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THE UNREASONABLE EXPRESSIVE POWER OF THE LIFTING PROPERTY IN ELEMENTARY MATHEMATICS

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> 2. Hawk/Goose effect. A baby chick does not have any built-in image of "deadly hawk" in its head but distinguishes frequent, hence, harmless shapes, sliding overhead from potentially dangerous ones that appear rarely. Similarly to "first", "frequent" and "rare" are universal concepts that were not specifically designed by evolution for distinguishing hawks from geese. This kind of universality is what, we believe, turns the hidden wheels of the human thinking machinery.
>
> Misha Gromov, *Math Currents in the Brain.*

Abstract

Playing with the lifting property, we observed that the lifting property and other categorical constructions from algebraic topology has the power to express concisely and uniformly textbook definitions across disparate domains, including topology, analysis, group theory, model theory, in terms of simple(st?) or archetypal (counter)examples, and that an apparently straightforward attempt to read line by line the text of these definitions and rephrase it in categorical language leads to this observation.

Examples of the lifting property in topology include the notions of: compact, contractible, discrete, connected, and extremally disconnected spaces, dense image, induced topology, subset, closed or open subsets, quotient, Lebesgue dimension, and separation axioms; perhaps acyclic fibration. Examples in algebra include: finite groups being nilpotent, solvable, p-groups, and prime-to-p groups; injective and projective modules; injective, surjective, and split homomorphisms. Each of these can be defined by iteratively applying the lifting property (i.e. taking either left or right orthogonal in a category) from simple and concrete examples of maps. Moreover, in topology such an example often is a map of finite spaces which is the simplest (counter)example to the property being defined, and this leads to a concise notation for basic topological notions in terms of maps of finite preorders (=finite topological spaces).

Rephrasing in simplicial language the definitions of topological and uniform spaces, in (Bourbaki, General Topology) led us to define a category of generalised topological spaces flexible enough to formulate categorically a number of standard basic elementary definitions in various fields, e.g. in analysis, limit, (uniform) continuity and convergence, equicontinuity of sequences of functions; in algebraic topology, being locally trivial and geometric realisation; in geometry, quasi-isomorphism; in model theory, stability and simplicity and several Shelah's dividing lines, e.g. NIP, NOP, NSOP, NSOP_i, NTP, NTP_i,NATP, NFCP, of a theory.

Our reformulations illustrate the generative power of the lifting property as a means of defining natural elementary mathematical concepts starting from their simplest or archetypal (counter)example.

We also offer a couple of brief speculations on cognitive and AI aspects of this observation,

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particularly that in point-set topology some arguments read as diagram chasing computations with finite preorders.

1. *Introduction. Structure of the Paper*

This note is written for *The De Morgan Gazette* to demonstrate that some natural definitions are lifting properties relative to the simplest counterexample, and to suggest a way to "extract" these lifting properties from the text of the usual definitions and proofs. The exposition is in the form of a story and aims to be self-contained and accessible to a first year student who has taken some first lectures in naive set theory, topology, and who has heard a definition of a category. A more sophisticated reader may find it more illuminating to recover our formulations herself from reading either the abstract, or the abstract and the opening sentence of the next section. The displayed formulae and Figure 1(a) defining the lifting property provide complete formulations of our theorems to such a reader.

Our approach naturally leads to a more general observation that in basic point-set topology, a number of arguments are computations based on symbolic diagram chasing with finite preorders; because of lack of space, we discuss it in a separate note [G0].

$\star$ The present text is an update of the original note written in 2014. We added a few examples and clarified some ideas. The old text is left as is, apart from a few typographical changes. All new material is marked by $\star$. Note [G0] of 2015 is superseded by §3(Examples in topology) and references in [LP2].

2. *Surjection and injection*

We try to find some "algebraic" notation to (re)write the *text* of the definitions of surjectivity and injectivity of a function, as found in any standard textbook. We want something very straightforward and syntactic—notation for what we (actually) say, for the text we write, and not for its meaning, for who knows what meaning is anyway?

$(*)_{\text{words}}$ "A function f from X to Y is *surjective* iff for every element y of Y there is an element x of X such that $f(x) = y$."

A function from X to Y is an arrow $X \longrightarrow Y$. Grothendieck taught us that a point, say "x of X", is (better viewed as) as $\{\bullet\}$-valued point, that is an arrow

$$\{\bullet\} \longrightarrow X$$

from a (the?) set with a unique element; similarly "y of Y" we denote by an arrow

$$\{\bullet\} \longrightarrow Y$$

Finally, make dashed the arrows required to "exist", $\star$ for they appear in the conclusion rather than the hypothesis. We get the diagram Fig. 1(b) without the upper left corner; there "{}" denotes the empty set with no elements listed inside of the brackets.

$(**)_{\text{words}}$ "A function f from X to Y is *injective* iff no pair of different points is sent to the same point of Y""

"A function f from X to Y" is an arrow $X \longrightarrow Y$. "A pair of points" is a $\{\bullet, \bullet\}$-valued point, that is an arrow

$$\{\bullet, \bullet\} \longrightarrow X$$

from a two element set; we ignore "different" for now. "the same point" is an arrow $\{\bullet\} \longrightarrow Y$. Represent "sent to" by an arrow

$$\{\bullet, \bullet\} \longrightarrow \{\bullet\}$$

What about "different"? If the points are not "different", then they are "the same" point, that is an arrow

$$\{\bullet\} \longrightarrow X.$$

Now all these arrows combine nicely into diagram Figure 1(c). How do we read it? We want this diagram to have the meaning of the sentence $(**)_{\text{words}}$ above, so we interpret such diagrams as follows:

($\rightthreetimes$) "for every commutative square (of solid arrows) as shown there is a diagonal (dashed) arrow making the total diagram commutative" (see Fig. 1(a)).

(recall that "commutative" in category theory means that the composition of the arrows along a directed path depends only on the endpoints of the path)

Property ($\rightthreetimes$) has a name and is in fact quite well-known [Qui]. It is called *the lifting property*, or sometimes *orthogonality of morphisms*, and is viewed as the property of the two downward arrows; we denote it by $\rightthreetimes$.

Now we rewrite $(*)_{\text{words}}$ and $(**)_{\text{words}}$ as:

$$(*)_{\rightthreetimes} \qquad \{\} \longrightarrow \{\bullet\} \rightthreetimes X \longrightarrow Y$$

$$(**)_{\rightthreetimes} \qquad \{\bullet, \bullet\} \longrightarrow \{\bullet\} \rightthreetimes X \longrightarrow Y$$

So we rewrote these definitions without any words at all. Our benefits? The usual little miracles happen:

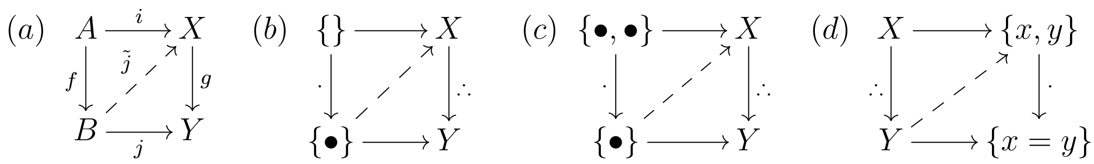


Figure 1: Lifting properties. Dots $\therefore$ indicate free variables, i.e. a property of what is being defined. (a) The definition of a lifting property $f \rightthreetimes g$: for each $i : A \longrightarrow X$ and $j : B \longrightarrow Y$ making the square commutative, i.e. $f \circ j = i \circ g$, there is a diagonal arrow $\tilde{j} : B \longrightarrow X$ making the total diagram $A \xrightarrow{f} B \xrightarrow{\tilde{j}} X \xrightarrow{g} Y, A \xrightarrow{i} X, B \xrightarrow{j} Y$ commutative, i.e. $f \circ \tilde{j} = i$ and $\tilde{j} \circ g = j$. (b) $X \longrightarrow Y$ is surjective (c) $X \longrightarrow Y$ is injective; $X \longrightarrow Y$ is an epicmorphism if we forget that $\{\bullet\}$ denotes a singleton (rather than an arbitrary object and thus $\{\bullet, \bullet\} \longrightarrow \{\bullet\}$ denotes an arbitrary morphism $Z \sqcup Z \xrightarrow{(id,id)} Z$) (d) $X \longrightarrow Y$ is injective, in the category of Sets; $\pi_0(X) \longrightarrow \pi_0(Y)$ is injective, in the category of topological spaces.

Notation makes apparent a similarity of $(*)_{\text{words}}$ and $(**)_{\text{words}}$: they are obtained, in the same purely formal way, from the two simplest arrows (maps, morphisms) in the category of Sets. More is true: it is also apparent that these arrows are the simplest *counterexamples* to the properties, and this suggests that we think of the lifting property as a category-theoretic (substitute for) negation. Note also that a non-trivial (=non-isomorphism) morphism never has the lifting property relative to itself, which fits with this interpretation.

Now that we have a formal notation and the little observation above, we start to play around looking at simple arrows in various categories, and also at not-so-simple arrows representing standard counterexamples.

You notice a few words from your first course on topology: (i) *connected,* (ii) *the separation axioms T_0 and T_1,* (iii) *dense,* (iv) *induced (pullback) topology*, and (v) *Hausdorff* are, respectively, [1]

(i):

$$X \longrightarrow \{\bullet\} \rightthreetimes \{\bullet, \bullet\} \longrightarrow \{\bullet\}$$

(ii):

$$\{\bullet \leftrightarrow \star\} \longrightarrow \{\bullet\} \rightthreetimes X \longrightarrow \{\bullet\}$$

and

$$\{\bullet \rightarrow \star\} \longrightarrow \{\bullet\} \rightthreetimes X \longrightarrow \{\bullet\}$$

(iii):

$$X \longrightarrow Y \rightthreetimes \{\bullet\} \longrightarrow \{\bullet \rightarrow \star\}$$

(iv):

$$X \longrightarrow Y \rightthreetimes \{\bullet \rightarrow \star\} \longrightarrow \{\bullet\}$$

(v):

$$\{\bullet,\bullet'\} \longrightarrow X \rightthreetimes \{\bullet \leftarrow \star \rightarrow \bullet'\} \longrightarrow \{\bullet\}$$

See the last two pages for illustrations how to read and draw on the blackboard these lifting properties in topology; here

$$\{\bullet \rightarrow \star\},\ \{\bullet \leftrightarrow \star\},\ \ldots$$

denote finite preorders, or, equivalently, finite categories with at most one arrow between any two objects, or finite topological spaces on their elements or objects, where a subset is closed iff it is downward closed (that is, together with each element, it contains all the smaller elements; our convention is that $\bullet \rightarrow \star$ iff $\bullet < \star$ iff $\star \in \mathrm{cl}\,\bullet$). Thus

$$\{\bullet \rightarrow \star\},\ \{\bullet \leftrightarrow \star\} \text{ and } \{\bullet \leftarrow \star \rightarrow \bullet'\} \longrightarrow \{\bullet\}$$

denote the connected spaces with only one open point $\bullet$, with no open points, and with two open points $\bullet, \bullet'$ and a closed point $\star$. Line (v) is to be interpreted somewhat differently: we consider *all* the arrows of form

$$\{\bullet,\bullet'\} \longrightarrow X.$$

A negation in category theory. We mentioned that the lifting property can be seen as a kind of negation. Confusingly, there are *two* negations, depending on whether the morphism appears on the left or right side of the square, that are quite different: for example, both the pullback topology and the separation axiom T_1 are negations of the same morphism, and the same goes for injectivity and injectivity on π_0 (see Figure 1(c,d)).

$\star$ Playing with these two negations in the category of topological spaces, a student shall easily compute a few more examples showing that *iteratively applying this category-theoretic negation to simple or archetypal (counter)examples often leads to meaningful notions.* For example, taking once left negation of the map $\{\} \longrightarrow \{\bullet\}$ is the class of maps with non-empty domain; taking left, then right, negation, is the domain being empty; taking negation once left and then twice right gets the property of (=the class of maps) admitting a section; and taking negation right twice gets the class (notion) of subsets. This may lead her to define a concise notation where, e.g., $\{\{\} \rightarrow \{\bullet\}\}^r$ means "surjection", $\{\{\} \rightarrow \{\bullet\}\}^{lrr}$ means "subset", $\{\{\} \rightarrow \{\bullet\}\}^{lrr}$ means "has a section", $\{\{\} \rightarrow \{\bullet\}\}^{lrrrll}$ means "disconnected union", $\{\{\} \rightarrow \{\bullet\}\}^{lrrr}$ "injective", $\{\{\} \rightarrow \{\bullet\}\}^{lrrr}$ "quotient", and $\{\{\} \rightarrow \{\bullet\}\}^{rll}$ means "the induced map of π_0's is surjective", for nice maps.

Playing with $\mathbb{Z}/p\mathbb{Z} \rightarrow 0$ in the category of finite groups gives the property of having kernel of order prime to p ($\{\mathbb{Z}/p\mathbb{Z} \to 1\}^r$) and of order a power of p ($\{\mathbb{Z}/p\mathbb{Z} \to 1\}^{rr}$).. Playing with abelian groups or communtant subgroups leads to "soluble"

$$0 \to S \in \{0 \to A : A \text{ abelian}\}^{\curlywedge \ell r} = \{[G,G] \to G : G \text{ arbitrary }\}^{\curlywedge \ell r}$$

And playing with simplest classes of morphisms might perhaps lead to "nilpotent"

$$H \to H \times H \in \{1 \to G : G \text{ arbitrary }\}\}^{\curlywedge \ell r}$$

See §5 and [**LP2**] for precise statements.

3. ⋆ *Examples in topology*

⋆ A notation for maps of finite topological spaces. Rewrite (i) using subscripts to indicate that the map on the right sends two points to the same point, and perhaps also to remind us of the decomposition $X = A \cup B$ into the union of two connected components mentioned in the definition of "connected".

(i)′ $$X \longrightarrow \{\bullet\} \curlywedge \{\bullet_A, \bullet_B\} \longrightarrow \{\bullet_{A=B}\}$$

Removing the unnecessary •'s, a reader may define a concise notation[2] such that

(i)$_=$ $$X \longrightarrow \{\bullet\} \curlywedge \{A, B\} \longrightarrow \{A = B\}$$

is a complete definition of the topological notion, and so are (ii)-(v). Such a notation will use the identification of a finite topological space and its specialisation preorder viewed as a category where all diagrams commute: write $x \searrow y$ iff $y \in \mathrm{cl}x$, i.e. point x lies in the closure of point y.

Compact. Now consider the standard example of something non-compact: the open covering

$$\mathbb{R} = \bigcup_{n\in\mathbb{N}} \{\, x \,:\, -n < x < n \,\}$$

of the real line by infinitely many increasing intervals. A related arrow in the category of topological spaces is

$$\bigsqcup_{n\in\mathbb{N}} \{\, x \,:\, -n < x < n\} \longrightarrow \mathbb{R}.$$

Does the lifting property relative to that arrow define compactness? Not quite but almost:

$$\{\} \longrightarrow X \curlywedge \bigsqcup_{n\in\mathbb{N}} \{x \,:\, -n < x < n\} \longrightarrow \mathbb{R}$$

reads, for X connected, as "Every continuous real-valued function on X is bounded", which is an early characterisation of compactness taught in a first course on analysis.

$\star$ Picking one of the simplest maps to play, $\{o\} \longrightarrow \{o{\to}c\}$, a student would notice[3] that

- A map g of finite spaces is closed iff $\{o\} \longrightarrow \{o{\to}c\} \rightthreetimes g$

Thinking of double negation as generalisation, and knowing that for maps of finite spaces being closed and proper are the same, she will find that a well-known theorem on extending maps to compact spaces is almost

- A Hausdorff space K is compact iff

$$K \to \{\bullet\} \text{ lies in class } \left(\left(\{\{o\} \longrightarrow \{o{\to}c\}\}^{\rightthreetimes r}\right)_{<5}\right)^{\rightthreetimes lr}$$

and wonder whether

- $\left(\left(\{\{o\} \longrightarrow \{o{\to}c\}\}^{\rightthreetimes r}\right)_{<5}\right)^{\rightthreetimes lr}$ is the class of proper maps?

She won't be surprised by this once she notices that $\{o\} \longrightarrow \{o{\to}c\}$ is the(?) simplest map which is not proper.

$\star$ *Contractible.* Playing with axiom T_4 shall lead a student to observe that

- A topological space is normal (=has axiom T_4) iff

$$\{\} \to X \rightthreetimes \{a{\leftarrow}u{\to}x{\leftarrow}v{\to}b\} \longrightarrow \{a{\leftarrow}u{=}x{=}\,v{\to}b\}$$

that Tiezte extension theorem is almost that

- $\mathbb{R} \to \{\bullet\}$ is in $\{\{a{\leftarrow}u{\to}x{\leftarrow}v{\to}b\} \longrightarrow \{a{\leftarrow}u{=}x{=}\,v{\to}b\}\}^{\rightthreetimes lr}$

hence

- A finite CW complex X is contractible iff

$$X \to \{\bullet\} \text{ is in } \{\{a{\leftarrow}u{\to}x{\leftarrow}v{\to}b\} \longrightarrow \{a{\leftarrow}u{=}x{=}\,v{\to}b\}\}^{\rightthreetimes lr}$$

Once a student notices that the map $\{a{\leftarrow}u{\to}x{\leftarrow}v{\to}b\} \longrightarrow \{a{\leftarrow}u{=}x{=}\,v{\to}b\}$ is a trivial Serre fibration (in fact, its geometric realisation is the barycentric subvisition of an interval) and hence its left-then-right orthogonal is a class of trivial Serre fibrations, she might wonder whether

- $\{\{a{\leftarrow}u{\to}x{\leftarrow}v{\to}b\} \longrightarrow \{a{\leftarrow}u{=}x{=}\,v{\to}b\}\}^{\rightthreetimes lr}$ is the class of all trivial fibrations (at least among "nice" maps) ?

and perhaps use Michael selection theory to prove this.

4. ⋆ *Axioms of topology "transcribed" in simplicial language*

Our goal was to suggest a way to "extract" the category-theoretic language (reformulation) "implicit" in the text of the usual definitions and proofs.

Below we "transcribe" in simplicial language the text of the definition of uniform structure, of a characterisation of topological structure in terms of neighbourhoods of points, and of limit, in (Bourbaki, General Topology). A mathematically inclined reader might want to skip our verbose textual analysis and go directly to Definitions 1-5 motivated by it; the exposition there is self-contained.

As before, the exposition is in the form of a story and aims to be self-contained and accessible to a first year student who has taken some first lectures in naive set theory, topology, and who has heard a definition of a simplicial set. A more sophisticated reader may find it more illuminating to recover our formulations herself by analysing the text of Bourbaki: Axioms $(V)_I$-$(V)_{IV}$ in [Bourbaki,I§1.2] and of Definition I in [ibid,II§1]. Rewriting in categorical language the definition of uniform space is particularly straightforward, and we do recommend trying to do so yourself first. Rewriting the Bourbaki definition of a limit of a filter might be a fun exercise, either before or after reading our definition of a generalised topological space.

⋆ *"Transcribing" simplicially a definition of topological structure.* A topology is a collection of (filters of) neighbourhoods of points compatible in some sense. We now show that it is "compatible" in the sense that it is "functorial", i.e. defines a functor from Non-emptyFiniteLinearOrders to a category of filters.

This is almost explicit in the axioms (V_I)-(V_{IV}) of [Bourbaki,I§1.2] of topology in terms of neighbourhoods. We now quote:

Let us denote by $\mathfrak{B}(x)$ the set of all neighbourhoods of x. The sets $\mathfrak{B}(x)$ have the following properties:

(V_I) *Every subset of X which contains a set belonging to $\mathfrak{B}(x)$ itself belongs to $\mathfrak{B}(x)$.*

(V_{II}) *Every finite intersection of sets of $\mathfrak{B}(x)$ belongs to $\mathfrak{B}(x)$.*

(V_{III}) *The element x is in every set of $\mathfrak{B}(x)$.*

Indeed, these three properties are immediate consequences of Definition 4 and axiom (O_{II}).

(V_{IV}) *If V belongs to $\mathfrak{B}(x)$, then there is a set W belonging to $\mathfrak{B}(x)$ such that, for each $y \in W$, V belongs to $\mathfrak{B}(y)$.*

By Proposition 1, we may take W to be any open set which contains x and is contained in V.

This property may be expressed in the form that *a neighbourhood of x is also a neighbourhood of all points sufficiently near to x.*

What is "the set $\mathfrak{B}(x)$ of all neighbourhoods of x"? $\mathfrak{B}(x)$ is a set of subsets of X parametrised by $x \in X$, thus it is natural to view $\mathfrak{B}(x)$ as a set of subsets of $\{x\} \times X$, and then view "the sets $\mathfrak{B}(x)$", $x \in X$, as a filter on $X \times X = \bigsqcup_{x \in X} \{x\} \times X$ consisting of subsets of form

$$\bigsqcup_{x \in X, U_x \in \mathfrak{B}(x)} \{x\} \times U_x$$

Axioms (V_I) and (V_{II}) say exactly that it is indeed a filter on $X \times X$.

Axiom (V_{III}) says that the filter induced on the diagonal $\{(x, x) : x \in X\} \subset X \times X$ is antidiscrete, i.e. the only large subset is the whole set itself. To view this category-theoretically, first consider the inclusion as the diagonal map

$$X \longrightarrow X \times X, \quad x \longmapsto (x, x).$$

Axiom (V_{III}) says that the preimage of any large subset contains the whole of X.

To express "the whole of X", make it part of structure: equip X with the indiscrete filter. Then Axiom (V_{III}) is expressed by saying that "the preimage of any large subset is large", which is a condition that makes sense for any map of sets equipped with filters.

This condition reminds us of the definition of a continuous map of topological spaces (the preimage of any open subset is open), and define a *continuous* map of filters to be a map such that the preimage of a large subset is large.

With these definitions, Axiom (V_{III}) says precisely that the diagonal map $X \longrightarrow X \times X$, $x \longmapsto (x, x)$ is continuous.

At last, consider Axiom (V_{IV}). The phrase "there is a set W belonging to $\mathfrak{B}(x)$ such that, for each $y \in W$, V belongs to $\mathfrak{B}(y)$" reads as a property of subsets of $X \times X$ or perhaps $\{x\} \times X \times X$: a subset $U \subset X \times X$ has this property iff there is a set W belonging to $\mathfrak{B}(x)$ such that, for each $y \in W$, the fibre $V_y := U \cap \{y\} \times X$ over y belongs to $\mathfrak{B}(y)$. This property depends on a parameter $x \in X$, and this leads us to define a filter on $X \times X \times X$: call a subset $U \subset X \times X \times X$ *large* iff

for all $x \in X$ there is a set W belonging to $\mathfrak{B}(x)$ such that, for each $y \in W$, the fibre $V_{(x,y)} := U \cap \{(x, y)\} \times X$ belongs to $\mathfrak{B}(y)$.

Equip $X \times X \times X$ with this filter. Then Axiom (V_{IV}) says that the map $X \times X \times X \to X \times X$, $(x, y, z) \mapsto (x, z)$, is continuous.

These considerations are summed up in Definition 3.

$\star$ *"Transcribing" simplicially a definition of uniform structure.* A uniform structure on a set X is a filter on $X \times X$ satisfying certain properties. We now see that properties mean it defines a functor from Non-emptyFiniteLinearOrders to a category of filters which factors via Non-emptyFiniteLinearOrders $\to$ FiniteNon-EmptySets.

This is almost explicit in Definition I of [Bourbaki,I§1.2] §2.1]. We now quote:

Definition I. ***A filter on a set X is a set $\mathfrak{F}$ of subsets of X which has the following properties:***

(F_I) ***Every subset of X which contains a set of $\mathfrak{F}$ belongs to $\mathfrak{F}$.***

(F_{II}) ***Every finite intersection of sets of $\mathfrak{F}$ belongs to $\mathfrak{F}$.***

(F_{III}) ***The empty set is not in $\mathfrak{F}$.***

Definition 1. *A uniform structure* (or *uniformity*) *on a set* X *is a structure given by a set* $\mathfrak{U}$ *of subsets of* X × X *which satisfies axioms* (F_I) *and* (F_{II}) *of Chapter* I, § 6, *no.* 1 *and also satisfies the following axioms:*

(U_I) *Every set belonging to* $\mathfrak{U}$ *contains the diagonal* Δ.

(U_{II}) *If* $V \in \mathfrak{U}$ *then* $\overset{-1}{V} \in \mathfrak{U}$.

(U_{III}) *For each* $V \in \mathfrak{U}$ *there exists* $W \in \mathfrak{U}$ *such that* $W \circ W \subset V$.

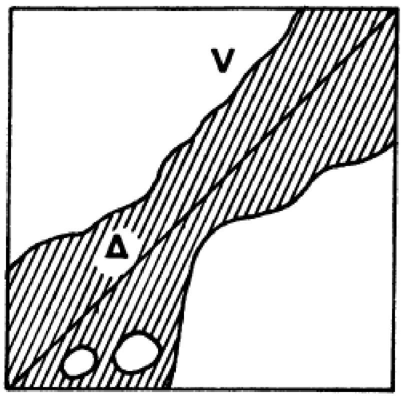


Figure 2.

The sets of $\mathfrak{U}$ *are called entourages of the uniformity defined on* X *by* $\mathfrak{U}$. *A set endowed with a uniformity is called a uniform space.*

(*) We recall (*Set Theory*, R, § 3, nos. 4 and 10) that if V and W are two subsets of X × X, then the set of pairs $(x, y) \in X \times X$, such that $(x, z) \in W$ and $(z, y) \in V$ for some $z \in X$, is denoted by $V \circ W$ or VW, and that the set of pairs $(x, y) \in X \times X$ such that $(y, x) \in V$ is denoted by $\overset{-1}{V}$.

Axioms (F_I) and (F_{II}) say that $\mathfrak{U}$ is a filter on $X \times X$ (but allowing $\emptyset \in \mathfrak{U}$).

To rephrase Axiom (U_I) in the categorical language, first consider the diagonal map

$$X \longrightarrow X \times X, x \mapsto (x, x)$$

Axiom (U_I) says that the preimage of any set beloning to $\mathfrak{U}$ is the whole of X, i.e. in other words, belongs to the indiscrete filter on X. Thus, if we equip X with the indiscrete filter, Axiom (U_I) simply says that "the preimage of a large set is necessarily large". This remind us of the definition of continuity, and so we call a map of sets equipped with filters *continuous* iff the preimage of a large is necessarily large.

This definition of a *continuous map of filters* makes translation to categorical language straightforward. *Axiom (U_I) says that the diagonal map is continuous*, and *Axiom (U_{II}) says that the map permuting coordinates $X \times X \longrightarrow X \times X, (x, y) \mapsto (y, x)$, is continuous.*

In Axiom (U_{III}), first note that "$(x,z) \in W$ and $(z,y) \in V$ for some $y \in X$" describes

$$p_{12}^{-1}(W) \cap p_{23}^{-1}(V) \subset X \times X \times X = \{(x,z,y) : x,z,y \in X\}$$

and thus $W \circ W \subset V$ means that

$$p_{12}^{-1}(W) \cap p_{23}^{-1}(W) \subset p_{13}(V)$$

where $p_{ij} : X \times X \times X \to X \times X, (x_1,x_2,x_3) \mapsto (x_i,x_j)$ are coordinate projections. Thus, *Axiom (U_{III}) says that $p_{12} : X \times X \times X \to X \times X$ is continuous if $X \times X \times X$ is equipped with the pullback of the filter $\mathfrak{U}$ on $X \times X$ along p_{12} and p_{23}.*

These considerations are summed up in Definition 4.

⋆ *"Transcribing" simplicially the definition of limit* Let us now express the Bourbaki definition of limit in terms of generalised topological spaces.

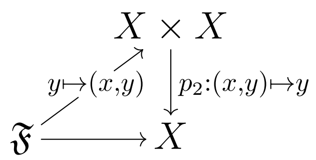
Definition 1. ***Let* X *be a topological space and* $\mathfrak{F}$ *a filter on* X. *A point* $x \in$ X *is said to be a limit point* (or simply a *limit*) *of* $\mathfrak{F}$, *if* $\mathfrak{F}$ *is finer than the neighbourhood filter* $\mathfrak{B}(x)$ *of* x; $\mathfrak{F}$ *is also said to converge* (or to be *convergent*) *to* x.**

View "neighourhood filter $\mathfrak{B}(x)$ as a filter on $\{x\} \times X$, to keep track of parameter "x". The phrase "$\mathfrak{F}$ is finer than the neighbourhood filter $\mathfrak{B}(x)$" means that the map $X \to \{x\} \times X, y \mapsto (x,y)$ is continuous when X is equipped with $\mathfrak{F}$ and $\{x\} \times X$ is equipped with $\mathfrak{B}(x)$. It can also be expressed saying that the map $X \to X \times X, y \mapsto (x,y)$ continuous when X is equipped with $\mathfrak{F}$ and $X \times X$ is equipped with the filter

$$\bigsqcup_{x \in X, U_x \in \mathfrak{B}(x)} \{x\} \times U_x$$

appearing in our reformulation of the definition of topological spaces.

"$\mathfrak{F}$ a filter on X" suggests we consider an arrow $\mathfrak{F} \to X$ and then a diagram, whose meaning is yet unclear



The arrow $p_2 : X \times X \longrightarrow X$ suggests the map of "forgetting the first coordinate" if we view $X, X \times X, ...$ as part of the simplicial set $X_\bullet := \mathrm{Hom}(n^{\leqslant}, X)$ represented by set X:

$$X_\bullet \circ [+1] \longrightarrow X_\bullet$$

where

$[+1]$: Non-emptyFiniteLinearOrders $\longrightarrow$ Non-emptyFiniteLinearOrders

$n \mapsto n{+}1,\ \ f : n \to m \ \longmapsto\ f' : n{+}1 \to m{+}1, f'(1) := 1; f'(i{+}1) := f(i)$

The simplicial set $X_\bullet \circ [+1]$ is a disconnected union of copies of $X_\bullet$ parametrised by $x \in X$

$$X_\bullet \circ [+1] = \bigsqcup_{x\in X} X_\bullet \text{ (as simplicial sets)}$$

and the map $X_\bullet \circ [+1] \longrightarrow X_\bullet$ is identity on each connected component. Hence, if $\mathfrak{F}$ is connected in some appropriate sense, the diagonal map $\mathfrak{F} \longrightarrow X \times X$ is necessary of the form shown (i.e. $y \mapsto (x, y)$).

Thus, we would want $\mathfrak{F}$ to denote a connected simplicial set $\mathfrak{F}_\bullet$ such that $\mathfrak{F}_\bullet(1^\leqslant)$ is the set X equipped with filter $\mathfrak{F}$. A simple way to ensure that is to set $\mathfrak{F}^{\text{diag}}_\bullet := \text{Hom}(n^\leqslant, X)$ where each X^n is equipped with the finest filter such that the map $X \to X^n$ is continuous.

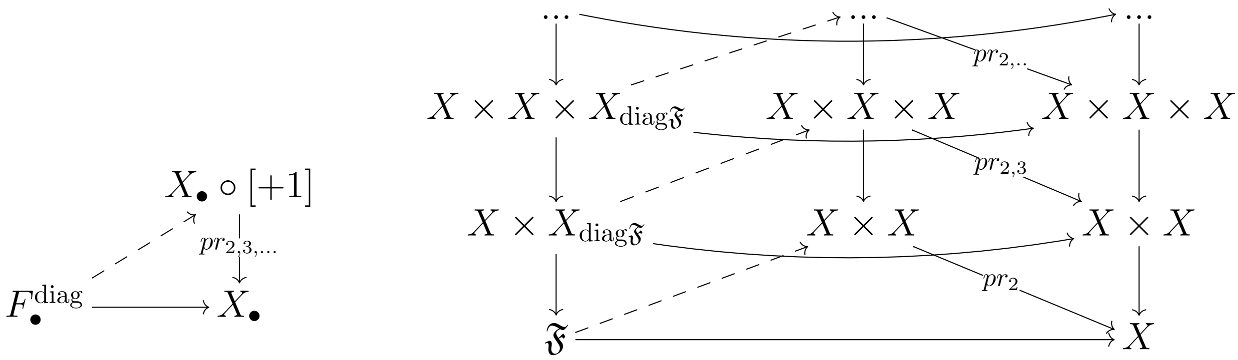


FIGURE 2. (a) The diagram in $s\mathcal{P}$. (b) The same diagram in $s\mathcal{P}$ expanded.

These considerations are summed up in Definition 5.

⋆ *The definition of generalised topological spaces.* The analysis above leads us to the following definitions. A mathematically inclined reader might want to skip the previous subsections motivating these definitions, and the exposition here is self-contained.

Below *a filter on a set* X means a set $\mathfrak{F}$ of subsets of X closed under finite intersection and such that every subset of X which contains a set in $\mathfrak{F}$, belongs to $\mathfrak{F}$. (Beware that Bourbaki also requires that $\emptyset \notin \mathfrak{F}$.) Subsets in $\mathfrak{F}$ are called *large* or *big* according to $\mathfrak{F}$.

DEFINITION 1 (Continuous maps of filters). *Let X and Y be sets equipped with filters (resp. measures). Call a map $f : X \to Y$* continuous *iff the preimage of a big (resp. full measure) set is necessary big (resp. has full measure).*

Definition 2 (Generalised topological spaces). *Let $\boldsymbol{\varphi}$ denote the category formed by sets equipped with filters, and their continuous maps. Its category of simplicial objects*

$$s\boldsymbol{\varphi} := Functors(Non\text{-}emptyFiniteLinearOrders, \boldsymbol{\varphi})$$

is our category of generalised topological spaces.

It contains, as full subcategories, the categories of topological and uniform spaces, as the following two definitions below show. It also contains, trivially, simplicial sets, which suggests that the geometric realisation $|\cdot| : sSets \longrightarrow Top$ is understood as an endofunctor of $s\boldsymbol{\varphi}$.

Definition 3 (A topological space as a generalised space). *Let X be a topological space. Let $X_\bullet : Non\text{-}emptyFiniteLinearOrders \to \boldsymbol{\varphi}$ denote*

$$X_\bullet(n^{\leqslant}) := Hom(n^{\leqslant}, |X|) = |X|^n$$

where

- $X = X_\bullet(1^{\leqslant})$ *is equipped with the antidiscrete filter*
- $X \times X = X_\bullet(2^{\leqslant})$ *is equipped with the filter of subsets of form*

$$\bigsqcup_{x\in X \text{ and } U_x \text{ is a neighbourhood of } x} \{x\} \times U_x$$

- *the filter on $X^n = X_\bullet(n^{\leqslant})$ is the coarsest filter such that all the simplicial maps $X^n \longrightarrow X \times X$ are continuous (in other words, the simplicial dimension of $X_\bullet$ is at most 1.)*

This defines a fully faithful embedding of the category of topological spaces into $s\boldsymbol{\varphi}$.

Definition 4 (A uniform space as a generalised space). *A uniform space is a symmetric simplicial object of dimension 1 in the category $\boldsymbol{\varphi}$ of filters whose underlying simplicial set is representable, i.e. an object $X_\bullet$ of $s\boldsymbol{\varphi}$ which factors via a functor*

$$Non\text{-}emptyFiniteLinearOrders \longrightarrow FiniteNon\text{-}emptySets \longrightarrow \boldsymbol{\varphi}$$

such that

- *there is a set X such that $X_\bullet(n^{\leqslant}) = Hom(n^{\leqslant}, \mathcal{X})$*
- *the filter on $X^n = X_\bullet(n^{\leqslant})$ is the coarsest filter such that all the simplicial maps $X^n \longrightarrow X \times X$ are continuous*

This defines a fully faithful embedding of the category of uniform spaces into $s\boldsymbol{\varphi}$.

Definition 5 (Limit in a generalised space). *Let $\mathfrak{F}_\bullet : F_\bullet \longrightarrow X_\bullet$ be a morphism in $s\mathcal{P}$. A morphism $x_\bullet$ is said to be a* limit morphism *(or simply* a limit*) of $\mathfrak{F}_\bullet$ iff the following diagram commutes:*

$$\begin{array}{ccc} & & X_\bullet \circ [+1] \\ & \overset{x_\bullet}{\nearrow} & \downarrow {\scriptstyle pr_{2,3,\ldots}} \\ F_\bullet & \xrightarrow{\mathfrak{F}_\bullet} & X_\bullet \end{array}$$

where $[+1] : \Delta \longrightarrow \Delta$ is the shift

$$n \mapsto n{+}1,\ \ f : n \to m \longmapsto f' : n{+}1 \to m{+}1, f'(0) := 0; f'(i{+}1) := f(i) \text{ for } i > 0,$$

and $X_\bullet \circ [+1] \longrightarrow X_\bullet$ is the expected map "forgetting the first coordinate".

To recover the Bourbaki definition of a limit of a filter $\mathfrak{F}$ on a topological space X, associate with $\mathfrak{F} \in \mathrm{Ob}\,\mathcal{P}$ the simplical set represented by X

$$\mathfrak{F}_\bullet(n^\leqslant) := \mathrm{Hom}(n^\leqslant, X)$$

equip $X_\bullet(1^\leqslant) = X$ with $\mathfrak{F}$, and equip each $\mathfrak{F}_\bullet(n^\leqslant) := \mathrm{Hom}(n^\leqslant, X)$ is equipped with the finest filter such that the diagonal map $\mathfrak{F}_\bullet(1^\leqslant) \longrightarrow \mathfrak{F}_\bullet(n^\leqslant)$ is continuous. A verification shows that (possibly discontinuous) liftings correspond to points of X, and the continuity requirement means precisely that they are limit points (see Fig. 2).

Remark 1 (Limit as homotopy). *A category theorist would interpret our "taking limit" in $s\mathcal{P}$ as "taking a contracting homotopy", as follows.*

To understand our definition of "limit" in $s\mathcal{P}$, a category theorist would note that homotopy contracting F in X to a point (i.e. a map $h : F \times [0,1]/F \times \{1\} \longrightarrow X$ from the cone of F to X), gives rise to a map*

$$\mathrm{sing}\, F_\bullet \longrightarrow \mathrm{sing}\, X_\bullet[+1]$$

of singular complexes lifting the map $\mathrm{sing}\, F_\bullet \longrightarrow \mathrm{sing}\, X_\bullet$*:*

*To define a limit(=lifting) $h_\bullet$, take each $\delta : \Delta^n \to F$ in $F_\bullet((n+1)^\leqslant)$ to $h_*(\delta) : \Delta^n \times [0,1]/\Delta^n \times \{1\} \to X$ in $X_\bullet((n+2)^\leqslant)$ defined by $h_*(\delta)(x,t) := h(\delta(x), t)$.

To see the other direction, note that $h_\bullet : F_\bullet \longrightarrow X_\bullet[+1]$ takes a singular simplex $\delta : \Delta^n \longrightarrow F$ into $h_\bullet(\delta) : \Delta^{n+1} = \Delta^n \times [0,1]/\Delta^n \times \{1\} \longrightarrow X$ such that $\delta \circ h_0 = h_\bullet(\delta)_{|\Delta^n \times \{0\}}$, i.e. each $\delta : \Delta^n \longrightarrow F \longrightarrow X$ factors through the cone of Δ^n. A verification using functoriality shows that the same factorisation holds for $\mathbb{S}^n = \partial\Delta^{n+1}$, which means exactly that h_0 is weakly contractible, and for "nice" topological spaces contractible and weakly contractible are equivalent.

$$\begin{array}{ccc} & & \operatorname{sing} X_\bullet \circ [+1] \\ & \overset{h_\bullet}{\nearrow} & \downarrow {\scriptstyle pr_{2,3,\ldots}} \\ \operatorname{sing} F_\bullet & \longrightarrow & \operatorname{sing} X_\bullet \end{array}$$

Recall that the singular complex is defined using simplices $\Delta^n = \mathrm{Hom}_{\mathrm{preorders}}([0,1]^{\leqslant}, (n+1)^{\leqslant})$ as "test spaces":

$$\operatorname{sing} F_\bullet((n+1)^{\leqslant}) := \mathrm{Hom}_{\mathrm{Top}}(\Delta^n, F),$$
$$\operatorname{sing} X_\bullet((n+1)^{\leqslant}) := \mathrm{Hom}_{\mathrm{Top}}(\Delta^n, X),$$
$$\operatorname{sing} X_\bullet \circ [+1]((n+1)^{\leqslant}) = \mathrm{Hom}_{\mathrm{Top}}(\Delta^n \times [0,1]/\Delta^n \times \{1\}, X)$$

where $n \geqslant 0$ and $\Delta^n \times [0,1]/\Delta^n \times \{1\}$ is the cone of n-simplex Δ^n.

Remark 2. *In $s\mathcal{P}$ a map $h : F \times [0,1]/F \times \{1\} \longrightarrow X$ continuous in a neighbourhood of "the top of the cone" point $F \times \{1\}$ is the same as a map*

$$F_\bullet \times ([0,1]_1)_\bullet \longrightarrow X_\bullet[+1]$$

$$(x,t) \mapsto (h(x,1), h(x,t))$$

where $[0,1]_1$ denotes the interval $[0,1]$ equipped with the filter of neighbourhoods of point 1.

In other words, $s\mathcal{P}$ can express infinitesimally/"sufficiently" short" homotopies: in an expressive language, we may say that $h_t : F \longrightarrow X$, $t \in [0,1]$, *converges at* 0 iff there is $\varepsilon > 0$ such that $h_{|[0,\varepsilon]}$ is a homotopy contracting F in X.

A proper understanding of this analogy is left to a category theorist.*

⋆ *Geometric realisation in $s\mathcal{P}$.* Recall that Besser-Drinfeld-Grayson construction of geometric realisation starws with an observation that the geometric simplex

$$\Delta^n = \{0 \leqslant x_1 \leqslant x_2 \leqslant \ddots x_n \leqslant 1\} \subset \mathbb{R}^n$$

is, up to almost everywhere, the space of monotone functions $[0,1]^{\leqslant} \longrightarrow (n+1)^{\leqslant}$ with a Skorokhod-type metric

$$\operatorname{dist}(f,g) := \inf\{\varepsilon : \forall x \exists y (|x-y| < \varepsilon \,\&\, |f(x) - g(y)| < \varepsilon)\}$$

*See [**Z2**, §22] for a speculation how this may lead to define a notion of homotopy for models of a first order theory in mathematical logic.

To render this in $s\boldsymbol{\varphi}$, we only need to define a space of maps in $s\boldsymbol{\varphi}$ from $[0,1]$, see [**GP**].

5. ⋆ *Examples of iterated negation*

⋆ For a class C of morphisms in a category, C^l and C^r denote the classes of those morphisms in the category which have the left, resp. right, lifting property with respect to each map in C:

$$C^l := \{f : f \rightthreetimes g \ \forall g \in C\}, \quad C^r := \{g : f \rightthreetimes g \ \forall f \in C\}$$

It is convenient to think of C as a property of maps (namely, the property defining the class C), and to think of C^l and C^r as a *left* and *right negation* of the property: taking C^l and C^r is a simple (simplest?) way to define *some* class of maps without a given property in a way useful in a category theoretic computation. Above we we saw that the notions of both surjectivity and injectivity are negations of the archetypal counterexample, namely $\{\emptyset \to \{\bullet\}\}^r$ is the class of all surjections, and $\{\{\bullet,\bullet\} \to \{\bullet\}\}^r$ is the class of all injections.

But a few more notions can be defined in this way. Here is a list, see [**LP1**, **LP2**] for more examples.

In the category of (all) topological spaces,

r=rrl: $(\emptyset \longrightarrow \{o\})^r$ is the class of surjections

l: $(\emptyset \longrightarrow \{o\})^l$ is the class of maps $A \longrightarrow B$ where $A \neq \emptyset$ or $A \xrightarrow{\text{id}} B$

rr=rllrr: $(\emptyset \longrightarrow \{o\})^{rr} = \{\{x \leftrightarrow y \to c\} \longrightarrow \{x = y = c\}\}^l = \{\{x \leftrightarrow y \leftarrow c\} \longrightarrow \{x = y = c\}\}^l$ is the class of subsets, i.e. injective maps $A \hookrightarrow B$ where the topology on A is induced from B

lr: $(\emptyset \longrightarrow \{o\})^{lr}$ is the class of maps $\emptyset \longrightarrow B$, B arbitrary, and $A \xrightarrow{\text{id}} B$

lrr=lrrrllr: $(\emptyset \longrightarrow \{o\})^{lrr}$ is the class of maps $A \longrightarrow B$ which admit a section

l: $(\emptyset \longrightarrow \{o\})^l$ consists of maps $f : A \longrightarrow B$ such that either $A \neq \emptyset$ or $A = B = \emptyset$

ll=rlll: $(\emptyset \longrightarrow \{o\})^{ll}$ consists of isomorphisms

rl: $(\emptyset \longrightarrow \{o\})^{rl}$ is the class of maps of form $A \longrightarrow A \sqcup D$ where D is discrete

rll=lrrlll: $(\emptyset \longrightarrow \{o\})^{rll}$ is the class of maps $A \to B$ such that each non-empty closed and open subset of B intersects the image of A; for "nice" spaces this means that $\pi_0(A) \to \pi_0(B)$ is surjective.

rllr: $(\emptyset \longrightarrow \{o\})^{rllr}$ is the class of maps $A \to B$ such that $\operatorname{Im} A$ is the intersection of all open closed subsets containing it

lrrrll: $(\emptyset \longrightarrow \{o\})^{lrrrll}$ is the class of maps of form $A \to A \sqcup B$ where $A \sqcup B$ denotes the disconnected union of A and B.

lrrr=lrrrrl=rllrr: $\{\emptyset \longrightarrow \{o\}\}^{lrrr}$ is the class of injective maps, i.e. such that $f(x) \neq f(y)$ whenever $x \neq y$, equiv. $\{a,b\} \to \{a{=}b\} \rightthreetimes f$

lrrrr: $\{\emptyset \longrightarrow \{o\}\}^{lrrrr}$ is the class of "coquetients", i.e. surjective maps $A \to B$ where the topology on A is pulled back from B

lrrrrr: $\{\emptyset \longrightarrow \{o\}\}^{lrrrrr}$ is the class of maps $A \to B$ such that no fibre has indistinguishable points, i.e. $\{a \leftrightarrow b\} \to \{a = b\} \rightthreetimes A \to B$

lrrrl: $\{\emptyset \longrightarrow \{o\}\}^{lrrrl}$ is the class of *quotients*, i.e. the maps $f : A \to B$ such that a subset $U \subset B$ is open in B iff its preimage $f^{-1}(U) \subset A$ is open in A.

In the category of finite groups,

- $f \in \{\mathbb{Z}/p\mathbb{Z} \to 1\}^r$ iff the order of $\operatorname{Ker} f$ is prime to p
- $f \in \{\mathbb{Z}/p\mathbb{Z} \to 1\}^{rr}$ iff the order of $\operatorname{Ker} f$ is a power of p.

- $f \in \{A \to 0 \,:\, A \text{ is abelian }\}^r$ iff $\operatorname{Ker} f$ is soluble
- $0 \to S \in \{0 \to A : A \ \text{ abelian}\}^{\not\lessdot\ell r} = \{[G,G] \to G : G \ \text{ arbitrary }\}^{\not\lessdot\ell r}$ iff S is soluble
- $H \to H \times H \in \{0 \to A \,:\, A \text{ arbitrary }\}^r$ iff H is nilpotent

In the category of R-modules,

- $0 \to P \in \{\{0 \to R\}\}^{lr}$ iff P is projective
- $I \to 0 \ \in \{\{R \to 0\}\}^{rr}$ iff I is injective

6. *A category theorist's view.*

A category theorist would rewrite $(**)_{\not\lessdot}$ as

$$(**)_{\mathbf{mono}} \qquad \bullet \vee \bullet \longrightarrow \bullet \not\lessdot X \longrightarrow Y$$

denoting by $\vee$ and $\bullet \vee \bullet \longrightarrow \bullet$ the coproduct and the codiagonal morphism, respectively, and then read it as follows: given two morphisms

$$\bullet \xrightarrow{\text{left}} X \quad \text{and} \quad \bullet \xrightarrow{\text{right}} X,$$

if the compositions

$$\bullet \xrightarrow{\text{left}} X \longrightarrow Y \;=\; \bullet \xrightarrow{\text{right}} X \longrightarrow Y$$

are equal (both to $\bullet \xrightarrow{\text{down}} Y$), then

$$\bullet \xrightarrow{\text{left}} X \;=\; \bullet \xrightarrow{\text{right}} X$$

are equal (both to $\bullet \xrightarrow{\text{down}} X$). Naturally her first assumption would be that $\bullet$ denotes an *arbitrary* object, as that spares the extra effort needed to invent the axioms particular to the category of sets (or topological spaces) that capture that $\bullet$ denotes a single element, i.e. allow to treat $\bullet$ as a single element. (A logician understands "arbitrary" as "we do not know", "make no assumptions", and that is how formal derivation systems treat "arbitrary" objects.) Thus she would read $(**)_{\not\lessdot}$ as the usual category theoretic definition of a monomorphism. Note this reading doesn't need that the underlying category has coproducts: a category theorist would think of working inside a larger category with formally added coproducts $\bullet \vee \bullet$, and a logician would think of working inside a formal derivation system where "$\bullet$" is either a built-in or "a new variable" symbol, and "$\bullet \vee \bullet \longrightarrow \bullet$" (or "$\{\bullet,\bullet\} \longrightarrow \{\bullet\}$") is (part of) a well-formed term or formula.

And of course, nothing prevents a category theorist to make a dual diagram

$$(**)_{\mathbf{epi}} \qquad X \longrightarrow Y \rightthreetimes \bullet \longrightarrow \bullet \times \bullet, \qquad \bullet \text{ runs through all the objects}$$

and read it as:

$$X \longrightarrow Y \xrightarrow{\text{left}} \bullet \;=\; X \longrightarrow Y \xrightarrow{\text{right}} \bullet \text{ implies } Y \xrightarrow{\text{left}} \bullet \;=\; Y \xrightarrow{\text{right}} \bullet$$

which is the definition of an epimorphism.

$\star$ A category theorist will perhaps find that the notions of limit and homotopy are defined by the same categorical construction, namely factoring through a shifted (decalage) simplicial object, see Remark 1.

7. *Speculations.*

Does your brain (or your kitten's) have the lifting property built-in? Note [G0] suggests a broader and more flexible context making contemplating an experiment possible. Namely, some standard arguments in point-set topology are computations with category-theoretic (not always) commutative diagrams of preorders, in the same way that lifting properties define injection and surjection. In that approach, the lifting property is viewed as a rule to add a new arrow, a computational recipe to modify diagrams.

Can one find an experiment to check whether humans *subconsciously* use diagram chasing to reason about topology?

Does it appear implicitly in old original papers and books on point-set topology?

Is diagram chasing with preorders too complex to have evolved? Perhaps; but note the self-similarity: preorders are categories as well, with the property that there is at most one arrow between any two objects; in fact sometimes these categories are thought of as 0-categories. So essentially your computations are in the category of (finite 0-) categories.

Is it universal enough? Diagram chasing and point-set topology, arguably a formalisation of "nearness", is used as a matter of course in many arguments in mathematics.

Finally, isn't it all a bit too obvious? Curiously, in my experience it's a party topic people often get stuck on. If asked, few if any can define a surjective or an injective map without words, by a diagram, or as a lifting property, even if given the opening sentence of the previous section as a hint. No textbooks seem to bother to mention these reformulations (why?). An early version of [GH-I] states $(*)_{\rightthreetimes}$ and $(**)_{\rightthreetimes}$ as the simplest examples of lifting properties we were able to think up; these examples were removed while preparing for publication.

$\star$ We argued at length that a straightforward attempt to rephrase textbook definitions into categorical language and then playing with examples (by calculating iterated negations(orthogonals) of interesting maps) shall lead to our observations (reformulations and definitions). But is it really so ? This can only be judged by an experiment. And an experiment is possible, as none of our reformulations appeared in print and therefore it is not hard to find a student who never heard of them. How much does one need to tell/explain to make a student able to reproduce our reformulations ?

No effort has been made to provide a complete bibliography; the author shall happily include any references suggested by readers in the next version [G].

Acknowledgments and historical remarks

It seems embarrassing to thank anyone for ideas so trivial, and we do that in the form of historical remarks. Ideas here have greatly influenced by extensive discussions with Grigori Mints, Martin Bays, and, later, with Alexander Luzgarev and Vladimir Sosnilo. At an early stage Ksenia Kuznetsova helped to realise an earlier reformulation of compactness was inadequate and that labels on arrows are necessaryf to formalise topological arguments. "A category theorist [that rewrote] (**) as" the usual category theoretic definition of a monomorphism, and $\star$ interpreted 'taking limit' .. as '...homotopy'", is Vladimir Sosnilo. Exposition has been polished in the numerous conversations with students at St. Peterburg and Yaroslavl'2014 summer school.

Reformulations $(*)_{\rightthreetimes}$ and $(**)_{\rightthreetimes}$ of surjectivity and injectivity, as well as connectedness and (not quite) compactness, appeared in early drafts of a paper [GH-I] with Assaf Hasson as trivial and somewhat curious examples of a lifting property but were removed during preparation for publication. After $(**)_{\rightthreetimes}$ came up in a conversation with Misha Gromov the author decided to try to think seriously about such lifting properties, and in fact gave talks at logic seminars in 2012 at Lviv and in 2013 at Munster and Freiburg, and 2014 at St. Petersburg. At a certain point the author realised that possibly a number of simple arguments in point-set topology may become diagram chasing computations with finite topological spaces, and Grigori Mints insisted these observations be written. Ideas of [ErgB] influenced this paper (and [GH-I] as well), and particularly our computational approach to category theory. Alexandre Borovik suggested to write a note for *The De Morgan Gazette* explaining the observation that 'some of human's "natural proofs" are expressions of lifting properties as applied to "simplest counterexample"'.

I thank Yuri Manin for several discussions motivated by [GH-I].

I thank Kurt Goedel Research Center, Vienna, and particulary Sy David Friedman, Jakob Kellner, Lyubomyr Zdomskyy, and Chebyshev Laboratory, St.Petersburg, for hospitality.

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I wish to express my deep thanks to Grigori Mints, to whose memory this paper is dedicated ...

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voice chat upon request at former online Oxford logic seminar tea room at Gathertown

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⋆ added in 2022:

mintsGE M. Gavrilovich, K.Pimenov. *A naive diagram-chasing approach to formalisation of tame topology.* `http://mishap.sdf.org/mintsGE.pdf` (esp. §2.1-2.3 and §4)

GP M. Gavrilovich, K.Pimenov. *Geometric realisation as the Skorokhod semi-continuous path space endofunctor.* `http://mishap.sdf.org/Skorokhod_Geometric_Realisation.pdf`

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S An overview of generalised topological spaces. Includes a sketch of reformulations of model theoretic notions. `ncatlab.com/show/situs`

LP1 Lifting property. `en.wikipedia.org/wiki/Lifting_property` `ncatlab.com/show/lift`

LP2 Lifting property. A list of examples of iterated lifting property. `ncatlab.com/show/lift`

Z1 M.Gavrilovich. Remarks on Shelah's classification theory and Quillen's negation. `mishap.sdf.org/yetanothernotanobfuscatedstudy.pdf` 2020.

Z2 M.Gavrilovich. A category-theoretic interpretation of Shelah's diving lines. `mishap.sdf.org/model_theory_poster.pdf` 2022. a poster. :

Notes

[1] In the original note, quoted below, we used $<$ instead of $\to$, and $\gtrless$ instead of $\leftrightarrow$.

(i):

$$X \longrightarrow \{\bullet\} \curlywedge \{\bullet, \bullet\} \longrightarrow \{\bullet\}$$

(ii):

$$\{\bullet \gtrless \star\} \longrightarrow \{\bullet\} \curlywedge X \longrightarrow \{\bullet\}$$

and

$$\{\bullet < \star\} \longrightarrow \{\bullet\} \curlywedge X \longrightarrow \{\bullet\}$$

(iii):

$$X \longrightarrow Y \curlywedge \{\bullet\} \longrightarrow \{\bullet \to \star\}$$

(iv):

$$X \longrightarrow Y \curlywedge \{\bullet < \star\} \longrightarrow \{\bullet\}$$

(v):

$$\{\bullet, \bullet'\} \longrightarrow X \curlywedge \{\bullet > \star < \bullet'\} \longrightarrow \{\bullet\}$$

See the last two pages for illustrations how to read and draw on the blackboard these lifting properties in topology; here

$$\{\bullet < \star\},\ \{\bullet \gtrless \star\},\ \ldots$$

denote finite preorders, or, equivalently, finite categories with at most one arrow between any two objects, or finite topological spaces on their elements or objects, where a subset is closed iff it is downward closed (that is, together with each element, it contains all the smaller elements). Thus

$$\{\bullet < \star\},\ \{\bullet \gtrless \star\} \text{ and } \{\bullet > \star < \bullet'\} \longrightarrow \{\bullet\}$$

denote the connected spaces with only one open point $\bullet$, with no open points, and with two open points $\bullet, \bullet'$ and a closed point $\star$. Line (v) is to be interpreted somewhat differently: we consider *all* the arrows of form

$$\{\bullet, \bullet'\} \longrightarrow X.$$

[2] For a definition of a concise notation for maps of finite topological spaces see `https://ncatlab.org/nlab/show/separation+axioms+in+terms+of+lifting+properties`. This takes care of (i-iv) but not (v). Item (v) needs more care for the following reason: we either need to let $\{\bullet, \bullet'\} \to X$ vary among all injective maps, or ensure that in the lifting property square, the top arrow $\{\bullet, \bullet'\} \longrightarrow \{\bullet \leftarrow \star \to \bullet'\}$ is not quantified over but rather is the map denoted, well, by "$\{\bullet, \bullet'\} \longrightarrow \{\bullet \leftarrow \star \to \bullet'\}$".

[3] This wasn't noticed for a number of years.... A version of this paper contains an amusingly misguided early attempt to understand compactness, see §3 there.

8. *Appendix: Transcribing "dense" and "T_0".*

We shall now transcribe the definitions of *dense* and *Kolmogoroff* T_0 spaces. An interested reader should read our exposition of compactness in [**mintsGE**, §2] from where this is taken.

8.0.1. *"A is a dense subset of X."* By definition [Bourbaki, I§1.6, Def.12],

Definition 12. ***A subset* A *of a topological space* X *is said to be dense in* X (or simply *dense*, if there is no ambiguity about X) *if* $\overline{A} = X$, *i.e. if every non-empty open set* U *of* X *meets* A.**

Let us transcribe this by means of the language of arrows.

A subset A of a topological space X is an arrow $A \longrightarrow X$. (Note we are making a choice here: there is an alternative translation analogous to the one used in the next sentence). An open subset U of X is an arrow $X \longrightarrow \{U \searrow U'\}$; here $\{U \searrow U'\}$ denotes the topological space consisting of one open point U and one closed point U'; by the arrow $\searrow$ we mean that that $U' \in cl(U)$. Non-empty: a subset U of X is *empty* iff the arrow $X \longrightarrow \{U \searrow U'\}$ factors as $X \longrightarrow \{U'\} \longrightarrow \{U \searrow U'\}$; here the map $\{U'\} \longrightarrow \{U \searrow U'\}$ is the obvious map sending U' to U'. set U of X meets A: $U \cap A = \emptyset$ iff the arrow $A \longrightarrow X \longrightarrow \{U \searrow U'\}$ factors as $A \longrightarrow \{U'\} \longrightarrow \{U \searrow U'\}$.

Collecting above (Figure 1c), we see that a map $A \xrightarrow{f} X$ has dense image iff

$$A \xrightarrow{f} X \rightthreetimes \{U'\} \longrightarrow \{U \searrow U'\}$$

Note a little miracle: $\{U'\} \longrightarrow \{U \searrow U'\}$ is the simplest map whose image isn't dense. We'll see it happen again.

8.0.2. *Kolmogoroff spaces, axiom T_0.* By definition [Bourbaki,I§1, Ex.2b; p.117/122],

b) A topological space is said to be a *Kolmogoroff space* if it satisfies the following condition : given any two distinct points x, x' of X, there is a neighbourhood of one of these points which does not contain the other. Show that an ordered set with the right topology is a Kolmogoroff space.

Let us transcribe this. given any two ... points x, x' of X: given a map $\{x, x'\} \xrightarrow{f} X$. two *distinct* points: the map $\{x, x'\} \xrightarrow{f} X$ does not factor through a single point, i.e. $\{x, x'\} \longrightarrow X$ does not factor as $\{x, x'\} \longrightarrow \{x = x'\} \longrightarrow X$. The negation of the sentence there is a neighbourhood which does not contain the other defines a topology on the set $\{x, x'\}$: indeed, the antidiscrete topology on the set $\{x, x'\}$ is the only topology with the property that there is [no] neighbourhood of one of these points which does not contain the other. Let us denote by $\{x \leftrightarrow x'\}$ the antidiscrete space consisting of x and x'. Now we note that the text implicitly defines the space $\{x \leftrightarrow x'\}$, and the only way to use it is to consider a map $\{x \leftrightarrow x'\} \xrightarrow{f} X$ instead of the map $\{x, x'\} \xrightarrow{f} X$.

Collecting above (see Figure 1d), we see that *a topological space X*

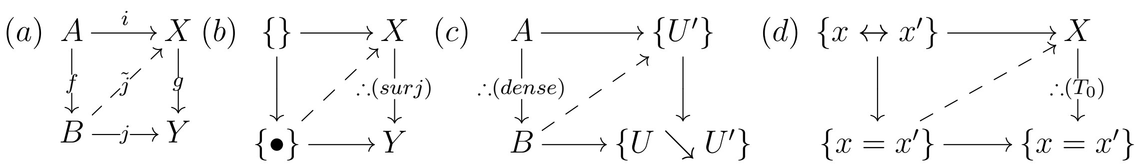

Figure 2: Lifting properties. Dots $\therefore$ indicate free variables and what property of these variables is being defined; in a diagram chasing calculation, "$\therefore$ (*dense*)" reads as: given a (valid) diagram, add label (*dense*) to the corresponding arrow.
(a) The definition of a lifting property $f \rightthreetimes g$: for each $i : A \longrightarrow X$ and $j : B \longrightarrow Y$ making the square commutative, i.e. $f \circ j = i \circ g$, there is a diagonal arrow $\tilde{j} : B \longrightarrow X$ making the total diagram $A \xrightarrow{f} B \xrightarrow{\tilde{j}} X \xrightarrow{g} Y, A \xrightarrow{i} X, B \xrightarrow{j} Y$ commutative, i.e. $f \circ \tilde{j} = i$ and $\tilde{j} \circ g = j$. (b) $X \longrightarrow Y$ is surjective (c) the image of $A \longrightarrow B$ is dense in B (d) X is Kolmogoroff/T_0

is said to be a Kolmogoroff *space iff any map* $\{x \leftrightarrow x'\} \xrightarrow{f} X$ *factors as* $\{x \leftrightarrow x'\} \longrightarrow \{x = x'\} \longrightarrow X$*.*

Note another little miracle: it also reduces to orthogonality of morphisms

$$\{x \leftrightarrow x'\} \longrightarrow \{x = x'\} \rightthreetimes X \longrightarrow \{x = x'\}$$

and $\{x \leftrightarrow x'\}$ is the simplest non-Kolmogoroff space.