDER THEORY IS STABLE IFF ITS TYPE SPACE IS SIMPLICIALLY CONTRACTIBLE

to Vita Kreps in memoriam

request for comments

Abstract. — We spell out simplicial diagrams in *sTop* representing several basic notions in model theory such as a parameter set and a type, a type being invariant, definable, and product of invariant types, and a theory being stable, and give pointers to the same diagrams in homotopy theory.

A definable type of a first-order theory is the same as a section (retraction) of the simplicial path space (decalage) of its space of types viewed as a simplicial topological space; as is well-known, in the category of simplicial sets such sections correspond to homotopies contracting each connected component. Without the simplicial language this is stated in [**Tent-Ziegler**, Exercise 8.3.3], which defines a bijection between the set of all 1-types definable over a parameter set B, and the set of all "coherent" families of continuous sections $\pi_n : S^T_n(B) \to S^T_{n+1}(B)$ where $S^T_n(B)$ is the Stone space of types with n variables of the theory T with parameters in B.

Thus the definition of stability "each type is definable" says that *a first order theory is stable iff its space of types is simplicially contractible*, in the precise sense that the simplicial type space functor $\mathbb{S}^T_\bullet(B) : \Delta^{op} \longrightarrow \text{Top},\ n \longmapsto \mathbb{S}_{n+1}(B)$ fits into a certain well-known simplicial diagram in the category of simplicial topological spaces which does define contractibility for fibrant simplicial sets.

In this note we rewrite several definitions in stability theory in terms of simplicial diagrams, and give pointers to similar diagrams in homotopy theory. We note that basic definitions of a *set of parameters in a model*, and *type*, an *invariant* or *definable* type, can all be tautologically rewritten as certain kinds of morphisms to the space of types of a first-order theory viewed as a simplicial set or topological space. Namely, we view the space of types as a functor $\mathbb{S}_\bullet : \Delta^{\text{op}} \longrightarrow \text{Top}$, $\mathbb{S}_\bullet : n^{\leqslant} \mapsto \mathbb{S}_{n+1}(\varnothing)$ from the category of non-empty linear orders to the category of topological spaces or sets. Then a subset of parameters, or rather its complete first-order diagram, is a morphism from a representable set, to the type space; a type is a morphism to the "shifted" (decalage) type space. Recall an intuition that a shifted simplicial object is the "simplicial path space" of it; from this point of view *a type is a homotopy contracting the parameters* or its limit. These and similar considerations lead to a

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tautological reformulations of the notions of an *invariant* or *definable* type, and *stability* ("each type is definable") as a simplicial diagram arguably defining something like contractibility of the space of types over a model.(1)

To what extent can one rephrase model theory in terms of the simplicial type space $\mathbb{S}_\bullet : \Delta^{\mathrm{op}} \longrightarrow \mathrm{Top}$? We do not try to answer this question, but remark that our reformulations is §1 do seem as if the type space, as a simplicial topological space, of a first order theory remembers almost everything about the theory.(2)

This note was started when we noted the simplicial language almost explicit in [**Tent-Ziegler**, Exercise 8.3.3] which establishes a bijection between the set of all global types definable over a set B and the set of all "coherent" families of continuous sections $\pi_n : \mathbb{S}_n(B) \longrightarrow \mathbb{S}_{n+1}(B)$, $n > 0$, where, as usual, $\mathbb{S}_n(B)$ denotes the Stone space of n-types over B of a theory T:

$$\pi_n : r(y_1, ..., y_n) \longmapsto \{\, \varphi(x, y_1, ..., y_n) : \mathrm{d}_p\, \varphi \in r\}$$

If $p(x/\mathfrak{C})$ is a global type invariant over B, this map can be described in terms of product of types as $\pi_n : r(y_1, ..., y_n) \longmapsto p(x) \otimes r(y_1, ..., y_n)$

In the simplicial language, such a "coherent" family of continuous sections is precisely the lifting map in the following diagram in the category of simplicial topological spaces or profinite sets, as explained in 2.2:

$$(1) \qquad \begin{array}{ccc} & & \mathbb{S}_\bullet(B) \circ [+1] \\ & \overset{\pi_\bullet}{\nearrow} & \downarrow \mathrm{pr}_{2,3,..} \\ \mathbb{S}_\bullet(B) & \xrightarrow{\mathrm{id}} & \mathbb{S}_\bullet(B) \end{array} \qquad \begin{array}{ccc} & & \mathfrak{C}_\bullet/\mathrm{Aut}(\mathfrak{C}/B) \circ [+1] \\ & \overset{\pi_\bullet}{\nearrow} & \downarrow \mathrm{pr}_{2,3,..} \\ \mathfrak{C}_\bullet/\mathrm{Aut}(\mathfrak{C}/B) & \xrightarrow{\mathrm{id}} & \mathfrak{C}_\bullet/\mathrm{Aut}(\mathfrak{C}/B) \end{array}$$

Here $\mathbb{S}_\bullet(B) : n \longmapsto \mathbb{S}_n(B)$ is the simplicial space of types of T over a parameter set $B \subset \mathfrak{C}$, and $[+1] : \Delta^{\mathrm{op}} \longrightarrow \Delta^{\mathrm{op}}, n \longmapsto n+1$ is the decalage shift endomorphism of Δ^{op} so that $\mathbb{S}_\bullet(B) \circ [+1] : n \longmapsto \mathbb{S}_{n+1}(B)$ is what is called *the simplicial path space of* $\mathbb{S}_\bullet(B)$. The diagram on the left expands the diagram on the right: $\mathfrak{C}_\bullet : n \longmapsto \mathfrak{C}^{n+1}$ is the simplicial set represented by the monster model $\mathfrak{C}$ of T where each $\mathfrak{C}^{n+1}$ is equipped with the topology generated by the solution sets $\{(x_0, ..., x_n) \in \mathfrak{C}^{n+1} : \mathfrak{C} \models \varphi(x_0, .., x_n, b_1, .., b_m), m \geq 0, b_1, ..., b_m \in B\}$ of formulas with parameters in B; the quotient is taken by the diagonal action. Dropping the continuity requirement (i.e. considering this diagram in *sSets*) leads to the notion of a *global type invariant over* B; in the category $s\boldsymbol{\Phi}$ of simplcial filters it defines a notion similar to non-forking. Two "coherent" families $\pi_n^p, \pi_n^q : \mathbb{S}_n(B) \longrightarrow \mathbb{S}_{n+1}(B)$ of sections can be composed in an obvious way $\mathbb{S}_n(B) \xrightarrow{\pi_n} \mathbb{S}_{n+1}(B) \xrightarrow{\pi_n^q} \mathbb{S}_{n+2}(B)$, and the composition is a coherent family of sections corresponding to the product $p(x) \otimes q(x)$ of types. In simplicial terms (§1.4) you say that given liftings $\pi_\bullet^p, \pi_\bullet^q : \mathbb{S}_\bullet(B) \longrightarrow \mathbb{S}_\bullet(B) \circ [+1]$, form the composition $\mathbb{S}_\bullet(B) \xrightarrow{\pi_\bullet^p} \mathbb{S}_\bullet(B) \circ [+1] \xrightarrow{\pi_\bullet^q[+1]} \mathbb{S}_\bullet(B) \circ [+2]$. Hence, a Morley

(1)Philipp Rothmaler informed us of a topological reformulation of another equivalent definition of stability "each type over a model has a unique coheir": namely, for a set $A \supset M$ of parameters containing a model M, each type over M has a unique coheir over A iff there is a continuous section of the restriction map $\mathbb{S}_1(A) \longrightarrow \mathbb{S}_1(M)$, see [**Rothmaler**, Exercises 11.3.3-7] in §6, also [**Pillay-Ziegler**].

(2)We know of no attempt to develop basic properties of a first-order theory in terms of its type space viewed simplicially and which uses explicit simplicial language.

sequence of an invariant type p is the infinite composition

$$\mathbb{S}_\bullet(B) \xrightarrow{\pi^p_\bullet} \mathbb{S}_\bullet(B)\circ[+1] \xrightarrow{\pi^p_\bullet[+1]} \mathbb{S}_\bullet(B)\circ[+2] \xrightarrow{\pi^p_\bullet[+2]} \dots$$

In *sSets* for fibrant simplicial sets these diagrams define the notion of a homotopy contracting each connected component, and in *Top* correspond to factorisations $X \longrightarrow \mathrm{Cone}(X) \longrightarrow X$ or $X \longrightarrow \mathbb{S}X \longrightarrow X$ of $\mathrm{id} : X \longrightarrow X$ though the cone or suspension(3) for a connected nice enough space X; for X not connected one needs to take the disjoint union of cones, resp. suspensions, of the connected components of X. In the category $s\boldsymbol{\varphi}$ of simplicial filters the same diagram captures the notion of convergence.

Recall that a theory is stable iff for any set (equiv., any model) each 1-type over the set is definable. Hence, in a certain precise sense given by the diagram (2) below,

> *a theory is stable iff its space of types over any set is simplicially contractible*, in the precise sense that it fits into the following diagram in the category *sTop* or its full subcategory *sProFiniteSets*

$$\begin{array}{ccc} & & \mathbb{S}_\bullet(B)\circ[+1] \\ & \overset{\pi_\bullet}{\nearrow} & \Big\downarrow{\scriptstyle \mathrm{pr}_1\times\mathrm{pr}_{2,3,..}} \\ \mathrm{const}_\bullet\,\mathbb{S}_1(B)\times\mathbb{S}_\bullet(B) & \xrightarrow{\mathrm{id}} & \mathrm{const}_\bullet\,\mathbb{S}_1(B)\times\mathbb{S}_\bullet(B) \end{array} \tag{2}$$

Here $\mathrm{const}_\bullet\,\mathbb{S}_1(B)$ denotes the constant functor $(\mathrm{const}_\bullet\,\mathbb{S}_1(B))_n := \mathbb{S}_1(B)$.

Unfortunately, it is not quite clear to us how fair is it to say that this diagram defines contractibility. Perhaps informally one may say that this diagram says that the space can be contracted to each of its points.

Simplicially, a parameter set $A \subset \mathfrak{C}$, or rather its complete diagram, resp. an 1-type over $A \subset \mathfrak{C}$, can be described as a map in *sSets* from the simplicial set $|A|_\bullet : n \longmapsto A^{n+1}$ represented by A, to the space $\mathbb{S}_\bullet(\varnothing)$ of types over the empty set, resp. to the decalage shifted space $\mathbb{S}_\bullet(\varnothing)\circ[+1]$, see §1.1.

$$\begin{array}{ccc} & & \mathbb{S}_\bullet(\varnothing)\circ[+1] \\ & \overset{p(x/A)}{\nearrow} & \Big\downarrow{\scriptstyle pr_{2,3,..}} \\ |A|_\bullet := \mathrm{Hom}_{sets}(-,A) & \xrightarrow{A\subset\mathfrak{C}} & \mathbb{S}_\bullet(\varnothing) \end{array}$$

a type over $A \subset \mathfrak{C}$

$$\begin{array}{ccc} |A|_\bullet := \mathrm{Hom}_{sets}(-,A) & \xrightarrow{p(x/A)} & \mathbb{S}_\bullet(\varnothing)\circ[+1] \\ \Big\downarrow & \nearrow & \Big\downarrow{\scriptstyle pr_{2,3,..}} \\ \mathbb{S}_\bullet(B) & \xrightarrow{A\subset\mathfrak{C}} & \mathbb{S}_\bullet(\varnothing) \end{array}$$

a type over A invariant over $B \subset A$

A little informal glossary of model theory vs topology. — This leads to the following little very informal glossary of model theory vs topology: notions on both sides fit into the same simplicial formulas. Below

$$\mathrm{Cone}_{c.c.}(X) := \bigsqcup_{X_{c.c.}\text{ a connected component of }X} \mathrm{Cone}(X_{c.c.})$$

$$\mathbb{S}_{c.c.}(X) := \bigsqcup_{X_{c.c.}\text{ a connected component of }X} \mathbb{S}(X_{c.c.})$$

(3) Recall $\mathrm{Cone}(X) := X\times[0,1]/X\times\{1\}$, and $\mathbb{S}(X) := X\star\{-1,1\} = X\times[-1,1]/\{X\times\{-1\}, X\times\{1\}\}$. Also, $\mathbb{S}S^n = S^{n+1}$ where S^n denotes the n-th sphere, $\pi_n(\mathbb{S}^k(X),x) = \pi_{n+k}(X,x)$, $0 \leqslant k \leqslant n$, and, more generally, $[\mathbb{S}X, Y] = [X, \Omega Y]$ where $[-,-]$ denote the homotopy classes of maps, and $\Omega Y := \mathrm{Hom}(S^1, Y)$ is the loop space of Y.

is the disjoint union of cones, resp. suspensions, of the connected components of X. This is well-defined in a useful way only for topological spaces "nice" enough.

- stable theories — contractible spaces
- an invariant or definable type — a homotopy $\mathrm{Cone}_{c.c.}(X) \longrightarrow X$ or $h : \mathbb{S}_{c.c.}X \longrightarrow X$ contracting $\mathrm{id} : X \longrightarrow X$
- product of invariant types $p(x) \otimes q(y)$ — composition of homotopies

$$h^p \circ \mathbb{S}_{c.c.}h^q : \mathbb{S}_{c.c.}\mathbb{S}_{c.c.}X \longrightarrow X$$

- a Morley sequence — a sequence somewhat reminiscent of a specturum in stable homotopy theory

$$h \circ \mathbb{S}_{c.c.}h \circ ... \circ \mathbb{S}^{n-1}_{c.c.}h : \mathbb{S}^n_{c.c.}X \longrightarrow X, n > 0$$

Note that a standard advice (cf. Remark 1.3.1) from homotopy theory would be to use a "better" standard construction of the quotient $\mathfrak{C}_\bullet/\mathrm{Aut}(\mathfrak{C}/B)$, called the classifying space or Borel construction of a group action.

Connection to stability theory arises if one considers the diagrams (1), (2) in the categories of simplicial topological spaces *sTop*, profinite sets *sProFiniteSets*, or filters $s\boldsymbol{\Phi}$. The homotopy theory for simplicial topological spaces *sTop* and for simplicial profinite sets *sProFiniteSets* is known; however, we were unable to understand how our diagrams relate to the notion of contracitibily/null-homotopy there. In $s\boldsymbol{\Phi}$ the same diagram defines the notion of convergence, but no theory of $s\boldsymbol{\Phi}$ exists, only examples of reformulations of various basic notions in topology, analysis, and model theory including a reformulation of stability as a lifting property [**Z1, Z2**].

Related work. — We know of no attempt to develop basic properties of a first-order theory in terms of its type space viewed simplicially and which uses explicit simplicial language. [**Morley, Knight, Levon, Kamsma, Eagle-Hamel-Tall**] consider the space of types as a functor on the category of (non-empty) finite sets, i.e. as an (augumented) symmetric simplicial topological space, and refer to them as *type space functor* or type category. [**Knight**] rewrites category-theoretically several notions in model theory, e.g. [**Knight**, Def.2.9] and [**Eagle-Hamel-Tall**, Def.4.3], defines a model and [**Knight**, Def.4.2] defines a Vaught tree. Neither of these references uses standard simplicial terminology, notation, or technique.

Structure of the paper. — In §1 we sketch how to view simplicially sets of parameters and types over parameters: a set of parameters (resp. a type) or rather its complete diagram, in a model of a theory is a morphism from a representable set to the (resp., shifted) simplicial Stone space of the theory. Following [**Tent-Ziegler**, Exercise 8.3.3], invariance and definability of types are then interpreted as lifting diagrams (retractions) in *sSets* and *sTop*.

In §2 we repeat in more detail some of §1 and explain in detail how to view [**Tent-Ziegler**, Exercise 8.3.3] in the simplicial language. Care is taken so that §2 can be read independently.

In §3.1 we explain that the simplicial formula (1) defines the usual notion of *a homotopy contracting each connected component*, in the category of topological spaces when applied to the singular complexes of sufficiently nice topological spaces.

Thus, in a certain precise sense, *a definable global type* is a homotopy contracting the simplicial Stone space of types. The product $p(x) \otimes q(x)$ of two global invariant types corresponds to a composition of such homotopie in *sSets*, and thus the type of a Morley sequence corresponds to iteratively composing in *sSets* an invariant type with itself shifted $...\pi_\bullet[n] \circ \pi_\bullet[n-1] \circ .. \circ \pi_\bullet$.

Recall that a theory is stable iff each type over any set (equiv., any model) is definable. Hence, (2) in *sTop* says, in a certain precise sense, that

> *a theory is stable iff its simplicial space of types over any parameter set B is contractible*

In §3.2 we give a reference saying that the same formula defines the notion of convergence in the category of simplicial objects of a category of filters.

In §4 we formulate hopefully easy problems which might be used to guide development of the simplicial reformulations in model theory.

The reader may want to skip §5 whose purpose is to formulate explicit requests for comments from readers, rather than be interesting in any way. By including such a section, paraphrasing [**RFC3**], we hope to promote the exchange and discussion of considerably less than authoritative ideas, and ease a natural hesitancy to publish something unpolished for the sole purpose of requesting comments and collaboration.

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1. Simplicial language as bookkeeping names of the variables

We demonstrate how to view parameters (rather, their complete diagrams) and types as morphisms in *sSets* or *sTop*.

Essentially, *simplicial/functoriality is a way of bookkeeping the names of variables or parameters* in a finitely consistent collection of formulas.

Preliminaries: fixing simplicial notation. — Let Δ denote the category of non-empty finite linear orders denoted by $\{1 < .. < n\}$. Let $[+1] : \Delta \longrightarrow \Delta$ be the *decalage* endomorphism adding a new least element to each finite liner order:

$$[+1] : \{1 < .. < n\} \longmapsto \{0 < 1 < ... < n\}$$

$$f : \{1 < ... < m\} \longrightarrow \{1 < ... < n\} \longmapsto f[+1](0) := 0, f[+1](l) =: l$$

We denote the finite linear order $\{1 < ... < n\}$ either by $n^\leqslant$, or by $[n-1]$, as is standard in simplicial literature. For a functor $X_\bullet : \Delta^{\mathrm{op}} \longrightarrow \mathcal{C}$ in a category $\mathcal{C}$, inclusions $\{1 < .. < n\} \subset \{0 < 1 < ... < n\}$ induce maps $X_\bullet((n+1)^\leqslant) \longrightarrow X_\bullet(n^\leqslant)$, and these form a natural transformation we denote by $\mathrm{pr}_{2,3,...} : X_\bullet \circ [+1] \longrightarrow X_\bullet$. Similarly, inclusions $\{0\} \subset \{0 < 1 < .. < n\}$ induce maps $X_\bullet((n+1)^\leqslant) \longrightarrow X_\bullet(1^\leqslant)$, and these form a natural transformation we denote by $\mathrm{pr}_1 : X_\bullet \circ [+1] \longrightarrow X(1^\leqslant)_\bullet$.

A simplicial object of a category $\mathcal{C}$ is by definition a functor $\Delta^{\mathrm{op}} \longrightarrow \mathcal{C}$. They form a category usually denoted as $s\mathcal{C}$. We shall work with categories *sTop* of simplicial topological spaces, simplicial profinite sets *sProFiniteSets*, and *sSets* of simplicial sets, and $s\boldsymbol{\Phi}$ of simplicial filters (defined in §5.2.5).

1.1. Talking simplicially about parameters and types over them. — Fix a theory T in a language L and a "monster" model $\mathfrak{C}$ of T. Recall that "monster" here means that we assume that $\mathfrak{C}$ is a model saturated and homogeneous with respect to all "small" subsets; I think these assumptions imply (mean?) that we can reconstruct $\mathfrak{C}$ using the simplicial space $\mathfrak{C}_\bullet / \operatorname{Aut}_L(\mathfrak{C}/B)$ described below. For a subset $B \subset \mathfrak{C}$, let $\mathbb{S}_n^T(B) := \mathfrak{C}^n / \operatorname{Aut}_L(\mathfrak{C}/B)$ denote the topological *Stone space of complete n-types over B*; we will often drop the superscript T. In model theory, orbits of $\operatorname{Aut}_L(\mathfrak{C}/B)$ are referred to as *types*. Recall the topology on $\mathbb{S}_n(B)$ is generated by open (and necessarily also closed) subsets $U_\phi = \{p(\bar{x}) \in S_n(B) : \phi(\bar{x}) \in p(\bar{x})\}$, where $\phi(\bar{x})$ varies though all the formulas in L with parameters in B. In other words, it is the weakest topology such that each $L(B)$-formula defines a continuous $\operatorname{Aut}_L(\mathfrak{C}/B)$-invariant function $\mathbb{S}_n(B) \longrightarrow \{0,1\}$ to the discrete two point set with the trivial action.

As a set, the Stone space $\mathbb{S}_n(B) = \mathfrak{C}^n / \operatorname{Aut}_L(\mathfrak{C}/B)$ is a quotient of $\mathfrak{C}^n$ by L-autmorphisms fixing B pointwise. We may equip $\mathfrak{C}^n$ with a topology in an obvious way so that this equality holds in the category of topological spaces.

The spaces $\mathbb{S}_n(B)$ form a functor $\mathbb{S}_\bullet(B) : \mathit{finiteSets}_{\neq\varnothing} \longrightarrow \mathit{Top}$, $\{1,2,..,n\} \longmapsto \mathbb{S}_n(B)$. It is a quotient of the functor $\mathfrak{C}_\bullet := \operatorname{Hom}_{\mathrm{Sets}}(\{1,..,n\}, \mathfrak{C}) = \mathfrak{C}^n$ by $\operatorname{Aut}(\mathfrak{C}/B)$ (however, note that here the topology on $\mathfrak{C}^n$ is not the product topology).

As $\Delta^{\mathrm{op}} \subset \mathit{finiteSets}_{\neq\varnothing}$ is a subcategory, these functors restrict to Δ^{op}, and thus can be considered as objects of *sTop* and *sSets*. In everything we say below about $\mathbb{S}_\bullet(B)$, it does not matter which category the functor is defined on.

1.1.1. Parameters as morphisms to the simplicial Stone space. — For a set A, let $|A|_\bullet := \operatorname{Hom}_{\mathrm{Sets}}(\{1,..,n\}, A) = |A|^n$ be the simplicial topological space represented by A; here we equip $|A|^n$ with the discrete topology.

Recall that *the complete diagram of a subset $A \subset M$ over parameters $B \subset M$* is the set of all formulas with parameters in B and variables indexed by elements of A, which became valid after replacing the variables by the corresponding elements of A.

To give a complete diagram of a subset $A \cup B \subset M$, $A \neq \varnothing$, of a model M of theory T is the same as to give a simplicial map in *sSets*

$$|A|_\bullet := \operatorname{Hom}_{\mathrm{Sets}}(-, A) \xrightarrow{\tau_A} \mathbb{S}_\bullet^T(B)$$

such that for $b \in A \cap B$ $\tau_A(b) = \{x_b = b\}$.

Indeed, for each n we get a map $A^n \longrightarrow S_n(B)$, i.e. we know/specify the complete type of each tuple in A over B. Functoriality ensures that these types are coherent, i.e. $\operatorname{tp}(ab/B)$ does extend $\operatorname{tp}(a/B)$ and $\operatorname{tp}(b/B)$.

Thus, *simplicial/functoriality is a way to keep bookkeeping (track) of the names of variables or parameters* in a type.

1.1.2. Types as morphisms to the shifted (decalage) simplicial Stone space. — To give a complete 1-type $p(x/AB)$ over a subset $A \cup B \subset M$, $A \neq \varnothing$, is the same as to give a lifting in *sSets*

$$\begin{array}{ccc} & & \mathbb{S}_\bullet(B) \circ [+1] \\ & \overset{p(x/AB)}{\nearrow} & \downarrow{\scriptstyle pr_{2,3,..}} \\ |A|_\bullet := \operatorname{Hom}_{sets}(-, A) & \xrightarrow{\tau_A} & \mathbb{S}_\bullet(B) \end{array}$$

Indeed, for each n we get a map $A^n \longrightarrow \mathbb{S}_{1+n}(B)$, i.e. we know/specify a type $p(-,\bar{a}/B)$ for each finite tuple $\bar{a} \subset A$.

To give a complete N-type $p(\bar{x}/AB)$ over a subset $A \cup B \subset M$, $A \neq \varnothing$, is the same as to give a simplicial map in *sSets*

$$\begin{array}{ccc} & & \mathbb{S}_\bullet(B)\circ[+N] \\ & \overset{p(\bar{x}/AB)}{\nearrow} & \big\downarrow pr_{N+1,N+2,,..} \\ |A|_\bullet := \mathrm{Hom}_{sets}(-,A) & \xrightarrow{\ \tau_A\ } & \mathbb{S}_\bullet(B) \end{array}$$

Indeed, for each n we get a map $A^n \longrightarrow S_{N+n}(T)$, i.e. we know/specify a type $p(-,-,..,-,\bar{a}/B)$ for each finite tuple $\bar{a} \subset A$.

Thus again we see that *simplicial/functoriality is a way to keep bookkeeping (track) of the names of variables...*

1.1.3. Recovering $\mathbb{S}_\bullet(A)$ *from* $\mathbb{S}_\bullet(\varnothing)$. — The diagrams above show that the simplicial space of types over the empty set contains the information of the simplicial space of types over arbitrary parameters. Namely, as a set,

$$\mathbb{S}_n(A) = \{\, p : |A|_\bullet \longrightarrow \mathbb{S}_\bullet(\varnothing)\circ[+n] \;:\; \tau_A = \mathrm{pr}_{n+1,n+2,...}\circ p \,\}$$

The topology is generated by the subsets corresponding to open subsets of the spaces of maps $|A|^m \longrightarrow \mathbb{S}_{n+m}(\varnothing)$ with the open-compact topology. This construction is somewhat reminiscent of the internal hom in *sSets*, and I would be thankful for a reference to it in literature, or perhaps to the Skorokhod space of maps used to define geometric realisation in $s\boldsymbol{\varphi}$ [**GP**].

1.1.4. Spaces of types over a model are fibrant ?— When is a set of parameters a model ? It is tempting to think that such a "always contains a witness" condition means $\mathbb{S}_\bullet(A)$ being fibrant in some sense. Can we recover the space of types from the space of types *realised* in a model, say in a well-understood model such as a standard model of the infinite cyclic group $(\mathbb{Z},+)$? Doing so seems to involve completition or compactification, which also makes it tempting to relate this to fibrant replacement. We should explicitly add that we see no technical reason for saying this, though.

1.2. Invariant and definable types. —

1.2.1. Invariant types. — [**Tent-Ziegler**, Exercise 8.3.3] says, as we explain below in §2.2, that a global type $p(\bar{x}/\mathfrak{C})$ *invariant over* B is the same as the following lifting diagram in *sSets*:

$$\begin{array}{ccc} & & \mathbb{S}_\bullet(B)\circ[+N] \\ & \nearrow & \big\downarrow pr_{N+1,N+2,,..} \\ \mathbb{S}_\bullet(B) & \xrightarrow{\ \mathrm{id}\ } & \mathbb{S}_\bullet(B) \end{array}$$

To see this, consider the diagram[4] in *sSets*

$$\begin{array}{ccc} |\mathfrak{C}|_\bullet := \mathrm{Hom}_{sets}(-,\mathfrak{C}) & \xrightarrow{p(\bar{x}/AB)} & \mathbb{S}_\bullet(B)\circ[+N] \\ \downarrow{\scriptstyle AB\subset\mathfrak{C}} & \nearrow & \downarrow{\scriptstyle pr_{N+1,N+2,..}} \\ \mathbb{S}_\bullet(B) & \xrightarrow{\mathrm{id}} & \mathbb{S}_\bullet(B) \end{array} \tag{3}$$

The model $\mathfrak{C}$ being saturated over B means precisely that the map on the right is surjective at each level (i.e. for each n the map of n-simplicies is surjective). Hence, there is at most one lifting map, and we only need to check it is well-defined. The lifting map is well-defined iff for a tuple $\bar{c}\subset\mathfrak{C}$, whether $\varphi(\bar{x},\bar{c})\in p(x/\mathfrak{C})$ depends only on the type $\mathrm{tp}(\bar{c}/B)$ of the parameters over B. This is precisely the definition of invariance over B.

It follows that a type $p(\bar{x}/AB)$ extends to a global type invariant over B iff there is a lifting diagram in *sSets*:

$$\begin{array}{ccc} |A|_\bullet := \mathrm{Hom}_{sets}(-,A) & \xrightarrow{p(\bar{x}/AB)} & \mathbb{S}_\bullet(B)\circ[+N] \\ \downarrow{\scriptstyle AB\subset\mathfrak{C}} & \nearrow & \downarrow{\scriptstyle pr_{N+1,N+2,..}} \\ \mathbb{S}_\bullet(B) & \xrightarrow{\mathrm{id}} & \mathbb{S}_\bullet(B) \end{array}$$

1.2.2. Definable types. — Further, [**Tent-Ziegler**, Exercise 8.3.3] says, as we explain below, that a global type $p(x/\mathfrak{C})$ *definable over* B is the diagonal map above is continuous, i.e. it is the same as the following lifting diagram in *sTop* or, equiv., in *sProFiniteSets*:

$$\begin{array}{ccc} & & \mathbb{S}_\bullet(B)\circ[+N] \\ & \nearrow & \downarrow{\scriptstyle pr_{N+1,N+2,..}} \\ \mathbb{S}_\bullet(B) & \xrightarrow{\mathrm{id}} & \mathbb{S}_\bullet(B) \end{array}$$

To see this, consider the diagram (3) in *sTop*. Continuity of the diagonal arrow says that for each formula $\varphi(\bar{x},\bar{c})$ there is a formula $\mathrm{d}_p\varphi(\bar{x},\bar{c})$ such that for each tuple $\bar{c}\subset\mathfrak{C}$ it holds $\varphi(\bar{x},\bar{c})\in p(\bar{x}/\mathfrak{C})$ iff $\mathrm{d}_p\varphi(\bar{x},\bar{c})\in\mathrm{tp}(\bar{c}/B)$. This means precisely that $\mathrm{d}_p\varphi(\bar{x},\bar{a})$ is a φ-definition of p over B.

It follows that a type $p(\bar{x}/AB)$ extends to a global type definable over B iff there is a lifting diagram in *sTop* or, equiv. its full subcategory *sProFiniteSets* of compact Hausdorff totally disconnected spaces

$$\begin{array}{ccc} |A|_\bullet := \mathrm{Hom}_{sets}(-,A) & \xrightarrow{p(\bar{x}/AB)} & \mathbb{S}_\bullet(B)\circ[+N] \\ \downarrow{\scriptstyle AB\subset\mathfrak{C}} & \nearrow & \downarrow{\scriptstyle pr_{N+1,N+2,..}} \\ \mathbb{S}_\bullet(B) & \xrightarrow{\mathrm{id}} & \mathbb{S}_\bullet(B) \end{array}$$

[4] A simplicially minded reader may consider this diagram as a definition of a global type invariant over B.

Because of its importance we rewrite the diagram expanding the notation for the type(=orbit) space:

$$\begin{array}{ccc}
|A|_\bullet := \mathrm{Hom}_{sets}(-,A) & \xrightarrow{\ p(\bar{x}/AB)\ } & \mathfrak{C}_\bullet/\mathrm{Aut}(\mathfrak{C},B)\circ[+N] \\
\downarrow{\scriptstyle AB\subset\mathfrak{C}} & \nearrow\text{(dashed)} & \downarrow{\scriptstyle pr_{N+1,N+2,..}} \\
\mathfrak{C}_\bullet/\mathrm{Aut}(\mathfrak{C},B) & \xrightarrow{\ \mathrm{id}\ } & \mathfrak{C}_\bullet/\mathrm{Aut}(\mathfrak{C},B)
\end{array}$$

1.2.3. Definably compact ?— [**Hrushovski-Loeser**, Def. 4.1.2] defines a *pro-definable topological space* to be *definablly compact* as "each definable type has a limit". Both notions of a limit and of a definable type are defined by simplicial diagrams of the same shape, although in different (but similar) categories, see §3.2.

It also seems that a *(pro-?)definable topological space* is something a quotient of the usual simplicial type space $\mathbb{S}.(M)$ by the relation of two types being infinitesimally close to each other defined by the topology [**Peterzil-Starchenko**, Abstract]. Precisely, [**Peterzil-Starchenko**, App.A] consider a quotient $S^{\mu}_{X}(M)$ by an type definable equivalence relation, i.e. with respect to a uniform strucuture defining the notion of "infinitesimally close to each other". It appears that the definition of a definable type applied to such a quotient defines convergent types, and then "definably compact" means that each continuous section (=definable type) $\mathbb{S}_\bullet(M) \longrightarrow \mathbb{S}_\bullet(M)\circ[+1]$ survives after taking the quotient. The analogy of this to topology is not yet clear.

For background, recall that a type definable equivalence relation is a uniform structure. Recall that a uniform structure is the same a 1-dimensional symmetric object in the category $s\boldsymbol{\varphi}$ of simplicial filters [**G**]. One wonders if a type definable equivalence relation is the same as (something like) a uniform structure whose underlying simplicial set is the space of types.

Hence, one might hope to be able to define simply and simplicially definably compact. Reformulating stable domination is less clear.

1.3. Various remarks. — We make a couple of remarks.

1.3.1. A notion of definable or invariant type "better" for homotopy theory. — A standard advice to improve the notion of type by a homotopy theorist, is to replace the quotient $\mathfrak{C}/\mathrm{Aut}(\mathfrak{C}/B))$ by a "better" and "standard" quotient which remembers more, the classifying space $\mathbb{B}(G,X)$ of a group $G := \mathrm{Aut}(\mathfrak{C}/B)$ acting on a space X.

I have not yet tried to interpret it, and there are immediate technical difficulties with the way I explain it below.

Remark 1.3.1 **($\mathbb{B}_\bullet(G,X)$ instead of $\mathbb{S}_\bullet(B)$).** — For a group G acting on a set X, there is an obvious canonical map

$$X\times G\times\ldots\times G \longrightarrow X\times X\times\ldots\times X$$

$$(x,g_1,\ldots,g_n)\longmapsto(x,xg_1^{-1},\ldots,xg_n^{-1})$$

invariant under the diagonal action

$$(x,g_1,\ldots,g_n)\longmapsto(xg,g_1g,\ldots,g_ng).$$

Recall[(5)] that the simplicial *Borel construction* $\mathbb{B}_\bullet(G,X) = X \times_G \mathbb{E}_\bullet(G)$ *of a group action* $\rho : G \times X \longrightarrow X$ is defined (explicitly given) by

$$\mathbb{B}_\bullet(G,X)(n^{\leqslant}) := (X \times G^n)/G$$

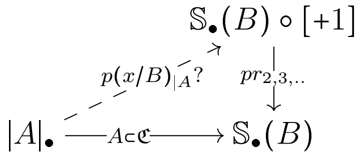
and this space is viewed as a "better", "right" quotient of X by G.

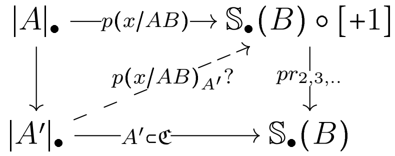
The above gives rise to a simplicial map $\mathbb{B}_\bullet(\mathfrak{C}, \mathrm{Aut}(\mathfrak{C}/B)) \longrightarrow \mathbb{S}_\bullet(B) \circ [+1]$ from the classifying space $\mathbb{B}_\bullet(\mathfrak{C}, \mathrm{Aut}(\mathfrak{C}/B))$. Note, however, that its image contains only tuples with all elements realising the same type.

Thus, a standard advice of a homotopy theorist would be to replace $\mathbb{S}_\bullet(B)$ in the diagrams above by something related to $\mathbb{B}(\mathfrak{C}, \mathrm{Aut}(\mathfrak{C}/B))$. In fact, perhaps it might be necessary to consider $\mathbb{B}_\bullet(\mathfrak{C}_\bullet, \mathrm{Aut}(\mathfrak{C}/B)_\bullet)$ for the simplicial group $\mathrm{Aut}(\mathfrak{C}/B)_\bullet$ where $\mathrm{Aut}(\mathfrak{C}/B)_\bullet(n^{\leqslant}) := \mathrm{Aut}(\mathfrak{C}^n/B^n)$...

Does this advice make any sense ? I have not yet thought about it, and there are immediate technical difficulties ...

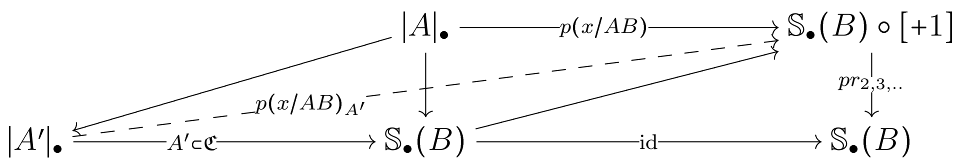
1.3.2. Extending the parameter set of a type. — A typical tool/problem in model theory is to extend/define *freely* a type to a larger parameter set. This corresponds to finding/defining a canonical way to define liftings





$p(x/B)$ to $p(x/B)_{|A}$ $\qquad$ $p(x/AB)$ to $p(x/AB)_{|A'}$

Note that if the type $p(x/AB)$ extends to a global B-invariant type, extending $p(x/AB)$ to $p(x/A'AB)$ is provided by taking the composition $|A'|_\bullet \longrightarrow \mathbb{S}_\bullet(B) \longrightarrow \mathbb{S}_\bullet(B) \circ [+1]$.



$p(x/AB)$ to $p(x/AB)_{|A'}$

1.4. Product of invariant types, and Morley sequences. —

(5) The following slightly paraphrased quote from [nlab,Borel construction] helps intuition: For X a topological space, G a topological group and $\rho : G \times X \longrightarrow X$ a continuous G-action (i.e. a topological G-space), the Borel construction of ρ is the topological space $X \times_G \mathbb{E}G$, hence quotient of the product of X with the total space of the G-universal principal bundle $\mathbb{E}(G)$ by the diagonal action of G on both.
Analogously, for $G_\bullet : \Delta^{\mathrm{op}} \longrightarrow$ Groups a simplicial group, $X_\bullet : \Delta^{\mathrm{op}} \longrightarrow$ Sets a simplicial set, and $G_\bullet \times X_\bullet \longrightarrow X_\bullet$ a simplicial group action, its Borel construction is the quotient $(X_\bullet \times \mathbb{E}_\bullet(G))/G_\bullet$ in *sSets* of the Cartesian product of $X_\bullet$ with the universal principal simplicial complex $\mathbb{E}G_\bullet$ by the diagonal action of $G_\bullet$ on these.

1.4.1. Product of invariant types. — Note that liftings $\pi^p_\bullet, \pi^q_\bullet : \mathbb{S}_\bullet(B) \longrightarrow \mathbb{S}_\bullet(B)\circ[+1]$ can be composed as

$$\mathbb{S}_\bullet(B) \xrightarrow{\pi^q_\bullet} \mathbb{S}_\bullet(B)\circ[+1] \xrightarrow{\pi^p_\bullet[+1]} \mathbb{S}_\bullet(B)\circ[+2]$$

This construction corresponds to the product $p(x)\otimes q(y)$ of invariant types [**Simon**, 2.2.1]. The product of types is transitive, i.e. $p(x)\otimes(q(y)\otimes s(z)) = (p(x)\otimes q(y))\otimes s(z)$: this corresponds to

$$\pi^s_\bullet\circ(\pi^q_\bullet\circ\pi^p[+1])[+1] = \pi^s_\bullet\circ\pi^q_\bullet[+1]\circ\pi^p[+2] = (\pi^s_\bullet\circ\pi^q_\bullet[+1])\circ\pi^p[+2]$$

$$\mathbb{S}_\bullet(B) \xrightarrow{\pi^s_\bullet} \mathbb{S}_\bullet(B)\circ[+1] \xrightarrow{\pi^p_\bullet[+1]} \mathbb{S}_\bullet(B)\circ[+2] \xrightarrow{\pi^p_\bullet[+2]} \mathbb{S}_\bullet(B)\circ[+3]$$

Recall [**Simon**, 2.2.1] that *product $p(x)\otimes p(y)$ of two B-invariant global types $p(x/\mathfrak{C}), q(y/\mathfrak{C}) \in S(\mathfrak{C})$* can be defined by the following property.

Given a formula $\varphi(x;y)\in L(C)$, where $B\subset C\subset\mathfrak{C}$, it holds $\varphi(x,y)\in p(x)\otimes q(y)$ iff $\varphi(x;c)\in p$ for some (equiv., any) $c\in\mathfrak{C}$ with $c\vDash q_{|C}$.

1.4.2. Morley sequence. — The n-type of a *Morley sequence of an invariant type* $p(x)$ is given by $p^{\otimes n}(x) := p(x_1)\otimes\ldots\otimes p(x_n)$, for $n>0$. Thus a Morley sequence corresponds to taking the self-composition

$$\mathbb{S}_\bullet(B) \xrightarrow{\pi^p_\bullet} \mathbb{S}_\bullet(B)\circ[+1] \xrightarrow{\pi^p_\bullet[+1]} \mathbb{S}_\bullet(B)\circ[+2] \longrightarrow \ldots \xrightarrow{\pi^p_\bullet[+n-1]} \mathbb{S}_\bullet(B)\circ[+n] \longrightarrow \ldots$$

Somewhat more precisely, *a Morley sequence of of a global B-invariant type $p(x/\mathfrak{C})$ over C* is the restriction to C of the following sequence of global B-invariant types:

$$p\in\mathbb{S}_1(\mathfrak{C}),\ \pi^p_\bullet(p)\in\mathbb{S}_2(\mathfrak{C}),\ \pi^p_\bullet[+1]\circ\pi^p_\bullet(p)\in\mathbb{S}_3(\mathfrak{C})\ ,\ldots,\ \pi^p_\bullet[n]\circ\ldots\circ\pi^p_\bullet(p)\in\mathbb{S}_{n+1}(\mathfrak{C}),\ldots$$

Indiscernability of a Morley sequence follows from associativity: we have that for each map $[i_1<\ldots<i_m] : m^{\leqslant}\longrightarrow n^{\leqslant}$ we have

$$\pi^p_\bullet[n]\circ\ldots\circ\pi^p_\bullet(p)[i_1<\ldots<i_m] = \pi^p_\bullet[m]\circ\ldots\circ\pi^p_\bullet(p)$$

The reader may wish to compare this with a model theoretic exposition [**Simon**, 2.2.1]:

If $p(x)$ is an A-invariant type, we define by induction on $n\in\mathbb{N}^*$:

$$p^{(1)}(x_0) = p(x_0) \quad\text{and}\quad p^{(n+1)}(x_0,\dots,x_n) = p(x_n)\otimes p^{(n)}(x_0,\dots,x_{n-1}).$$

Let also $p^{(\omega)}(x_0,x_1,\dots) = \bigcup p^{(n)}$. For any $B\supseteq A$, a realization $(a_i : i<\omega)$ of $p^{(\omega)}|_B$ is called a *Morley sequence* of p over B (indexed by ω). It follows from associativity of $\otimes$ that such a sequence $(a_i : i<\omega)$ is indiscernible over B (indeed for any $i_1<\cdots<i_n\in\omega$, we have $\operatorname{tp}(a_{i_1},\dots,a_{i_n}/B) = p^{(n)}|_B$).

More generally, any sequence indiscernible over B whose EM-type over B is given by $\{p^{(n)}|_B : 1\le n<\omega\}$ is called a Morley sequence of p over B.

1.4.3. Generically stable types. — A permutation $\sigma : N\longrightarrow N$ acts on $\mathbb{S}_\bullet[+N]$ by permuting variables, $p(x_1,..,x_n,y_1,..,y_m)\longmapsto p(x_{\sigma(1)},...,x_{\sigma(n)},y_1,..,y_n)$. A type is *generically stable* iff $p(x)\otimes p(y) = p(y)\otimes p(x)$, i.e. iff $\pi^p_\bullet[+1]\circ\pi^p_\bullet$ commutes with the permutation $\sigma : \mathbb{S}_\bullet(B)\circ[+2]\longrightarrow\mathbb{S}_\bullet(B)\circ[+2]$ permuting the two variables [**Simon**, 2.2.2,Theorem 2.29]. In fact, if $p(x)$ is generically stable, this holds for any lifting $\pi^q_\bullet$ [**Simon**, 2.2.2,Proposition 2.33].

1.4.4. Product of types in a stable theory. — Recall that for a definable type $p(-/B)$ [**Tent-Ziegler**, Def. 8.1.4] and any L-formula $\phi(\bar{x}, \bar{y})$ with parameters in B [**Tent-Ziegler**, Def. 8.1.4] defines the formula $d_p\,\bar{x}\phi(\bar{x}, \bar{b})$ by

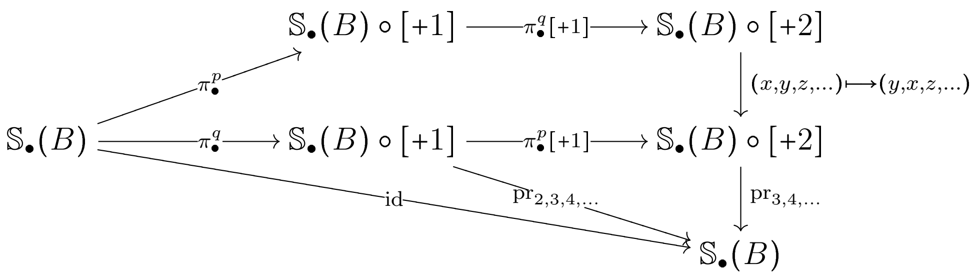
$$\varphi(\overline{x}, \overline{b}) \in p \text{ if and only if } \models \mathrm{d}_p\,\overline{x}\varphi(\overline{x}, \overline{b}).$$

In [**Tent-Ziegler**], the fact that in a stable theory $p(x) \otimes p(y) = p(y) \otimes p(x)$ is expressed as

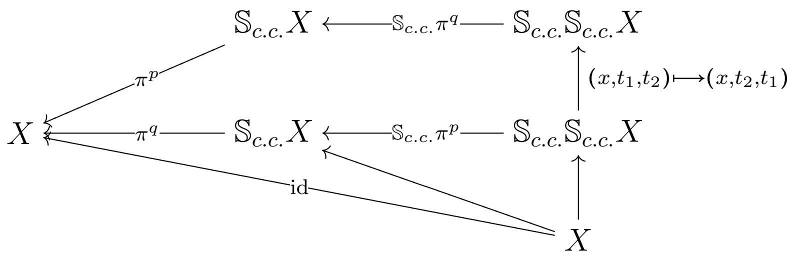
LEMMA 8.3.4 (Harrington). *Let T be stable and let $p(x)$ and $q(y)$ be global types. Then for every formula $\varphi(x, y)$ with parameters*

$$\mathrm{d}_p\, x\varphi(x, y) \in q(y) \;\Leftrightarrow\; \mathrm{d}_q\, y\varphi(x, y) \in p(x).$$

In simplicial notation, this is represented by the following diagram:



This corresponds to the following topological picture involving suspension (or cone):



1.4.5. Definability patterns (speculation). — It is tempting to view definability patterns as some kind of analogue of structure (e.g., compact-open topology) on possible liftings $\mathbb{S}_\bullet(M) \longrightarrow \mathbb{S}_\bullet(M) \circ [+1]$, i.e., by analogy, on the set(space...) of homotopies $\mathrm{Hom}(\mathbb{S}X, X)$ which are identity on X.... I cannot say more at this stage.

1.5. Interpretations. —

1.5.1. Reducts as morphisms. — A sublanguage $L_0 \subset L$ and a subset $B_0 \subset B$ of parameters defines the obvious forgetful morphism $\mathbb{S}^T_\bullet(B) \longrightarrow \mathbb{S}^{T(L_0)}_\bullet(B_0)$ remembering only the $L_0(B_0)$-formulas of the types.

1.5.2. Contractibility and Shelah's expansion by externally definable sets. — This map being contractible (i.e. fitting into diagram (4)) would mean that each $T(L_0)$-type over B is definable in $L(B)$. Roughly, for $B = M$ a model, this means that T

contains the Shelah expansion of $T_0 \coloneqq T(L_0)$.

$$
(4)\qquad
\begin{array}{ccc}
 & & \mathbb{S}_\bullet^{T(L_0)}(M) \circ [+1] \\
 & \overset{\pi_\bullet}{\nearrow} & \big\downarrow {\scriptstyle \mathrm{pr}_1 \times \mathrm{pr}_{2,3,..}} \\
\mathrm{const}_\bullet \mathbb{S}_1^T(M) \times \mathbb{S}_\bullet^T(M) & \xrightarrow{\mathrm{id}} & \mathrm{const}_\bullet \mathbb{S}_1^{T(L_0)}(M) \times \mathbb{S}_\bullet^{T(L_0)}(M)
\end{array}
$$

If $T_0 = T(L_0)$ is unstable, and we take B_0 and B to be large enough of the same cardinality, this has to fail: there are more $T(L_0)$-types over B_0 than T-formulas over B.

1.5.3. Proving non-intepretability using homotopy theory?— Hence, if methods of homotopy theory were able to prove that each map between certain simplicial (not contractible) type spaces is contractible, then we perhaps were able to prove a non-interpretability result in model theory....

1.5.4. Characterising interpretations simplicially?— I have nothing to say. [**Morley**, Theorem 3.1] characterises simplicial type spaces arising from $L_{\omega_1\omega}$-theories (without explicitly using the words "functor" or "category"). See also [**Levon**, §3] and [**Kamsma**] for a modern exposition of type space functors in a different context, especially [**Levon**, §3(The type space functor and interpretations of theories)] and [**Kamsma**, Defs. 4.19-20] which I have not yet read.

1.6. Shelah's representability. — The meaning of a morphism between two generalised Stone spaces is reminiscent of the notion of one structure *representing* another introduced by Shelah [CoSh:919] (we quote [Sh:1043]) to 'try to formalise the intuition that "the class of models of a stable first order theory is not much more complicated than the class of models $M = (A, \ldots, E_t, \ldots)_{s \in I}$ where E_t^M is an equivalence relation on A refining E_s^M for $s < t$; and I is a linear order of cardinality $\leq |T|$".' In [**Z1**, §3.2.4] we reformulate a corollary of a characterisation of stable theories in [CoSh:919] and give a more literal formalisation of this intuition: a theory is stable iff there is κ such that for each model of the theory there is a surjective morphism to its generalised Stone space from a structure whose language consists of at most κ equivalence relations and unary predicates (and nothing else). Based on this reformulation we suggest a conjecture with a category-theoretic characterisation of classes of models of stable theories.

It will be interesting to compare this to [Boney, Erdos-Rado Classes, Thm 6.8].

1.7. Stability as a lifting property. — In [**Z1**, §3.3.2] we observe that stability can defined by a lifting property. Let us very briefly sketch this observation adapted to our current context. Let $I \coloneqq \mathbb{Q}$ be a countable dense linear order, and let $I_\bullet^{\leqslant}$ and $|I|_\bullet$ be the simplicial sets represented by the linear order, resp. the set of its elements:

$$I_\bullet^{\leqslant} : n \longmapsto \mathrm{Hom}_{preorders}(n^{\leqslant}, I)$$

$$|I|_\bullet : n \longmapsto \mathrm{Hom}_{Sets}(n, I) = I^n$$

Recall that a theory is stable iff any infinite indiscernible sequence of n-tuples is necessarily an indiscernible set, for each $n > 0$. This definition is captured for $n = 1$

in *sSets* by the following diagram (where we only consider horizontal arrows whose image has unbounded dimension):

$$\begin{array}{ccc} I^{\leqslant}_{\bullet}/\mathrm{Aut}(I^{\leqslant}) & \xrightarrow{\text{unbounded dimension}} & \mathbb{S}_{\bullet}(\varnothing) \\ \downarrow & \nearrow & \\ |I|_{\bullet}/\mathrm{Aut}(I) & & \end{array}$$

One way to turn this diagram into an actual lifting property is to consider it in the category $s\mathfrak{P}$ of simplicial filters with continuous maps defined almost everywhere (see Definition 5.2.5), equip I with the filter of cofinite subsets, and equip each $I^{\leqslant}_{\bullet}(n)$ and $|I|_{\bullet}(n)$ with the coursest filter such that all the simplicial maps induced by $1^{\leqslant} \longrightarrow n^{\leqslant}$ are continuous.

2. Transcribing Exercise 8.3.3 as simplicial diagram chasing

We transcribe the Exercise 8.3.3 into the simplicial language. This section is self-contained and may be read first.

2.1. Exercise 8.3.3 in model theoretic language. — We start by quoting in full the Exercise 8.3.3, its solution, and the (only) required definition of a definable type. Fix a theory T and a monster model $\mathfrak{C}$ of T. For a subset $B \subset \mathfrak{C}$, let $S_n(B) := \mathfrak{C}^n/\,\mathrm{Aut}_L(\mathfrak{C}/B)$ denote the space of complete n-types over B. Recall the topology on $\mathbb{S}_n(B)$ is generated by open sets $U_\phi = \{p(\bar{x}) \in S_n(B) : \phi(\bar{x}) \in p(\bar{x})\}$ where ϕ varies though arbitrary formulas with parameters in B.

Definition 8.1.4. A type $p(\overline{x}) \in \mathrm{S}_n(B)$ is *definable* over C if the following holds: for any L-formula $\varphi(\overline{x}, \overline{y})$ there is an $L(C)$-formula $\psi(\overline{y})$ such that for all $\overline{b} \in B$

$$\varphi(\overline{x}, \overline{b}) \in p \text{ if and only if } \models \psi(\overline{b}).$$

We say p is *definable* if it is definable over its domain B.

We write $\psi(\overline{y})$ as $\mathrm{d}_p\, \overline{x}\varphi(\overline{x}, \overline{y})$ to indicate the dependence on p, $\varphi(\overline{x}, \overline{y})$ and the choice of the variable tuple $\overline{x}$. (So d_p has the syntax of a generalised quantifier, see [59].) Thus, we have

$$\varphi(\overline{x}, \overline{b}) \in p \text{ if and only if } \models \mathrm{d}_p\, \overline{x}\varphi(\overline{x}, \overline{b}).$$

Note that $\mathrm{d}_p\, \overline{x}\varphi(\overline{x}, \overline{y})$ is also meaningful for formulas φ with parameters in B.

Exercise 8.3.3. Let $p(x) \in \mathrm{S}(\mathfrak{C})$ be definable over B. Then, for any n, the map

$$r(y_1, \dots, y_n) \mapsto \{\varphi(x, y_1, \dots, y_n) \mid \mathrm{d}_p\, x\varphi \in r\}$$

defines a continuous section $\pi_n \colon \mathrm{S}_n(B) \to \mathrm{S}_{n+1}(B)$. Show that this defines a bijection between all types definable over B and all "coherent" families (π_n) of continuous sections $\mathrm{S}_n(B) \to \mathrm{S}_{n+1}(B)$.

Exercise 8.3.3. Let $\pi_n : \mathrm{S}_n(B) \to \mathrm{S}_{n+1}(B)$ be a continuous section. For any n-tuple c if π_n maps $\mathrm{tp}(c/B)$ to $p(x, y)$, then $p_c = p(x, c)$ is a type over cB. Continuity implies that for every $\varphi(x, y)$ there is a B–formula $\psi(y)$ such that for all c we have $\varphi(x, c) \in p_c \;\Leftrightarrow \models \psi(c)$. The following coherence condition ensures that $p = \bigcup_{c \in \mathfrak{C}} p_c$ is consistent. For any map $s : \{1, \dots, m\} \to \{1, \dots, n\}$ let $s^\# : \mathrm{S}_n(B) \to \mathrm{S}_m(B)$ and $s^* : \mathrm{S}_{n+1}(B) \to \mathrm{S}_{m+1}(B)$ be the associated natural restriction maps. Then coherence means $\pi_m \circ s^\# = s^* \circ \pi_n$.

2.2. Exercise 8.3.3 in simplicial language. — Let Δ denote the category of non-empty finite linear orders denoted by $\{1 < .. < n\}$. Let $[+1] : \Delta \longrightarrow \Delta$ be the *decalage* endomorphism adding a new least element to each finite liner order:

$$[+1] : \{1 < .. < n\} \longmapsto \{0 < 1 < ... < n\}$$

$$f : \{1 < ... < m\} \longrightarrow \{1 < ... < n\} \longmapsto f[+1](0) := 0, f[+1](l) =: l$$

We denote the finite linear order $\{1 < ... < n\}$ either by $n^{\leqslant}$, or by $[n-1]$, as is standard in simplicial literature.

Let $\mathbb{S}_\bullet(B) : \Delta^{op} \longrightarrow Top$ be the functor which sends each finite linear order $\{1 < ... < n\}$ to $S_n(B) = \mathfrak{C}/\operatorname{Aut}_L(\mathfrak{C}/B)$. Recall that functors $\Delta^{op} \longrightarrow Top$ are called *simplicial objects of the category Top*, or sometimes *simplicial topological spaces.* In fact, the functor $\mathbb{S}_\bullet(B) : \Delta^{op} \longrightarrow Top$ factors though the embedding of the category Δ^{op} into the opposite of the category $finiteSets_{\neq\varnothing}$ of non-empty finite sets, and let $\tilde{\mathbb{S}}_\bullet(B) : finiteSets_{\neq\varnothing} \longrightarrow Top$ be the corresponding functor.

A "coherent" family of sections $\pi_n : S_n(B) \longrightarrow S_{n+1}(B)$ defines a natural transformation $\tilde{\mathbb{S}}_\bullet(B) \longrightarrow \tilde{\mathbb{S}}_\bullet(B) \circ [+1]$. Indeed, the coherence condition $\pi_m \circ s^\# = s^* \circ \pi_n$ is precisely the defining property of a natural transformation $\tilde{\mathbb{S}}_\bullet(B) \longrightarrow \tilde{\mathbb{S}}_\bullet(B) \circ [+1]$. In fact, because of symmetry it is equivalent to require the coherence conditions only for non-decreasing maps $s : \{1, ..., m\} \longrightarrow \{1, ..., n\}$: for any permutation $\sigma : \{1, ..., m\} \longrightarrow \{1, ..., m\}$, the equality $\pi_m \circ s^\#(q(y_1, ..., y_n) = s^* \circ \pi_n(q(y_1, ..., y_n)$ is equivalent to $\pi_m \circ s^\#(q(y_{\sigma(1)}, ..., y_{\sigma(n)}) = s^* \circ \pi_n(q(y_{\sigma(1)}, ..., y_{\sigma(n)})$. Hence, it is equivalent to say that a "coherent" family of sections $\pi_n : S_n(B) \longrightarrow S_{n+1}(B)$ defines a natural transformation $\mathbb{S}_\bullet(B) \longrightarrow \mathbb{S}_\bullet(B) \circ [+1]$.

These coherence conditions is equivalent to the consistency of the global type

$$p(x/\mathfrak{C}) = \bigcup_{c \in \mathfrak{C}^n, n>0} p_c = \{\varphi(x, \bar{c}) : \phi(x, \bar{y}) \in \pi_n(\mathrm{tp}(\bar{c}/B))\}$$

Indeed, consider a finite collection $\varphi_i(x, \bar{c}_i) \in p(x/\mathfrak{C})$, $i < n$ of formulas. Consider the type $\pi_N\, \mathrm{tp}(\bar{c}_1, ..., \bar{c}_n/B)$ of the joint N-tuple $(\bar{c}_1, ..., \bar{c}_n)$. For an appropriate map $s : \mathrm{length}(\bar{c}_i)^\leqslant \longrightarrow N^\leqslant$, the coherence conditions (=funtoriality) implies that $\pi_{\mathrm{length}(\bar{c}_i)}(\bar{c}_i) = \pi(s)(\pi_N(\bar{c}_1, ..., \bar{c}_n))$, hence $\varphi_i(x, \bar{c}_i) \in \pi_N(\bar{c}_1, ..., \bar{c}_n)$, which is consistent.

By construction, the global type $p(x/\mathfrak{C})$ is B-invariant.

Hence, there is a bijection between global B-invariant types and "coherent" families of (possibly discontinuous) sections (π_n), or, equivalently, (possibly discontinuous) sections $\mathbb{S}_\bullet(B) \longrightarrow \mathbb{S}_\bullet(B) \circ [+1]$ of simplicial sets.

Therefore, we can reformulate Exercise 8.3.3 as follows:

Exercise 8.3.3. Let $p(x) \in S(\mathfrak{C})$ be definable over B. Then, for any n, the map

$$r(y_1, ..., y_n) \longmapsto \{\phi(x, y_1, ..., y_n) | d_p x\phi \in r\}$$

or, equivalently,

$$r \longmapsto p \otimes r$$

defines a continuous section $\pi_n : S_n(B) \longrightarrow S_{n+1}(B)$. Show that this defines a bijection between

- all types definable over B
- all "coherent" families (π_n) of continuous sections $S_n(B) \longrightarrow S_{n+1}(B)$.
- lifting arrows in the diagram of simplicial topological spaces

(5)
$$\begin{array}{ccc} & & \mathbb{S}_\bullet(B) \circ [+1] \\ & \overset{\pi_\bullet}{\nearrow} & \downarrow {\scriptstyle \mathrm{pr}_{2,3,..}} \\ \mathbb{S}_\bullet(B) & \xrightarrow{\mathrm{id}} & \mathbb{S}_\bullet(B) \end{array} \qquad \begin{array}{ccc} & & \mathfrak{C}_\bullet/\mathrm{Aut}(\mathfrak{C}/B) \circ [+1] \\ & \overset{\pi_\bullet}{\nearrow} & \downarrow {\scriptstyle \mathrm{pr}_{2,3,..}} \\ \mathfrak{C}_\bullet/\mathrm{Aut}(\mathfrak{C}/B) & \xrightarrow{\mathrm{id}} & \mathfrak{C}_\bullet/\mathrm{Aut}(\mathfrak{C}/B) \end{array}$$

Moreover, show that this defines a bijection between

- all global types invariant over B
- all "coherent" families (π_n) of possibly discontinuous sections $S_n(B) \longrightarrow S_{n+1}(B)$.
- lifting arrows in the diagram of simplicial sets

(6)
$$\begin{array}{ccc} & & \mathbb{S}_\bullet(B) \circ [+1] \\ & \overset{\pi_\bullet}{\nearrow} & \downarrow {\scriptstyle \mathrm{pr}_{2,3,..}} \\ \mathbb{S}_\bullet(B) & \xrightarrow{\mathrm{id}} & \mathbb{S}_\bullet(B) \end{array}$$

Note that by [**Simon**, 2.2,Example 2.17] any finitely consistent global type over B is necessarily B-invariant. As every type over a model is necessarily finitely consistent, it implies that each global type over a model gives rise to a diagram (6) in *sSets*. Hence, a notion of continuity is essential to be able to define definability.

3. Background for a homotopy theoretic interpretation of Exercise 8.3.3

It is well-known that the simplicial formula (5) defines homotopy triviality in *sSets*, as was pointed to us by V.Sosnilo. Note that the diagram (5) can be written in the category of simplicial objects of an arbitrary category.

Unfortunately, we understand very little about this formula, and request comments on it from our homotopy theory readers.

Below in §3.1 we explain that the formulas (5) and (2) defines the usual notion of a map being contractible in the category of topological spaces when applied to the singular complexes of sufficiently nice topological spaces.

In §3.2 we say that the same formula defines the notion of convergence in the category of simplicial objects of a category of filters.

3.1. Simplicial homotopy in the category of topological spaces. — Indeed, in the category *Top* of topological spaces, a homotopy contracting a space F in a space X to a point (i.e. a map $h : F\times[0,1]/F\times\{1\}\longrightarrow X$ from the cone of F to X), gives rise to a map

$$h_\bullet : \mathrm{Sing}_\bullet F \longrightarrow \mathrm{Sing}_\bullet X[+1]$$

of singular complexes lifting the map $(h_{|F\times\{0\}})_\bullet : \mathrm{Sing}_\bullet F \longrightarrow \mathrm{Sing}_\bullet X$, defined as follows. This map takes each $\delta : \Delta^n \to F$ in $\mathrm{Sing}_\bullet F((n+1)^\leqslant)$ to $h_*(\delta) : \Delta^n\times[0,1]/\Delta^n\times\{1\}\to X$ in $\mathrm{Sing}_\bullet X((n+2)^\leqslant)$ defined by $h_*(\delta)(x,t) := h(\delta(x),t)$.

$$\begin{array}{ccc} & & \mathrm{Sing}_\bullet X\circ[+1] \\ & \overset{h_\bullet}{\nearrow} & \big\downarrow{\scriptstyle pr_{2,3,...}} \\ \mathrm{Sing}_\bullet F & \longrightarrow & \mathrm{Sing}_\bullet X \end{array} \tag{7}$$

By a nice topological space we mean any class of spaces such a map $h : F\longrightarrow X$ is contractible (=null-homotopic) iff it is weakly contractible; Whitehead's theorem says that CW-spaces are nice in this sense.

Proposition 1. — *If F and X are nice, and F is connected, than a map $h_0 : F\longrightarrow X$ factors though the cone of F as*

$$F\longrightarrow F\times[0,1]/F\times\{1\}\longrightarrow X$$

iff the induced map $\mathrm{Sing}_\bullet X\longrightarrow\mathrm{Sing}_\bullet X$ *of singular complexes factors through the decalage* $\mathrm{pr}_{2,3,..} : \mathrm{Sing}_\bullet X\circ[+1]\longrightarrow\mathrm{Sing}\, X_\bullet$.

Proof. — Recall that the singular complex is defined using simplices

$$\Delta^n = \mathrm{Hom}_{\mathrm{preorders}}([0,1]^\leqslant,(n+1)^\leqslant)$$

as "test spaces":

$$\begin{aligned}
\mathrm{Sing}_\bullet F((n+1)^{\leqslant}) &:= \mathrm{Hom}_{\mathrm{Top}}(\Delta^n, F),\\
\mathrm{Sing}_\bullet X((n+1)^{\leqslant}) &:= \mathrm{Hom}_{\mathrm{Top}}(\Delta^n, X),\\
\mathrm{Sing}_\bullet X \circ [+1]((n+1)^{\leqslant}) &= \mathrm{Hom}_{\mathrm{Top}}(\Delta^n \times [0,1]/\Delta^n \times \{1\}, X)
\end{aligned}$$

where $n \geqslant 0$ and $\Delta^n \times [0,1]/\Delta^n \times \{1\}$ is the cone of n-simplex Δ^n.

To define a lifting $h_\bullet$ given a map $h : F\times[0,1]/F\times\{1\} \longrightarrow X$, take each $\delta : \Delta^n \to F$ in $F_\bullet((n+1)^{\leqslant})$ to $h_*(\delta) : \Delta^n \times [0,1]/\Delta^n \times \{1\} \to X$ in $X_\bullet((n+2)^{\leqslant})$ defined by $h_*(\delta)(x,t) := h(\delta(x),t)$.

To see the other direction, note that a map $h_\bullet : F_\bullet \longrightarrow X_\bullet[+1]$ takes a singular simplex $\delta : \Delta^n \longrightarrow F$ of F into a singular simplex $h_\bullet(\delta) : \Delta^{n+1} = \Delta^n \times [0,1]/\Delta^n \times \{1\} \longrightarrow X$ of X such that $\delta = \mathrm{pr}_{2<3<..<n} h_\bullet(\delta)$, i.e. $\delta = h_\bullet(\delta)_{|\Delta^n\times\{0\}}$, and thereby each $\delta : \Delta^n \longrightarrow F \longrightarrow X$ factors through the cone of Δ^n. A verification using functoriality shows that the same factorisation holds for a map $\delta' : \mathbb{S}^n = \partial\Delta^{n+1}$ from any connected sphere $\mathbb{S}^n = \partial\Delta^{n+1}$, $n > 0$, which means exactly that h_0 is weakly contractible, and for nice topological spaces contractible and weakly contractible are equivalent. □

Let $\mathrm{const}_\bullet F$ denote the constant functor $\Delta^{op} \longrightarrow Top$, $\mathrm{const}_\bullet F(n^{\leqslant}) := F$ for all $n > 0$.

Proposition 2. — *Assume topological spaces F and X are nice.*

A map $h_0 : F \longrightarrow X$ is homotopic to a constant map, i.e. factors though the cone of F as

$$F \longrightarrow F \times [0,1]/F \times \{1\} \longrightarrow X$$

iff the induced map $\mathrm{const}_\bullet F \times \mathrm{Sing}_\bullet F \longrightarrow \mathrm{const}_\bullet X \times \mathrm{Sing}_\bullet X$ *of singular complexes factors through the decalage* $\mathrm{pr}_{2,3,..} : \mathrm{Sing}_\bullet X \circ [+1] \longrightarrow \mathrm{const}_\bullet X \times \mathrm{Sing}\, X_\bullet$.

$$\begin{array}{ccc}
 & & \mathrm{Sing}_\bullet X \circ [+1] \\
 & \overset{h_\bullet}{\nearrow} & \big\downarrow{\scriptstyle \mathrm{pr}_0 \times \mathrm{pr}_{2,3,\ldots}} \\
\mathrm{const}_\bullet F \times \mathrm{Sing}_\bullet F & \longrightarrow & \mathrm{const}_\bullet X \times \mathrm{Sing}_\bullet X
\end{array}$$

In particular, a nice (possibly disconnected) space X is contractible iff

$$\begin{array}{ccc}
 & & \mathrm{Sing}_\bullet X \circ [+1] \\
 & \nearrow & \big\downarrow{\scriptstyle \mathrm{pr}_{2,3,\ldots}} \\
\mathrm{const}_\bullet X \times \mathrm{Sing}_\bullet X & \xrightarrow{\mathrm{id}} & \mathrm{const}_\bullet X \times \mathrm{Sing}_\bullet X
\end{array}$$

3.2. Convergence as being contractible. — In [**L**] we observe we associate with a sequence $(a_i)_i$ of points of a topological space X a morphism in the category $s\boldsymbol{\varphi}$ of simplicial objects in the category $\boldsymbol{\varphi}$ of filters (cf. Definition 5.2.5) such that it factors as in formula (7) iff the sequence is convergent; moreover, limits of the sequence correspond precisely to the liftings. In fact, we first wrote [**mintsGE**, §3.2] the simplicial diagram (7) when transcribing the definition of a limit of a filter on a topological space in [**?**], but were sadly unaware of its connection to contractibility before a remark by V.Sosnilo.

4. Research directions. Test problems.

We suggest a couple of test problems which might be used to guide development of the simplicial reformulations of model theory.

4.1. State and prove simplicially the Fundamental Theorem of Stability Theory. — The Fundamental Theorem of Stability Theory claims equivalence of two definitions of stability theory admitting simplicial reformulations: (*) each type over a set is definable, and (**) no formula has the order property. The first definition is what this note is about, and the second defitition is quite close to the reformulations in terms of the lifting property discussed in [**Z1, Z2**], esp. [**Z1**, §3.3.2],[**Z2**, §17] see also §1.7. Moreover, it follows from a general theorem about compactness, namely the Grothendieck's double limit theorem [**Groth**, Thm.6], as explained in [**BenYaacov, Starchenko**]. Note that there is a definition of compactness involving diagram similar to that used to define definability [**L**].

Can one give a purely simplicial or homotopy theoretic proof of this theorem ?

One immediate difficulty is that it is not quite clear what category one should work in: the reformulations in [**Z1, Z2**] use the category $s\mathfrak{F}$ of simplicial filters and continuous maps (though §1.7 suggests it might be better to use almost everywhere continuous maps), whereas here we use the category *sTop* of simplicial topological spaces or its full subcategory *sProFiniteSets* of simplicial profinite Hausdorff compact spaces. See also speculations in §4.2.1(Forking as a notion of continuity?).

4.2. Kim-Pillay characterisation of non-forking in terms of independence relation. — Rewrite simplicially the characterisation of simple theories [**Kim-Pillay**, §4, Def. 4.1] and stable theories [**Harnik-Harrington**, Axioms 0-4] in terms of parameter sets and types as morphisms to $\mathbb{S}^T_\bullet(\varnothing)$. Doing so appears straightforward, and the difficulty lies in identifying the category-theoretic notion that such a translation would lead to. Note that [**Lieberman-Rosicky-Vasey**, Def. 2.1] and [**Kamsma2**] use a different approach to reformulate these notions category-theoretically.

4.2.1. Forking as a notion of continuity?— [**Harnik-Harrington**] observed stability of a first-order theory T can be characterised in terms of a class of distinguished extensions $p \subset q$ of its complete types (here $p \subset q$ means that each formula in p is also in q) over arbitrary sets of parameters, see also [**Tent-Ziegler**, Theorem 8.5.10(Characterisation of Forking)]. They denote this relation by $p \sqsubset q$ between the complete types of a theory T, and the intuition is that $p \sqsubset q$ means that q is a *free* extension of p to a larger set of parameters. They prove that a theory T is stable iff there is a relation $p \sqsubset q$ on its complete types over arbitrary parameters satisfying the following axioms:

Axiom 0. If $p \sqsubset q$ then $p \subset q$ and if f is elementary then $p \sqsubset q$ iff $fp \sqsubset fq$

Axiom 1. If $p \subset q \subset r$, then:
(a) $p \sqsubset q \sqsubset r$ implies that $p \sqsubset r$,
(b) $p \sqsubset r$ implies that $p \sqsubset q$,
(c) $p \sqsubset r$ implies that $q \sqsubset r$.

Axiom 2. If $p \in S(A)$ and $A \subset B$, then $p \sqsubset q$ for some $q \in S(B)$.

Axiom $3_{\aleph_0}$. If $p \in S(A)$, then $p \restriction A_0 \sqsubset p$ for some finite $A_0 \subset A$.

Axiom 4. For any p there is a cardinal λ such that there are at most λ mutually contradictory types q s.t. $p \sqsubset q$.

Axiom 3. There is a cardinal κ such that if $p \in S(A)$, then $p \restriction A_0 \sqsubset p$ for some $A_0 \subset A$ with $|A_0| < \kappa$.

In simplicial terms, a relation $p \sqsubset q$ is a class of distinguished diagrams in *sSets*

$$\begin{array}{ccc} |A|_\bullet & \xrightarrow{\ p\ } & \mathbb{S}_\bullet(\varnothing) \circ [+N] \\ \downarrow{\scriptstyle A\subseteq B} & \nearrow{\scriptstyle q} & \downarrow{\scriptstyle \mathrm{pr}_{N+1,N+2,\dots}} \\ |B|_\bullet & \xrightarrow{\ \mathrm{tp}(B/\varnothing)\ } & \mathbb{S}_\bullet(\varnothing) \end{array} \qquad \begin{array}{ccc} |A|_\bullet & \xrightarrow{\ p\ } & \mathbb{S}_\bullet(B) \circ [+N] \\ \downarrow{\scriptstyle A\subseteq B} & \nearrow{\scriptstyle q} & \downarrow{\scriptstyle \mathrm{pr}_{N+1,N+2,\dots}} \\ |B|_\bullet & \xrightarrow{\ \mathrm{tp}(B/B)\ } & \mathbb{S}_\bullet(B) \end{array}$$

A notion of continuity defines an extra structure on *sSets* which does provide a class of distinguished diagrams, and one wonders if this is a useful point view on non-forking. A standard way to define something like a topology on a category is provided by the notion of a Grothendieck topology on a category, i.e. a choice of distinguished families $\{f_i : U_i \longrightarrow U\}_i$ of morhpisms called *coverings*. Note that Axiom 3 reminds of accessible categories.

4.2.2. Stability in terms of an independence relation.— We have nothing to say but only quote some definitions with a hope that the reader may recognize the diagrams involved. [**Kim-Pillay**, Theorem 5.2, cf. also Theorem 3.2 and Def. 4.1] characterise the class of simple theories. We shall quote the axioms of an independence relations. [**Kim-Pillay**, Theorem 3.2(Independence Theorem over a model)] states a non-triviality condition of the independence relation characterising simple theories. Setting $tp(a/A) \sqsubset tp(a/BA)$ iff (a, B, A) is independent relates the independence relation and the non-forking extensions above.

Definition 4.1. Let T be an arbitrary theory, and let Γ be a collection of triples (a, B, A) (where a is a tuple from $\mathscr{C}$ and $A \subseteq B$ are small subsets of $\mathscr{C}$). We will say that Γ is a notion of independence if the following hold:

(i) (invariance) Γ is invariant under automorphisms of $\mathscr{C}$.
(ii) (local character) For any a, B there is $A \subseteq B$ such that the cardinality of A is at most the cardinality of T and $(a, B, A) \in \Gamma$.
(iii) (finite character) $(a, B, A) \in \Gamma$ if and only if for every finite tuple b from B, $(a, A \cup \{b\}, A) \in \Gamma$.
(iv) (extension) For any a, A and $B \supseteq A$ there is a' such that $tp(a'/A) = tp(a/A)$ and $(a', B, A) \in \Gamma$.
(v) (symmetry) If $(a, A \cup \{b\}, A) \in \Gamma$ then $(b, A \cup \{a\}, A) \in \Gamma$.
(vi) (transitivity) Suppose $A \subseteq B \subseteq C$. Then $(a, C, B) \in \Gamma$ and $(a, B, A) \in \Gamma$ if and only if $(a, C, A) \in \Gamma$.

In simplicial terms, this describes a collection of distinguished diagrams of form

$$\begin{array}{ccc} |a| \vee |B|_\bullet & \longleftarrow & |a|_\bullet \\ \uparrow & \searrow & \downarrow \\ |B|_\bullet & \longrightarrow & \mathbb{S}_\bullet(A) \end{array}$$

4.3. Reformulate superstability, NIP, categoricity, simplicity, properties of an independence relation and non-forking, Kim-dividing and Kim-forking, excellence...— Reformulate some of the classical theory. In particular, reformulate simplicially model theoretic properties of structures related to combinatorics or algebraic geometry, such as those related to Elkes-Szabo or peudoexponentiation.

5. Appendix A. Requests for comments

In the appendix I take the liberty to present questions which I would ask in a private conversation or email.

5.1. Requests for comments from a homotopy theorist. — I do not know much about the simplicial formula (5) used to define contractibility, convergence, and stability of a theory. Essentially, any reference to a general theory would be welcome.

Does homotopy theory suggest a point of view or technique for dealing with stable theories ? Particularly in view of the connection between the Borel construction of a group action and the simplicial Stone space mentioned in Remark 1.3.1. For example, did anyone consider the Borel construction $\mathbb{B}_\bullet(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}), \mathbb{Q})$ of the Galois action on $\bar{\mathbb{Q}}$? Probably $\mathbb{B}_\bullet(\mathrm{GL}(V), V)$ for a vector space V, is standard to consider, but how would it relate to the Stone space $\mathbb{S}_\bullet^{\mathrm{VectorSpaces}}$ of the theory of vector spaces...

I remark that stability (and a number of other properties of theories) can be defined by a lifting property with respect to an explicitly given morphism, in a somewhat similar category $s\boldsymbol{\varphi}$ of simplicial objects in the category $\boldsymbol{\varphi}$ of filters [**Z1, Z2**].

I should explicitly say that I am no expert in homotopy theory, and solicit collaboration.

Question 1 **(Background on our simplicial formula for contractibility)**

– *Find a good reference discussing this simplicial formula and decalage..*

- *This simplicial formula can be interpreted in the category of simplicial objects of an arbitrary category, and thus defines a notion of a map being contractible. Did anyone study this formula as a definition of contractibility ? Is it well-behaved ? We do know that in sSets it does define a standard notion of contractibility for fibrant simplicial sets.*
- *Is there a similar formula using endomorphisms of Δ which defines when two maps are homotopy equivalent in the category of simplicial objects of an arbitrary category ?*

5.2. Requests for comments from a model theorist. — Where to take this further ? What notions in model theory look as if they might be added to our little glossary ?

An obvious wish is to apply methods or intuitions of homotopy theory in model theory. Say, make a homotopy theory calculation in model theory.

5.2.1. Morley sequences as spectra in stable homotopy theory ?— Both notions involve a sequence and taking a suspension at each step. Is there any analogy ?

5.2.2. Does the classifying space

$$\mathbb{B}_\bullet(\mathfrak{C}, \mathrm{Aut}(\mathfrak{C}/B))$$

appear in model theory?— Does Remark 1.3.1 provoke any associations in model theory ?

5.2.3. A technical question: the same diagram with different notion of continuity. — [**Z1, Z2**] show that stability and a number of other notions are defined by lifting properties not in in the category *sTop* of simplicial topological spaces, but in the category $s\boldsymbol{\varphi}$ of simplicial objects in the category of filters. Interpreting (5) in that category leads to a different property of type which I state below. Is it familiar ?

Question 2. — *Is the following property of types familiar ? It seems very much as an analogue of non-dividing but only for elements, not tuples,* [**Tent-Ziegler**, Cor.7.1.5]. *Note it is expressed as a lifting property in $s\boldsymbol{\varphi}$ in* [**Z1**, 5.3.2].

- *(oversimplified) A global type $p(x/\mathfrak{C})$ invariant over B such that if $\bar{c}$ is indiscernible over B, then $p(x/\mathfrak{C})$ contains all the formulas saying that $\bar{c}$ is indiscernible over x.*
- *A global type $p(x/\mathfrak{C})$ invariant over B such that*
  - *for each length $l > 0$, for each formula $\varphi(x, \bar{y}, \bar{b})$, $\bar{b} \subset B$, there are finitely many formulas $\psi_i(-, \bar{b}_i)$, $\bar{b}_i \subset B$, $0 < i < n$, such that*
    - *for any tuple $\bar{c} \subset \mathfrak{C}$ of length l, if $\bar{c}$ is indiscernible wrt each $\psi(-, \bar{b}_i)$, $0 < i < n$, then $p(x/\mathfrak{C})$ contains the formula saying that $\bar{c}$ is indiscernible wrt $\phi(x, -, \bar{b})$.*
- *same as above, but instead of indiscernibility wrt finitely many formulas require extending to an arbitrary long finite tuple indiscernible wrt finitely many formulas. Thus, it now reads:*

  A global type $p(x/\mathfrak{C})$ invariant over B such that
  - *for each lengths $l < l_1$, for each finite set Θ of formulas over B there are $l_2 > 0$ and finite set Δ of formulas over B such that*

* *for any tuple $\bar{c} \subset \mathfrak{C}$ of length l, if $\bar{c}$ extends to some finite tuple of length l_2 indiscernible wrt Δ, then $p(x/\mathfrak{C})$ contains the formula saying that $\bar{c}$ extends to a tuple $\bar{c}\bar{c}'$ of length l_1 indiscernible wrt Θ*

5.2.4. References to simplicial type spaces in model theory?— The only three references to simplicial type spaces I know, are by (Michael Morley. Applications of topology to $L_{\omega_1\omega}$. 1974) [**Morley**], and by (Levon Haykazyan. *Spaces of Types in Positive Model Theory.* J. symb. log. 84 (2019) 833-848.) [**Levon**], and (Mark Kamsma. *Type space functors and interpretations in positive logic.* 2020). The latter two [**Levon, Kamsma**] mention simplicial type spaces under the name of *type space functors* and consider them in the context of positive logic. We particularly draw attention to [**Levon**, §3(The type space functor and interpretations of theories)] and [**Kamsma**, Defs. 4.19-20] which I have not yet read.

[**Morley**] calls them *type structures* associated to an $L_{\omega_1\omega}$-theory, but never uses words "functor" or "category" explicitly. Consider the following wording by Morley used to introduce notions necessary to charactercise simplicial spaces (called "type structures") associated with an $L_{\omega_1\omega}$-theories.

> In considering these spaces and maps between them some complexities are introduced by the substitution properties of equality. We shall, therefore, usually consider not S_n but the closed-open subset Q_n consisting of the types of sequences of distinct elements. Similarly, we shall consider only the maps f^* which are induced by one-one f. The collection of all the Q_n's together with the maps f^* will be called the *type structure* for T.
>
> We shall always use Π to denote f^* where f is the identity map of n into $n+1$. (The value of n will be clear from context.) Similarly λ will always denote f^* where $f: n \to n+1$ is defined by $f(i) = i$ for $i < n-1$ and $f(n-1) = n$.

Is there anything else ? In particular, about interpretations as maps of type spaces.

5.2.5. A concise definition of simplicial Stone spaces of types in the category of filters. — Let me now define the category $\mathfrak{P}$ of filters with continuous maps, and the category $\mathfrak{P}$ of filters with continuous maps defined almost everywhere.

Definition 5.2.1. — An object of $\mathfrak{P}$ is a set equipped with a filter. A morphism $f : (X, \mathcal{F}) \longrightarrow (Y, \mathcal{G})$ is a map $f : X \longrightarrow Y$ of the underlying sets such that the preimage of a big set is big, i.e. $\{f^{-1}(U) : U \in \mathcal{G}\} \subset \mathcal{F}$. We call such maps of filters *continuous*, as it enables us to say that a map of topological spaces is continuous iff the induced maps of neighbourhoods filters are continuous.

Let $\mathfrak{P}$ denote a category of filters where morphisms are defined only on big subsets, where we identify maps which coincide on a big subset. That is, $\mathfrak{P}$ and $\mathfrak{P}$ have the same objects, and in $\mathfrak{P}$ a morphism $f : (X, \mathcal{F}) \longrightarrow (Y, \mathcal{G})$ is a map $f : U_X \longrightarrow Y$ defined on a big subset $U_X \in \mathcal{F}$ such that the preimage of a big set is big, i.e. $\{f^{-1}(U) : U \in \mathcal{G}\} \subset \mathcal{F}$. Two such morphisms are considered identical iff they coincide on a big subset. We call such maps of filters *continuous defined almost everywhere.* Note that we may still say that a map of topological spaces is continuous iff the induced maps of neighbourhoods filters are almost everywhere continuous.

We may consider a type space $S_n(B)$ to be objects of $\mathfrak{P}$ if we equip $S_n(B)$ with the following *indiscernability* filter generated by sets of types containing a formula over B of the form

$$\bigwedge_{0<l<k} (x_{i_l} \neq x_{i_{l+1}} \,\&\, x_{j_l} \neq x_{j_{l+1}}) \implies (\varphi(x_{i_1}, ..., x_{i_k}) \leftrightarrow \varphi(x_{j_1}, ..., x_{j_k}))$$

Perhaps it is more reasonable to define these filters slightly different by taking the formulas of the form, for each $k < N$ and a finite collection of formulas φ_s over B:

$$\bigwedge_{0<l<k} (x_{i_l} \neq x_{i_{l+1}} \,\&\, x_{j_l} \neq x_{j_{l+1}}) \implies \exists x_{n+1} ... x_N (\bigwedge_{n<r<s\leqslant N} x_r \neq x_s \;\; \&$$

$$\bigwedge_{\substack{i_k<i_{k+1}<...<i_r\leqslant N, \\ j_k<j_{k+1}<...<j_r\leqslant N}} (\bigwedge_s \varphi_s(x_{i_1}, ..., x_{i_r}) \leftrightarrow \varphi_s(x_{j_1}, ..., x_{j_r}))$$

The formula is meant to say that the tuple $x_1, .., x_n$ can be extended to an arbitrary long finite tuple indiscernible with respect to arbitrary finitely many formulas over B.

6. Appendix B. Related Work: a topological reformulation of stability by Herzog and Rothmaler

We quote from [**Rothmaler**, Exercises 11.3.4-7], for convenience of the reader.

Exercise 11.3.3. Let $\mathcal{M}$ be a structure and $A \subseteq B \subseteq M$.

Show that $\mathrm{tp}(\bar{a}/B) \mapsto \mathrm{tp}(\bar{a}/A)$, where $\bar{a}$ runs over the n-tuples in elementary extensions of $\mathcal{M}$, defines a continuous map from $\mathrm{S}_n^{\mathcal{M}}(B)$ onto $\mathrm{S}_n^{\mathcal{M}}(A)$, called **restriction** onto A and denoted by $\rho_{B,A}$.

The series of exercises following deals with continuous sections of this restriction map in a situation as above, but with A a model, where a **section** of a map $f : X \to Y$ is, as usual, a map $g : Y \to X$ such that $fg = \mathrm{id}_Y$. So suppose, for the remaining exercises, $\mathcal{N} \preccurlyeq \mathcal{M}$ and $N \subseteq B \subseteq M$. Note that *any* section of $\rho_{B,N}$ brings an algebraic type $\mathrm{tp}(\bar{a}/N)$ (where $\bar{a}$ is in $\mathcal{N}$) to $\mathrm{tp}(\bar{a}/B)$: the former contains the formula $\bar{x} = \bar{a}$, hence, being an extension, its image also has to contain it.

In the next exercise we deal with material due to Daniel Lascar and Bruno Poizat (see their books). Given a type $\Phi \in \mathrm{S}_n^{\mathcal{M}}(N)$, a **coheir** of Φ over B is a type $\Psi \in \mathrm{S}_n^{\mathcal{M}}(B)$ extending Φ which is **finitely satisfied** in $\mathcal{N}$, i. e., every finite subset of Ψ is satisfied in $\mathcal{N}$ (more precisely—as Ψ contains parameters not in $\mathcal{N}$—satisfied in $\mathcal{M}$ by some tuple from N). Note that this is equivalent to saying that every *single* formula in Ψ is satisfied in $\mathcal{N}$, for Ψ is complete and thus closed under taking finite conjunctions.

Exercise 11.3.4. Prove that the set of types from $\mathrm{S}_n^{\mathcal{M}}(B)$ that are coheirs of their restrictions to N is the closure in $\mathrm{S}_n^{\mathcal{M}}(B)$ of the set $\{\mathrm{tp}(\bar{a}/B) : \bar{a} \in N^n\}$.

The remainder of this section contains material from unpublished joint work of Ivo Herzog and the author.

Exercise 11.3.5. Show that there can be at most one continuous section of $\rho_{B,N}$.

Exercise 11.3.6. Prove that if $\sigma : \mathrm{S}_n^{\mathcal{M}}(N) \to \mathrm{S}_n^{\mathcal{M}}(B)$ is a continuous section of $\rho_{B,N}$, then, for each $\Phi \in \mathrm{S}_n^{\mathcal{M}}(N)$, the type $\sigma(\Phi)$ is a coheir of Φ.

Exercise 11.3.7. Show that the image of $\mathrm{S}_n^{\mathcal{M}}(N)$ under a continuous section of $\rho_{B,N}$ contains the set of types from $\mathrm{S}_n^{\mathcal{M}}(B)$ that are coheirs of their restrictions to N. Derive that, if there is such a continuous section, every type in $\mathrm{S}_n^{\mathcal{M}}(N)$ has (at most) one coheir.

This yields a new definition of stability (see §15.4 below): a theory is stable if and only if , for any two models $\mathcal{N} \preccurlyeq \mathcal{M}$, the restriction map $\rho_{M,N}$ has a (n automatically unique) continuous section. One direction follows from the last exercise and the known fact that uniqueness of coheirs implies stability, see Theorème 12.29 in Poizat's *Cours.* The converse is mentioned explicitely before Corollaire 16.07 of the same text.

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