...in memoriam...

CONVERGENCE AND HOMOTOPICAL TRIVIALITY ARE DEFINED BY THE SAME SIMPLICIAL FORMULA

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Abstract

We observe that convergence can be defined by a well-known simplicial construction which also defines homotopical triviality, in an appropriate simplicial category of generalised topological spaces, and also spell out how to obtain this observation by transcribing into simplicial language the Bourbaki definition of a limit point of a filter on a topological space. In particular, a metric space is complete iff all maps from 0-coskeletons to the associated generalised topological space are homotopically trivial.

1. *Summary*

A sequence $(a_i)_i$ *converges* to a iff the map $i \mapsto a_i$ is *homotopic* to the constant sequence (map) $i \mapsto a$: both convergence and homotopical triviality are defined by the same simplicial formula in a simplicial category of generalised topological spaces we introduce. This formula involving the décalage simplicial path space is implicit in the Bourbaki definition of convergence in terms of filters [**Bourbaki**, I§7.1]: it can be "read off" (and in fact it was) by rewriting the text of this definition in simplicial notation. Our category of generalised topological spaces is also implicit in [**Bourbaki**]: it is the category of simplicial objects of the category of filters (=sets equipped with a finitely additive probabilistic measure taking only values 0, 1, and "not measurable"), which is a basic notion Bourbaki choose to formulate notions of topology. This suggests that Bourbaki formulate notions of general topology in "right" generality appropriate for category theory.

Our construction is easy to describe explicitly: with a filter on a topological or uniform space, e.g. given by a sequence of points in a topological or metric space, we associate a morphism in the category of simplicial objects in a category of filters, and that morphism factors through the (décalage) simplicial path space if and only if the filter converges, and, moreover, liftings to the simplicial path space correspond precisely to the limit points. Recall that the *(décalage) simplicial path space* of a simplicial object is defined as the composition with the shift endofunctor $\Delta^{\rm op} \longrightarrow \Delta^{\rm op}$ adding a new minimal element to each finite linear order, and that in sSets homotopical triviality is defined by the same factorisation condition on a morphism with a connected domain and a fibrant codomain.

Hence, we see a homotopy theoretical point of view on the notion of convergence: we may say that *a filter converges on a topological or uniform space iff its associated morphism is homotopically trivial.* In particular, this allows us to say that a uniform

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... instances of human and animal behavior which are, on one hand, [...] miraculously complicated, on the other hand [of] little, if any, pragmatic (survival/reproduction) value, [thus they] were not the primarily targets specifically selected for by the evolution, [and therefore] are due to internal constraints on possible architectures of unknown to us functional "mental structures". —Gromov, Ergobrain.

space is *complete* iff each map from a 0-coskeleton to the space is homotopically trivial, and a topological space is *compact* iff any map to the space from an object associated with an ultrafilter is homotopically trivial, and propose definitions that conjecturally define compactness and completeness in terms of simplest examples of the properties, the discrete space with two points, and the real line.

Our category of generalised topological spaces is defined simply as the category of simplicial objects in the category of filters on sets, or, equivalently, in the category of sets equipped with finitely additive probabilistic measures taking values 0 and 1 only (but note that some subsets may be unmeasurable); a morphism of filters is a map of the underlying sets such that the preimage of a large subset is necessarily large. Thus *a generalised topological space is a simplicial set equipped, for each $n \geq 0$, with a filter, equiv. such (not quite) a measure, on the set of n-simplices such that under any face or degeneration map the preimage of a large set is large.*

These spaces generalise uniform and topological spaces, filters, and simplicial sets, and the concept is designed to be flexible enough to formulate categorically a number of standard basic elementary definitions in various fields [**S**], e.g. in analysis, limit, (uniform) continuity and convergence, equicontinuity of sequences of functions; in algebraic topology, being locally trivial and geometric realisation [**GP**]; in geometry, quasi-isomorphism; in model theory, stability, simplicity and several of the Shelah's dividing lines [**Z1**, **Z2**].

A logician should find amusing a metamathematical aspect—*how* trivial *is all we do here*: *take the text of the definition of a limit of a filter on a topological space in a standand textbook [***Bourbaki***, I§7.1]: and "transcribe" it line by line into the simplicial language* in an *oversimplified* manner. Moreover, this is exactly how the author did in fact wrote the formula this way back in 2018 [**mintsGE**, **6a6ywke**] *before* being made aware of its obvious interpretation in homotopy theory by V.Sosnilo in 2022, but *after* understanding how to reformulate the definition of a topological space. Thus, the analysis in §4.3 is historically accurate but not in §§4.1-4.2. One may view this historical incident as evidence that Bourbaki define notions of topology in terms appropriate for category theory, and the oversimplified method of transcribing text in terms of category theroy explained in §§4.1-4.3 does sometimes work.

[**G**] tries to explore this aspect by transcribing rather more basic textbook definitions including that of surjectivity, injectivity, and some others, and exploring the role of iterated lifting property as a useful negation in category theory allowing to concisely define notions in terms of simplest or archetypal (counter)examples.

Bourbaki and category theory. If one believes that Bourbaki defines the notions of general topology in generality appropriate for category theory, and tries to rewrite their definitions using simplicial notation as much as possible, perhaps our reformulation is not hard to notice following this line of thought.

Bourbaki defines basic topological notions in terms of filters (=sets equipped with filters) so we'd want to consider a category of filters and its category of objects. A filter $\mathfrak{F}$ on the set of points of a space X converges to a point $b \in X$ iff the map $X \to X \times X, x \mapsto (b, x)$ respects the filter $\mathfrak{F}$ on X and the filter of neighbourhoods of the diagonal on $X \times X$ in the sense that the preimage of a large subset is necessarily large. (To recover the definition of the limit of a sequence, take $\mathfrak{F}$ to be the filter of subsets containing cofinitely many elements of the sequence.) A map $X \to X \times X$ reminds of the representable simplicial set $X_\bullet := \mathrm{Hom}_{Sets}(-, X)$ and its dècalage endomorphism $X_\bullet \to X_\bullet \circ [+1]$, $(x_1, ..., x_n) \mapsto (a, x_1, ..., x_n)$ increasing dimension; here $[+1] : \Delta \longrightarrow \Delta, n \mapsto n+1$ denotes the endomorphism adding a new least element to each linear order. Equip each $X_{n-1} := X^n$ with the filter of neighbourhoods of the diagonal. Then *the filter* $\mathfrak{F}$ *converges to* $a \in X$ *iff the map* $a_\bullet : \mathrm{const}_\bullet \mathfrak{F} \longrightarrow X_\bullet \circ [+1]$, $x \mapsto (a, x, x, ..., x)$, *respects the filters*, where $\mathrm{const}_\bullet \mathfrak{F} = (\mathfrak{F}, \mathfrak{F}, ...)$ is the contant discrete object, or, equivalently, iff the obvious map $\mathrm{const}_\bullet \mathfrak{F} \longrightarrow X_\bullet$ decomposes as $\mathrm{const}_\bullet \mathfrak{F} \xrightarrow{a_\bullet} X_\bullet \circ [+1] \longrightarrow X_\bullet$.

Now, it is only left to notice that the same decomposition appears in homotopy theory and defines homotopy triviality. This relation is particularly easy to see if one notes that a map $\mathrm{Cone}\, A \longrightarrow X$ from the cone of a topological space determines a "shifted" map $\mathrm{sing}_\bullet A \longrightarrow \mathrm{sing}_\bullet X \circ [+1]$ of singluar complexes.

$X_\bullet \circ [+1]$ is known as the simplicial path space of $X_\bullet$, and it comes equipped with maps $X_\bullet \circ [+1] \longrightarrow X_\bullet$ (end of path), and $\mathrm{const}_\bullet(X_0) \longrightarrow X_\bullet \circ [+1] \longrightarrow \mathrm{const}_\bullet(X_0)$ (constant path, and start of path). In sSets for a fibrant $X_\bullet$ there is a sequence $\mathrm{const}_\bullet(X_0) \xrightarrow{(wc)} X_\bullet \circ [+1] \xrightarrow{(f)} X_\bullet$ (constant path, and start of path). where the first map is a (w)eak equivalence and a (co)fibration, and the second map is a (f)ibration. In sSets a map $F_\bullet \longrightarrow X_\bullet$ is homotopy equivalent to some map of form $F_\bullet \longrightarrow \mathrm{const}_\bullet(X_0) \longrightarrow X_\bullet$ iff it factors via $X_\bullet \circ [+1] \longrightarrow X_\bullet$, for fibrant $X_\bullet$. In particular, if $F_\bullet$ is also connected, we may say that a map $F_\bullet \longrightarrow X_\bullet$ is homotopically trivial iff it factors via the simplicial path space $X_\bullet \circ [+1] \longrightarrow X_\bullet$.

Thus, convergence and homotopical triviality are indeed defined by the same formula.

Some details. Now let us formulate our observation in more detail.

The *(décalage) simplicial path space* of a simplicial object $Y_\bullet : \Delta^{\mathrm{op}} \longrightarrow \boldsymbol{\varphi}$ in a category $\boldsymbol{\varphi}$, is defined as the composition $Y_\bullet \circ [+1]$ of $Y_\bullet$ with the shift endofunctor $[+1] : \Delta^{\mathrm{op}} \longrightarrow \Delta^{\mathrm{op}}$ which takes a linear order $[1 < ... < n]$ to $[0 < 1 < .. < n]$ by adding a new minimal element. The topological fact that a path space deforms into the subspace of constant paths has the following well known simplicial analogue, namely that in sSets the inclusion $\mathrm{sk}_0\, Y_0 \longrightarrow Y_\bullet$ decomposes as

$$\mathrm{sk}_0\, Y_0 \xrightarrow{(wc)} Y_\bullet \circ [+1] \xrightarrow{(f)} Y_\bullet$$

where the first map is a (w)eak equivalence and a (c)ofibration, and the second map is a (f)ibration whenever $Y_\bullet$ is fibrant [**Waldhausen**, Lemma 1.5.1]. In particular, in sSets for a fibrant $Y_\bullet$ a map $X_\bullet \longrightarrow Y_\bullet$ factors though the map $Y_\bullet \circ [+1] \longrightarrow Y_\bullet$ iff it is homotopy equivalent to a map $X_\bullet \to \mathrm{sk}_0\, Y_0$. In other words, as $\mathrm{sk}_0\, Y_0$ is a discrete object, this means that the map is homotopically trivial on each connected component of its domain.

The (décalage) simplicial path space is defined in an arbitrary category of simplicial objects, hence so is this notion of homotopical triviality.

Let $s\boldsymbol{\varphi} := \mathrm{Functors}(\Delta^{\mathrm{op}}, \boldsymbol{\varphi})$ denote our category of generalised topological spaces; here $\boldsymbol{\varphi}$ denotes the category of filters, or, equivalently, the category of finitely additive probabilistic measures taking values 0 and 1 only, where a morphism is a map of underlying sets such that the preimage of a large subset is necessarily large. With a filter $\mathfrak{F}$ on a topological or uniform space X we associate a morphism $\mathfrak{F}_\bullet \longrightarrow X_\bullet$ in $s\boldsymbol{\varphi}$ such that $\mathfrak{F}_\bullet \longrightarrow X_\bullet$ factors as $\mathfrak{F}_\bullet \longrightarrow X_\bullet \circ [+1] \longrightarrow X_\bullet$ iff the filter $\mathfrak{F}$ on X converges. Moreover, such factorisations correspond to limit points of $\mathfrak{F}$. The construction is particularly easy to describe explicitly in case of the limit of a sequence $(a_i)_{i\in\mathbb{N}} \in X$ in a metric space X. Let $\mathbb{N}$ denote the set of natural numbers equipped with the filter of cofinite subsets. Take $\mathfrak{F} := \mathbb{N}$ and let $\mathfrak{F}_\bullet = \mathbb{N}_\bullet := \operatorname{cosk}_0 \mathbb{N}_\bullet = (\mathbb{N}, \mathbb{N}\times\mathbb{N}, ...)$ to be 0-coskeleton of $\mathbb{N}$, where by 0-coskeleton we mean the right adjoint of the forgetful functor $s\boldsymbol{\varphi} \longrightarrow \boldsymbol{\varphi}$, $F_\bullet \mapsto F_0$. The underlying simplicial set of $X_\bullet$ is represented by the set $|X|$ of points of X; the filter on $X_{n-1} = X^n$, $n \geqslant 1$, is generated by ε-neighbourhoods of the diagonal $\{(x_1, ..., x_n) \in X^n : \operatorname{dist}(x_i, x_{i+1}) < \varepsilon, 1 \leqslant i \leqslant n\}$, $\varepsilon > 0$. The underlying simplicial set of $\operatorname{cosk}_0 \mathbb{N}_\bullet$ is connected, and that of $X_\bullet[+1]$ is a union of connected componets parametrised by $a \in X$; hence a lifting $\operatorname{cosk}_0 \mathbb{N}_\bullet \longrightarrow X_\bullet[+1]$ of $\operatorname{cosk}_0 \mathbb{N}_\bullet \longrightarrow X_\bullet$ picks a connected component, i.e. is determined by an element of $a \in X$. By definition, continuity of the map $\mathbb{N} \to X_1 = X \times X$, $x \mapsto (a, a_i)$ means that for each $\varepsilon > 0$ there is a cofinite subset $\delta \subset \mathbb{N}$ such that $\delta \subset \{(a, x) \in X \times X : \operatorname{dist}(a, x) < \varepsilon\}$, i.e. $\operatorname{dist}(a, a_i) < \varepsilon$ whenever $i \in \mathbb{N}$ is large enough, which is precisely the definition of the limit of a sequence. The same argument shows the map $\mathbb{N}_n \longrightarrow X_\bullet((n+1)^{\leqslant}), (i_1, ..., i_n) \mapsto (a, a_{i_1}, .., a_{i_n})$ is continuous for each n, and this gives a lifting $\mathbb{N}_\bullet \to X_\bullet \circ [+1]$. Continuity of the map $\mathbb{N} \times \mathbb{N} \to X_1 = X \times X$, $(i, j) \mapsto (a_i, a_j)$ means that for each $\varepsilon > 0$ there is $\delta \subset \mathbb{N}$ cofinite such that $\delta \times \delta \subset \{(x, y) \in X \times X : \operatorname{dist}(x, y) < \varepsilon\}$, i.e. $\operatorname{dist}(x, y) < \varepsilon$ whenever $i, j \in \mathbb{N}$ are large enough, which is precisely the definition of a Cauchy sequence. The same argument shows that the sequence $(a_i)_{i\in\mathbb{N}}$ is Cauchy iff the map $\mathbb{N}_\bullet \longrightarrow X_\bullet$ is continuous. In particular, a metric space is *complete* iff each map $\mathbb{N}_\bullet \longrightarrow X_\bullet$ lifts to the simplicial path space $X_\bullet \circ [+1] \longrightarrow X_\bullet$.

Structure of the paper. The goal of this note is rather limited: in §2 to state our observations about the convergence, compactness and completeness in a compact but complete manner, and in §4, to argue our observations can be easily obtained by transcribing the text of [**Bourbaki**, I§1.2,II§1.1,I§7.1] into simplicial language. In §3 we state a couple of questions about our category of generalised topological spaces asking whether our observations are part of a theory. In §5 we try to demonstrate the idea of transcribing in §4 on a simple example.

No attempt to study our category of generalised topological spaces is made, but rather we hope our reformulations in terms of $s\boldsymbol{\varphi}$ may help experts to see whether it would be worthwhile.

2. *Definitions and Reformulations.*

> Die Mathematiker sind eine Art Franzosen: Redet man zu ihnen, so übersetzen sie es in ihre Sprache, und dann ist es alsobald ganz etwas anderes.
>
> J.W. von Goethe. Maximen und Reflexionen.*

* Mathematicians are like Frenchmen: whatever you say to them they translate into their own language, and forthwith it is something entirely different. In: Johann Wolfgang von Goethe. Aphorismen und Aufzeichnungen. Nach den Handschriften des Goethe- und Schiller-Archivs hg. von Max Hecker, Verlag der Goethe-Gesellschaft, Weimar 1907. Aus dem Nachlass, Nr. 1005, Uber Natur und Naturwissenschaft. Maximen und Reflexionen.

2.1. *A definition of generalised topological spaces.* Below *a filter on a set X* means a set $\mathfrak{F}$ of subsets of X closed under finite intersection and such that every subset of X which contains a set in $\mathfrak{F}$, belongs to $\mathfrak{F}$. (Beware that Bourbaki also requires that $\varnothing \notin \mathfrak{F}$.) Subsets in $\mathfrak{F}$ are called *large* or *big* according to $\mathfrak{F}$. The category of finite linear orders is denoted by Δ, and the finite linear order with $n+1$ elements is denoted either by $[n]$ or $(n+1)^{\leqslant}$.[†]

Definition 1 (Continuous maps of filters). *Let X and Y be sets equipped with filters (resp. measures). Call a map $f : X \to Y$* continuous *iff the preimage of a big (resp. full measure) set is necessary big (resp. has full measure).*

Definition 2 (Generalised topological spaces). *Let $\boldsymbol{\varphi}$ denote the category formed by sets equipped with filters, and their continuous maps. Its category of simplicial objects*

$$s\boldsymbol{\varphi} := \mathit{Functors}(\Delta^{\mathrm{op}}, \boldsymbol{\varphi})$$

is our category of generalised topological spaces.

For brevity, it is convenient to refer to objects of $s\boldsymbol{\varphi}$ as situses. The category $s\boldsymbol{\varphi}$ of situses contains, as full subcategories, the categories of topological and uniform spaces: in $s\boldsymbol{\varphi}$ a topological, resp. uniform, space X is the simplicial set represented by the set of points of X, where $X \times X$ is equipped with the filter of non-uniform neighbourhoods of the diagonal of form $\bigsqcup_{U_x \text{ a neighbourhood of } x \in X} \{x\} \times U_x$, resp. the uniformity filter, and each X^n is equipped with the coarsest filter such that all the simplicial maps $X^n \to X \times X$ are continuous. The following two definitions express this in detail.

Definition 3 (A topological space as a generalised space). *Let X be a topological space. Let $X_\bullet : \Delta^{\mathrm{op}} \to \boldsymbol{\varphi}$ denote*

$$X_n := \mathit{Hom}_{\mathit{Sets}}([n], |X|) = |X|^{n+1}$$

where

- *$X = X_0$ is equipped with the indiscreet filter*
- *$X \times X = X_1$ is equipped with the filter of subsets of form*

$$\bigsqcup_{x \in X \text{ and } U_x \text{ is a neighbourhood of } x} \{x\} \times U_x$$

- *the filter on $X^n = X_{n-1}$ is the coarsest filter such that all the simplicial maps $X^n \longrightarrow X \times X$ are continuous (in other words, the simplicial dimension of $X_\bullet$ is at most 1.)*

This defines a fully faithful embedding of the category of topological spaces into $s\boldsymbol{\varphi}$.

Definition 4 (A uniform space as a generalised space). *A uniform space is a*

[†] Readers less familiar with the simplicial language might find it confusing that $[n]$ denotes something with $\neq n$ elements; for these readers we produced a version where the linear order with n elements is denoted by $n^{\leqslant}$.

symmetric simplicial object of dimension 1 *in the category* $\boldsymbol{\varphi}$ *of filters whose underlying simplicial set is representable, i.e. an object* $X_\bullet$ *of* $s\boldsymbol{\varphi}$ *which factors via a functor*

$$\Delta^{\mathrm{op}} \longrightarrow \textit{FiniteNon-emptySets} \longrightarrow \boldsymbol{\varphi}$$

such that

- *there is a set* X *such that* $X_n = \mathit{Hom}([n], X) = X^{n+1}$
- $X = X_0$ *is equipped with the indiscreet filter*
- *the filter on* $X^n = X_{n-1}$ *is the coarsest filter such that all the simplicial maps* $X^n \longrightarrow X \times X$ *are continuous, for each* $n > 0$

This defines a fully faithful embedding of the category of uniform spaces into $s\boldsymbol{\varphi}$. For a metric space X considered as a uniform space, the filter on $X_{n-1} = X^n$ can be described explicitly as generated by ε-neighbourhoods $\{(x_1, ..., x_n) : \forall 1 \le i, j \le n\, \mathrm{dist}(x_i, x_j) < \varepsilon\}$ of the diagonal where $\varepsilon > 0$.

2.1.1. *Intuition: A precise meaning for "n-tuple being sufficiently small" for* $n > 2$. A topological structure on a set enables one to give an exact meaning to the phrase "whenever x is *sufficiently near* a, x has the property $P(x)$", whereas *a generalised topological space enables one to give an exact meaning to the phase every n-tuple of* sufficiently similar *points* $x_1, x_2, ..., x_n$ *has property* $P(x_1, ..., x_n)$ *for* $n > 1$. (Uniform spaces were introduced to do this for $n = 2$ and "similar" meaning "at small distance", as explained in [**Bourbaki**, Introduction], "since *a priori* we have no means of comparing the neighbourhoods of two different points" of a topological space, yet "the notion of a pair of points near each other arises frequently in classical analysis (for example, in propositions involving uniform continuity)".) In a topological space, this exact meaning is that the set $\{x \,|\, P(x)\}$ belongs to the neighbourhood filter of a point a. Similarly, in a generalised topological space, it is that the set $\{(x_1, ..., x_n) \,|\, P(x_1, ..., x_n)\}$ belongs to the "neighbourhood" filter defined on n-simplices.

The case $n > 2$ is important in model theoretic examples [**Z1**, **Z2**](not discussed here) where similarity may mean either indiscernability or realising sufficiently many instances of a formula: n-simplices are tuples of elements of a model, and the "neighbourhood filter" on n-tuples consists of all subsets containing all "sufficiently" indiscernible tuples or realising "sufficiently many" instances of a formula. This exact meaning enables us to bring the standard intuition of topology to model theory.

2.2. *Convergence as contractability.*

Our main observation reformulates "convergence" in terms of the simplicial path space endofunctor, and various embeddings of $\boldsymbol{\varphi}$ into $s\boldsymbol{\varphi}$. To state it we need to fix a bit of notation.

Let $\mathfrak{F}$ be a filter on the set $|X|$ of points of a topological space X.

Let $X_\bullet^{\mathfrak{F}\text{-diag}}$ denote the simplicial set represented by set $|X|$ of points of X

$$X_\bullet^{\mathfrak{F}\text{-diag}}([n]) := \mathrm{Hom}([n], |X|)$$

where $X_0 = |X|$ is equipped with $\mathfrak{F}$, and each $X_\bullet^{\mathfrak{F}\text{-diag}}([n]) := \mathrm{Hom}([n], |X|) = |X|^{n+1}$ is equipped with the finest filter such that the diagonal map $X_\bullet^{\mathfrak{F}\text{-diag}}([0]) \to X_\bullet^{\mathfrak{F}\text{-diag}}([n])$ is continuous. Set-theoretically this means that a subset of X^n is big iff it contains $\{(x, ..., x) : x \in U\}$ for some $U \in \mathfrak{F}$ big.

Let $X_\bullet^{\mathfrak{F}\text{-diag}} \longrightarrow X_\bullet$ be the map which is identity on the underlying simplicial sets.

Let $\mathrm{sk}_0 : \boldsymbol{\varphi} \longrightarrow s\boldsymbol{\varphi}$ and $\mathrm{cosk}_0 : \boldsymbol{\varphi} \longrightarrow s\boldsymbol{\varphi}$ be the two embeddings $\boldsymbol{\varphi}$ into $s\boldsymbol{\varphi}$ given by the 0-skeleton and 0-coskeleton functors, i.e. by the left and right adjoint to the forgetful functor $s\boldsymbol{\varphi} \longrightarrow \boldsymbol{\varphi}$, $X_\bullet \mapsto X_0$. Recall that $\mathrm{sk}_0\,\mathfrak{F}_\bullet = (\mathfrak{F}, \mathfrak{F}, ...)$ is a constant (discrete) object, and $\mathrm{cosk}_0\,\mathfrak{F}_\bullet = (\mathfrak{F}, \mathfrak{F} \times \mathfrak{F}, ...)$ is formed by Cartesian powers of $\mathfrak{F}$.

Let $\{\bullet\}$ denote the set with a single element equipped with the degenerate filter where the empty set is large, and let $\{\bullet\}_\bullet := \mathrm{sk}_0\{\bullet\}_\bullet = \mathrm{cosk}_0\{\bullet\}_\bullet$. In $s\boldsymbol{\varphi}$ $\mathrm{Hom}(\{\bullet\}_\bullet, X_\bullet) = |X_0|$ is the set of 0-simplices $|X_0|$.

Recall that for a simplicial object $X_\bullet : \Delta^{\mathrm{op}} \longrightarrow \boldsymbol{\varphi}$ in a category, its associated *simplicial path space* object is defined as the composition of $X_\bullet$ with the shift functor $[+1] : \Delta^{\mathrm{op}} \longrightarrow \Delta^{\mathrm{op}}$ which takes $[1 < ... < n]$ to $[0 < 1 < .. < n]$ (by "sending i to i" — this fixes the behaviour on morphisms). The fact that a path space deforms into the subspace of constant paths has the following well known simplicial analogue, namely that in sSets $X_\bullet \circ [+1] \longrightarrow \mathrm{sk}_0\,X_0$ is a homotopy equivalence [**Waldhausen**, Lemma 1.5.1].

$X_\bullet \circ [+1]$ comes equipped with two projections $X_\bullet \circ [+1] \longrightarrow X_\bullet$, (one is induced by the 0-face map of $X_\bullet$ which is not otherwise used in $X_\bullet \circ [+1]$, and the other one is induced by the inclusions $[1 < ... < n] \subset [0 < ... < n]$), and there is an inclusion of X_1 considered as a constant simplicial object (because $X_\bullet \circ [+1]([0]) = X_1$). There results a sequence $\mathrm{sk}_0\,X_{1\bullet} \longrightarrow X_\bullet \circ [+1] \longrightarrow X_\bullet \times X_\bullet$. [**Waldhausen**, §1.5].

Theorem 1 (Convergence as homotopical triviality).

(i) *To give a limit point of a filter $\mathfrak{F}$ on a topological or uniform space X is the same as to give a lifting of the obvious map $X_\bullet^{\mathfrak{F}\text{-diag}} \longrightarrow X_\bullet$ through the simplicial path space $X \circ [+1]$*

$$\begin{tikzcd} & X_\bullet \circ [+1] \arrow[d, "pr_{1,2,3,...}"] \\ X_\bullet^{\mathfrak{F}\text{-diag}} \arrow[ur, dashed, "x_\bullet"] \arrow[r] & X_\bullet \end{tikzcd}$$

(ii) *A constant map $x_\bullet : \{\bullet\}_\bullet \to X_\bullet$ corresponds to a limit point of $\mathfrak{F}$ on a topological or uniform space X iff it fits into a commutative diagram*

$$\begin{tikzcd} & X_\bullet \circ [+1] \arrow[d, "pr_{1,2,3,...}"] \arrow[drrr, "pr_0"] & & & \\ \mathrm{sk}_0\,\mathfrak{F}_\bullet \arrow[ur, dashed] \arrow[r] \arrow[rr, bend right] & X_\bullet & \{\bullet\}_\bullet \arrow[rr, "x_\bullet"] & & X_\bullet \end{tikzcd}$$

(iii) *A filter $\mathfrak{F}$ on a uniform space X is Cauchy iff the obvious map* $\mathrm{cosk}_0\,\mathfrak{F}_\bullet \longrightarrow X_\bullet$ *is continuous.*

(iv) *To give a limit point of a filter $\mathfrak{F}$ on a uniform space X is the same as to give a lifting of the obvious map* $\mathrm{cosk}_0\,\mathfrak{F}_\bullet \longrightarrow X_\bullet$ *through the simplicial path space $X \circ [+1]$*

$$\begin{tikzcd} & X_\bullet \circ [+1] \arrow[d, "pr_{1,2,3,...}"] \\ \mathrm{cosk}_0\,\mathfrak{F}_\bullet \arrow[ur, dashed, "x_\bullet"] \arrow[r] & X_\bullet \end{tikzcd}$$

Proof. (i). Consider the case when X is a topological space. The other case is similar to (iv). First note that the underlying simplicial set $X_\bullet \circ [+1]$ is a disjoint union of copies of the underlying simplicial set of $X_\bullet$ indexed by the points of X

$$|X|_\bullet \circ [+1] = \bigsqcup_{x\in X} |X|_\bullet$$

and that the underlying simplicial set of $\mathfrak{F}_\bullet$ is connected. Hence, any possibly discontinuous map $X_\bullet^{\mathfrak{F}\text{-diag}} \longrightarrow X_\bullet \circ [+1]$ sends everything in one of the copies, and that copy can be arbitrary.

That is, the map $X_\bullet^{\mathfrak{F}\text{-diag}} \longrightarrow X_\bullet \circ [+1]$ is of the form

$$(x_1, ..., x_n) \in X^n \longmapsto (a, x_1, ..., x_n) \in X^{n+1}, n > 0$$

for some point $a \in X$. Take $n = 1$. By definition of continuity and the filter on $X \times X$, the map $x \in X \longmapsto (a, x) \in X \times X$ is continuous iff each neighbourhood $U_a \ni a$ of a is $\mathfrak{F}$-big, i.e. point a is a limit point of $\mathfrak{F}$. Now take $n > 1$. It is enough to show that continuity of $x \in X \longmapsto (a, x) \in X \times X$ implies that of the map $(x_1, ..., x_n) \in X^n \longmapsto (a, x_1, ..., x_n) \in X^{n+1}$ for each $n > 1$. By definition it is continuous iff for each big subset $U \subset X^{n+1}$ there is a subset $U_a \in \mathfrak{F}$ big such that $\{(a, x, ..., x) : x \in U_a\} \subset U$. This follows from the following description of the filter on X^{n+1}: a subset $U \subset X^{n+1}$ is big iff for each $x_0 \in X$ there is a neighbourhood $U_{x_0} \ni x_0$ such that for each $x_1 \in X$ there is a neighbourhood $U_{x_1} \ni x_1$ such that .... for each $x_n \in X$ there is a neighbourhood $U_{x_n} \ni x_n$ it holds $(x_0, x_1, ..., x_n) \in U$.

(ii) Same as (i) but the arrow $\{\bullet\}_\bullet \longrightarrow X_\bullet$ is used instead of connectivity of $X_\bullet^{\mathfrak{F}\text{-diag}}$.

(iii) Recall that a filter $\mathfrak{F}$ is Cauchy iff for each subset $\varepsilon \subset X \times X$ big in the uniformity filter $\mathfrak{U}$ on $X\times X$ there is a subset $\delta \in \mathfrak{F}$ such that $\delta\times\delta \subset \varepsilon$. This is precisely the definition of continuity of the inclusion map $\mathfrak{F} \times \mathfrak{F} \longrightarrow \mathfrak{U}$. Hence, continuity of $\operatorname{cosk}_0 \mathfrak{F} \longrightarrow X_\bullet$ is necessary. To verify that it is sufficient, note that the filter on X^n is generated by subsets $\{(x_1, ..., x_n) \in X^n : \forall 1 \leqslant i \leqslant j \leqslant n\, (x_i, x_j) \in \varepsilon\}, \varepsilon \in \mathfrak{U}$.

(iv) Recall that a limit point of a filter $\mathfrak{F}$ on the set $|X|$ of points of a uniform space X is defined to be a limit point of $\mathfrak{F}$ on the set $|X|$ of points of the topological space associated with the uniform space X. Let $\mathfrak{U}$ denote the uniformity filter on $X \times X$. Recall that in the associated topological space the neighbourhood filter of a point $a \in X$ is defined to be $U \cap \{a\} \times X$, $U \in \mathfrak{U}$. Thus $a \in X$ is a limit point of a filter $\mathfrak{F}$ on the set $|X|$ of points of a uniform space X iff for each neighbourhood $U \in \mathfrak{U}$ of the diagonal $U \cap \{a\} \times X \in \mathfrak{F}$.

This is equivalent to continuity of $\mathfrak{F} \longrightarrow \mathfrak{U}, x \mapsto (a, x)$ by the definition of continuity.

Now it is only left to show that if a is a limit point of $\mathfrak{F}$, then maps $\mathfrak{F} \times ... \times \mathfrak{F} \longrightarrow X\times X\times....\times X$, $(x_1, ..., x_n) \mapsto (a, x_1, ..., x_n)$ are continuous. The filter on $X\times X\times....\times X$ is generated by subsets $\{(x_0, x_1, ..., x_n) : (x_i, x_{i+1}) \in U\}, U \in \mathfrak{U}$.

Recall that by definition of uniform structure we may pick a neighbourhood $V \in \mathfrak{U}$ of the diagonal such that $\{(x, z) : \exists y((y, x) \in V \& (y, z) \in V))\} \subset U$. This follows from the continuity of maps $X \times X \times X \longrightarrow X \times X, (x, y, z) \mapsto (x, z)$ and $X \times X \longrightarrow X\times X, (x, y) \mapsto (y, x)$. Taking neighbourhood $V_a := V \cap \{a\}\times X$ we see that $V_a\times V_a \subset U$ and that $V_a \in \mathfrak{F}$. Then $V_a \times ... \times V_a \subset \{(x_0, x_1, ..., x_n) : \forall 1 \leqslant i \leqslant n\, (x_i, x_{i+1}) \in U\}$, and thus the map in question is continuous. □

2.3. *A homotopical interpretation of convergence.* As mentioned above, it is standard to think of $X_\bullet \circ [+1]$ as a *simplicial model of the path space* of $X_\bullet$. In sSets, $X_\bullet \circ [+1]$ is simplicially homotopic to the constant simplicial object $[n] \mapsto X_0$, i.e. to $\mathrm{sk}_0 X_{0\bullet}$ [**Waldhausen**, Lemma 1.5.1].

This and Theorem 1 justifies the terminology in the following definition.

Definition 5. *We say that a morphism $A_\bullet \xrightarrow{f} X_\bullet$* converges *or* contracts *to a morphism $A_\bullet \xrightarrow{\tilde f} X_\bullet \circ [+1]$ iff $f = \tilde f \circ pr_0$.*

If the underlying simplicial set of the domain $A_\bullet$ is connected, we may also say that a morphism $A_\bullet \xrightarrow{f} X_\bullet$ converges *or is* contractible *iff there is a morphism it converges or contracts to.*

In this terminology, Theorem 1 can be expressed as:

- A filter $\mathfrak{F}$ on a topological or uniform space $X_\bullet$ converges iff
$$X_\bullet^{\mathfrak{F}\text{-diag}} \to X_\bullet \text{ is contractible.}$$
- A filter $\mathfrak{F}$ on a topological or uniform space $X_\bullet$ is converges to a point $a \in X$ iff $\mathrm{sk}_0 \mathfrak{F} \to X_\bullet$ contracts to $\mathrm{sk}_0 \mathfrak{F} \longrightarrow \{\bullet\}_\bullet \xrightarrow{a_\bullet} X_\bullet$ where $\{\bullet\}_\bullet \xrightarrow{a_\bullet} X_\bullet$ is the constant map sending each $(\bullet, ..., \bullet) \mapsto (a, .., a)$.
- A Cauchy filter $\mathfrak{F}$ on a uniform space $X_\bullet$ is convergent iff $\mathrm{cosk}_0 \mathfrak{F} \to X_\bullet$ is contractible.

2.3.1. *Homotopy of topological spaces in terms of $s\boldsymbol{\varphi}$.* The definition above allows us to consider convergence of families of topological or uniform spaces rather than points, i.e. we may ask whether a map $X_\bullet \times \mathfrak{F}_\bullet \longrightarrow Y_\bullet$ converges.

Take $\mathfrak{F}$ to be the filter $[0,1]_0$ of neighbourhoods of point $0 \in [0,1]$, let X be a locally compact topological space, and let Y be a metric uniform space. A verification shows that maps $h_\bullet : X_\bullet \times ([0,1]_0)_\bullet \longrightarrow Y_\bullet$ correspond to (possibly discontinuous!) maps $h : X \times [0,1] \longrightarrow Y$ such that there is $\delta > 0$ such that $h_{|[0,\delta]} : X \times [0,\delta] \longrightarrow Y$ is a (continuous) homotopy. (For non-locally compact spaces we'd need to also require that for each $\varepsilon > 0$ for each $x \in X$ there is a neighbourhood $U_x \ni x$ and $\delta = \delta(x,\varepsilon) > 0$ such that the diameter of $h(U_x \times [0,\delta])$ is less than ε. If we were to consider here X as a uniform space, in this last condition we'd take $\delta = \delta(\varepsilon)$ to be independent of x.)

A map $h : X_\bullet \times ([0,1]_0)_\bullet \longrightarrow Y_\bullet$ converges iff there is $\varepsilon > 0$ such that $h_{|[0,\varepsilon]} : X \times [0,\varepsilon] \longrightarrow Y$ is a homotopy contracting X in Y to a point $y_0 \in Y$, and, moreover, $h : X \times [0,1] \longrightarrow Y$ is continuous in the preimage of an open neighbourhood of y_0. In an expressive language, we may say that *a family $h_t : X \longrightarrow Y$, $t \in [0,1]$, converges at* 0. For a subspace $B \subset Y$, maps $h_\bullet : X_\bullet \times ([0,1]_0)_\bullet \longrightarrow Y_\bullet \circ [+1] \times_{Y_\bullet} B_\bullet$ correspond to (possibly discontinuous!) maps $h : X \times [0,1] \longrightarrow Y$ such that $h(0,X) \subset B$ and there is $\varepsilon > 0$ such that $h_{|[0,\varepsilon]} : X \times [0,\varepsilon] \longrightarrow Y$ is a (continuous) homotopy.

$s\boldsymbol{\varphi}$ allows to reformulate a few other notions. To verify the reformulations below one only needs to rewrite the definitions in the usual language.

Let $\mathbb{N}$ denote the set of natural numbers equipped with the filter of cofinite subsets.

Let X, Y be topological spaces, and let M be a metric space.

Recall that a family $f_i : X \longrightarrow Y$ of functions is *equicontinuous* iff for every $x \in X$ and $\varepsilon > 0$, there exists a neighbourhood U of x such that $d_Y(f_i(x'), f_i(x)) \le \varepsilon$ for all

$i \in \mathbb{N}$ and $x' \in U$. In $s\boldsymbol{\varphi}$ this means exactly that the map

$$\mathrm{sk}_0\,\mathbb{N}_\bullet \times X_\bullet \longrightarrow M_\bullet$$

$$(i, x_1, ..., x_n) \longmapsto (f_i(x_1), ..., f_i(x_n))$$

is continuous.

Recall that a family $f_i : X \longrightarrow M$ of functions on a metric space N to a metric space M is *uniformly equicontinuous* iff for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $d_Y(f_i(x'), f_i(x)) \leq \varepsilon$ for all $i \in \mathbb{N}$ and $x', x \in N$ with $d_X(x, x') \leq \delta$. In $s\boldsymbol{\varphi}$ this means exactly continuity of the map

$$\mathrm{sk}_0\,\mathbb{N}_\bullet \times N_\bullet \longrightarrow M_\bullet$$

$$(i, x_1, ..., x_n) \longmapsto (f_i(x_1), ..., f_i(x_n))$$

An equicontinuous sequence $f_i : X \to M$ of functions converges uniformly to a function $f : X \to M$ iff the map

$$\mathrm{sk}_0\,\mathbb{N}_\bullet \times X_\bullet \circ [+1] \longrightarrow M_\bullet \circ [+1]$$

$$(i, x_0, x_1, ..., x_n) \longmapsto (f(x_0), f_i(x_1), ..., f_i(x_n))$$

is continuous.

If $X = (X, d_X)$ is also a metric space and in the $s\boldsymbol{\varphi}$ expression we use $X_\bullet$ to denote the $s\boldsymbol{\varphi}$ object corresponding to the metric space, we get the definition of *uniformly equicontinuity* and *uniform equicontinuous convergence.*

A map $f : X \to Y$ of topological or metric spaces is *locally trivial* with fibre F iff in $s\boldsymbol{\varphi}$ it becomes a direct product with $F_\bullet$ (="globally trivial") after base-change to the simplicial path space $Y_\bullet \circ [+1] \to Y_\bullet$. That is, $f_\bullet : (Y_\bullet \circ [+1]) \times_{Y_\bullet} X_\bullet \to Y_\bullet \circ [+1]$ is of form $(Y_\bullet \circ [+1]) \times F_\bullet \to Y_\bullet \circ [+1]$. Note that in the category of topological spaces the same condition using the (usual) covering space needs to assume that the fibre is discrete.

2.4. *Compact and complete.* This terminology allows to use homotopy theory language to reformulate in $s\boldsymbol{\varphi}$ the definition of completeness in terms of Cauchy filters[**Bourbaki**, II§3.1,Def.2], and the characterisation of compactness in terms of ultrafilters [**Bourbaki**, I§10.2,Th.1(d)].

THEOREM 2.

(i) *A uniform space M is complete iff any map $\mathrm{cosk}_0\,\mathfrak{F}_\bullet \to M_\bullet$ is contractible for any $\mathfrak{F} \in \boldsymbol{\varphi}$.*

(ii) *A topological or uniform space K is compact iff for each set X and each ultrafilter $\mathfrak{U} \in \boldsymbol{\varphi}$ on X any map $X_\bullet^{\mathfrak{U}\text{-diag}} \to K_\bullet$ is contractible.*

(iii) *A topological or uniform space K is compact iff for each ultrafilter $\mathfrak{U} \in \boldsymbol{\varphi}$ any map $\mathrm{sk}_0\,\mathfrak{U}_\bullet \to K_\bullet$ contracts to a map factoring through the point $\{\bullet\}_\bullet$.*

Proof. (i) For a map $f : |X| \longrightarrow |K|$, define a filter $f^{-1}(\mathfrak{F})$ on a set $|K|$ of points of a space K by $\mathfrak{F}' := \{U \subset |X| : f^{-1}(U) \in \mathfrak{F}\}$. One can check that existence of lifting for a map $X_\bullet^{\mathfrak{U}\text{-diag}} \longrightarrow K_\bullet$ induced by a map $f : |X| \longrightarrow |K|$ is equivalent to existence of lifting for the map $K_\bullet^{f^{-1}(\mathfrak{F})\text{-diag}} \longrightarrow K_\bullet$, which is claimed by the previous theorem. (ii-iii) is similar. □

2.4.1. *Compact and complete in terms of the lifting property.* We need to first introduce notations and basics on the lifting property [**LP1**, **LP2**].

Recall that a morphism i in a category has the *left lifting property* with respect to a morphism p, and p also has the *right lifting property* with respect to i, denoted $i \rightthreetimes p$, iff for each $f : A \to X$ and $g : B \to Y$ such that $p \circ f = g \circ i$ there exists $h : B \to X$ such that $h \circ i = f$ and $p \circ h = g$.

For a class P of morphisms in a category, its *left orthogonal* $P^{\rightthreetimes l}$ with respect to the lifting property, respectively its *right orthogonal* $P^{\rightthreetimes r}$, is the class of all morphisms which have the left, respectively right, lifting property with respect to each morphism in the class P. In notation,

$$P^{\rightthreetimes l} := \{i \;:\; \forall p \in P \; i \rightthreetimes p\}, P^{\rightthreetimes r} := \{p \;:\; \forall i \in P \; i \rightthreetimes p\}, P^{\rightthreetimes lr} := (P^{\rightthreetimes l})^{\rightthreetimes r}, ..$$

Taking the orthogonal of a class P is a simple way to define a class of morphisms excluding non-isomorphisms from P, in a way which is useful in a diagram chasing computation, and is often used to define properties of morphisms starting from an explicitly given class of (counter)examples. For this reason, it is convenient and intuitive to refer to $P^{\rightthreetimes l}$ and $P^{\rightthreetimes r}$ as *left, resp. right, Quillen negation* of property P.

THEOREM 3.
(i) *A topological space K is compact iff for each ultrafilter $\mathfrak{U}$ on each set X it holds*

$$\bot \longrightarrow X_\bullet^{\mathfrak{U}\text{-}diag} \rightthreetimes K_\bullet \circ [+1] \longrightarrow K_\bullet$$

(ii) *A uniform space M is complete iff for each filter $\mathfrak{F} \in \mathrm{Ob}\, s\boldsymbol{\varphi}$*

$$\bot \longrightarrow \mathrm{cosk}_0\, \mathfrak{F}_\bullet \rightthreetimes M_\bullet \circ [+1] \longrightarrow M_\bullet$$

The theorem implies that for any compact space K' it holds that

$$K_\bullet \circ [+1] \longrightarrow K_\bullet \in \{K'_\bullet \circ [+1] \longrightarrow K'_\bullet\}^{\rightthreetimes lr} \text{ implies } K \text{ is compact}$$

and that for any complete uniform space M' it holds that

$$M_\bullet \circ [+1] \longrightarrow M_\bullet \in \{M'_\bullet \circ [+1] \longrightarrow M'_\bullet\}^{\rightthreetimes lr} \text{ implies } M \text{ is complete.}$$

The intuition behind the terminology “Quillen negation” leads to the following (oversimplified?) conjecture defining compactness and completeness in terms of the “double negation” of the simplest examples of the properties.

CONJECTURE 1. *Let $\{a, b\}$ denote the discrete topological space with two points, and let $\mathbb{R}$ denote the real line with the usual metric.*

- *A topological or uniform K is compact iff*

$$K_\bullet \circ [+1] \longrightarrow K_\bullet \in \{\{a, b\}_\bullet \circ [+1] \longrightarrow \{a, b\}_\bullet\}^{\rightthreetimes lr}$$

- *A uniform space M is complete iff*

$$M_\bullet \circ [+1] \longrightarrow M_\bullet \in \{\mathbb{R}_\bullet \circ [+1] \longrightarrow \mathbb{R}_\bullet\}^{\rightthreetimes lr}$$

As evidence, we mention that the lifting property

$$\bot \longrightarrow X_\bullet^{\mathfrak{U}\text{-diag}} \rightthreetimes \{a, b\}_\bullet \circ [+1] \longrightarrow \{a, b\}_\bullet$$

is a concise way to write the usual definition of an ultrafilter: to take an arbitrary map $X_\bullet^{\mathfrak{U}\text{-diag}} \longrightarrow \{a,b\}_\bullet$ is to split X into a subset and its complement (preimages of a and b), and the lifting map picks which of them is large in $\mathfrak{U}$.

2.4.2. *Compact and contractible in terms of finite topological spaces* We need to introduce a notation for maps of finite topological spaces. Our notation represents finite topological space as preorders or finite categories with each diagram commuting, and is hopefully self-explanatory; see [**LP2**] for details. In short, a short arrow $o \to c$ indicates that $c \in \mathrm{cl}\, o$, and each point goes to "itself"; the list in $\{..\}$ after the longer arrow indicates new relations/morphisms added, thus in $\{o \to c\} \longrightarrow \{o = c\}$ the equality indicates that the two points are glued together or that we added an identity morphism between o and c. The expression

$$\{o, c\} \longrightarrow \{o \to c\} \longrightarrow \{o \leftrightarrow c\} \longrightarrow \{o = c\}$$

denotes the sequence of maps from the discrete space with two points, to the space with one point o open and one point c closed, to the indiscreet space, and then finally gluing the two points together.

[**V**, Lemma 3.2.1] characterises compactness for Hausdorff topological spaces in terms of the iterated lifting property and finite topological spaces:

- *a Hausdorff space K is compact iff*

$$K \to \{o\} \in (\{\{o\} \longrightarrow \{o \to c\}\}^r_{<5})^{\measuredangle lr}$$

Conjecturally, $(\{\{o\} \longrightarrow \{o \to c\}\}^r_{<5})^{\measuredangle lr}$ is the class af all proper maps (recall that proper means closed with compact fibres). $\{o\} \longrightarrow \{o \to c\}$ denotes the inclusion of the open point o into the two point space with one point open and one closed, and is an archetypal example of a non-proper map (=not closed, for maps of finite spaces).

[**V**, Lemma 3.3.1] characterises contractability for finite CW complexes:

- *A finite CW complex W is contractible iff*

$$W \to \{o\} \in \{\{a \leftarrow u \to x \leftarrow v \to b\} \longrightarrow \{a \leftarrow u = x = v \to b\}\}^{lr}$$

The map above is a trivial Serre fibration (and is a finite model of the barycentric subdivision of the interval), hence this orthogonal is contained in the class of trivial Serre fibrations, and conjecturally a cellular map of finite CW complexes is a trivial fibration iff it lies in this orthogonal.

In fact, the same map is used to reformulate separation axiom T4 (normal), and a number of other basic topological properties can be defined in terms of similar iterated orthogonals (lifting property) starting with simple(st?) (counter)examples of topological properties which are maps of finite topological spaces [**V**, Lemma 3.1.1],[**LP2**].

It is unclear how this relates to the conjecture above. An interesting question is how to interpret this concise notation for iterated orthogonals of maps of finite topological spaces (=preorders) in $s\mathfrak{P}$.

3. *Further questions*

3.1. *Homotopy theory for $s\mathfrak{P}$?*

We saw that $s\mathfrak{P}$ allows one to see a homotopy theory point of view on the definition of convergence, and below we mention a few other examples. But as far as we are aware, no homotopy theory for $s\boldsymbol{\varphi}$ has been developed.

QUESTION 1. *Develop a homotopy theory for $s\boldsymbol{\varphi}$ which captures both convergence and the (usual) homotopy theory for the category of topological spaces.*

In particular, give a homotopy-theoretic meaning to various Arzela-Ascoli theorems [**Grothendieck**, **benYaacov**]. In our treatment of compactness and completeness, we only care about simplicial dimension ≤ 1, and in Theorem 1 nothing chances if we replace the endomorphism $[+1] : \Delta^{\mathrm{op}} \longrightarrow \Delta^{\mathrm{op}}$ by any power $[+1]\circ\ldots\circ[+1] : \Delta^{\mathrm{op}} \longrightarrow \Delta^{\mathrm{op}}$. It should say something non-trivial about spaces associated with models of a first-order logic [**Z1**, **Z2**], local triviality, and quasi-isometries. In $s\boldsymbol{\varphi}$ the latter two notions are reformulated as follows.

A map $f : X \to Y$ of topological or metric spaces is locally trivial with fibre F iff in $s\boldsymbol{\varphi}$ it becomes a direct product with $F_\bullet$ (="globally trivial") after base-change to the simplicial path space $Y_\bullet \circ [+1] \to Y_\bullet$. That is, $f_\bullet : (Y_\bullet \circ [+1]) \times_{Y_\bullet} X_\bullet \to Y_\bullet \circ [+1]$ is of form $(Y_\bullet \circ [+1]) \times F_\bullet \to Y_\bullet \circ [+1]$.

Take a metric space M and equip each M^n with the filter such that a subset of M^n is large iff it contains all n-tuples such that the distance between distinct points is at least D, for some $D \geqslant 0$. For quasi-geodesic metric spaces M and N, with $M_\bullet$ and $N_\bullet$ so defined, a map $f_\bullet : M_\bullet \to N_\bullet$ is an isomorphism in $s\boldsymbol{\varphi}$ iff $f : M \to N$ is a quasi-isometry.

[**GP**] equips the internal hom of the underlying simplicial sets $\mathrm{Hom}_{sSets}(|X|_\bullet, |Y|_\bullet)$ with filters in a manner depending functorially on the filters on $Y_\bullet$ but not $X_\bullet$. The constructions is reminiscent of the Levi-Prokhorov or Skorokhod metric on the space of semi-continuous functions. Recall that the topological simplex Δ_N can be defined as the Skorokhod space of upper semi-continuous (=order preserving, up to measure 0, in this case) functions $[0,1]^{\leqslant} \longrightarrow [0 < 1 < \ldots < N]$ with the Levi-Prokhorov metric [**Grayson**, Remark 2.4-1.6], and this observation leads to an endofunctor $\mathrm{Hom}([0,1]^{\leqslant}_\bullet, -) : s\boldsymbol{\varphi} \longrightarrow s\boldsymbol{\varphi}$ rephrasing the Besser-Drinfeld-Grayson construction of geometric realisation in terms of $s\boldsymbol{\varphi}$.

The following question contains a suggestion towards a model category structure on $s\boldsymbol{\varphi}$.

QUESTION 2. *Can one define "contractible" and, more generally, "trivial fibration", or "compact" and "complete", in terms of a simple example starting from a finite simplicial object, similarly to the characterisation of contractible in [**V**, Lemma 3.3.1] ? Is being compact, complete, and proper related to being fibrant ?*

The following is probably not too hard to calculate, and the answer may be revealing if non-trivial.

EXERCISE 1. *Calculate in $s\boldsymbol{\varphi}$*

$$\{\{a \leftarrow u = x = v \rightarrow b\}_\bullet \circ [+1] \longrightarrow \{a \leftarrow u = x = v \rightarrow b\}_\bullet\}^{\prec lr}$$

$$\{\{a, b\}_\bullet \circ [+1] \longrightarrow \{a, b\}_\bullet\}^{\prec lr}$$

$$\{\perp \to \{o\}_\bullet\}^{\prec rll}$$

$$\{\perp \to \{o\}_\bullet\}^{\rightthreetimes\, lrrrl}$$

$$\{\mathbb{R}_\bullet \circ [+1] \longrightarrow \mathbb{R}_\bullet\}^{\rightthreetimes lr}$$

$$\begin{array}{rcl}
\{K_\bullet \circ [+1] \longrightarrow K_\bullet & : & K \textit{ is a compact topological space}\}^{\rightthreetimes lr} \\
\{M_\bullet \circ [+1] \longrightarrow M_\bullet & : & M \textit{ is a complete uniform space}\}^{\rightthreetimes lr} \\
\{S_\bullet \circ [+1] \longrightarrow S_\bullet & : & S \textit{ is a contractible fibrant simplicial set}\}^{\rightthreetimes lr} \\
\{\operatorname{sing} X_\bullet \circ [+1] \longrightarrow \operatorname{sing} X_\bullet & : & X \textit{ is a nice enough contractible space}\}^{\rightthreetimes lr}
\end{array}$$

$$\left\{\begin{array}{rcl}
K_\bullet \circ [+1] \longrightarrow K_\bullet & : & K \textit{ is a compact topological space} \\
M_\bullet \circ [+1] \longrightarrow M_\bullet & : & M \textit{ is a complete uniform space} \\
S_\bullet \circ [+1] \longrightarrow S_\bullet & : & S \textit{ is a contractible fibrant simplicial set} \\
\operatorname{sing} X_\bullet \circ [+1] \longrightarrow \operatorname{sing} X_\bullet & : & X \textit{ is a nice enough contractible space}
\end{array}\right\}^{\rightthreetimes lr}$$

For a simplicial set S, here $S_\bullet \in s\boldsymbol{\varphi}$ denotes one of the several embeddings sSets $\longrightarrow s\boldsymbol{\varphi}$, e.g. when each S_{n-1} is equipped with the indiscreet filter, the degenerate filter where $\varnothing$ is large, or S_0 is equipped with the indiscreet filter, and each S_n is equipped with the finest filter such that the simplicial diagonal map $S_0 \longrightarrow S_n$ is continuous. For a topological or uniform space X, here $\operatorname{sing} X_\bullet$ denotes its singular complex, possibly equipped with some filters coming from the topological or uniform structure on X; "nice enough" means that the space is such that sing is well-behaved. The motivation for $\{\perp \to \{o\}_\bullet\}^{\rightthreetimes\, rll}$ and $\{\perp \to \{o\}_\bullet\}^{\rightthreetimes\, lrrrl}$ is that in the category of topological spaces similar expressions are meaningful: $\{\varnothing \to \{o\}\}^{\rightthreetimes\, rll}$ defines connectedness, and $\{\varnothing \to \{o\}\}^{\rightthreetimes\, lrrrl}$ is the class of quotient maps [**V**, Lemma 3.3.1]. Here $\{o\}$ denotes the singleton equipped with either the degenerate or the non-degenerate filter, and $\{o\}_\bullet$ the corresponding constant object.

3.2. *Model theory in* $s\boldsymbol{\varphi}$ *?* A generalised topological space enables one to give an exact meaning to the phase every n-tuple of *sufficiently similar* points $x_1, x_2, ..., x_n$ has property $P(x_1, ..., x_n)$ for $n > 1$, as mentioned in §2.1.1, and this allows one to construct $s\boldsymbol{\varphi}$ spaces which talk about important properties of tuples in model theory (indiscernablity, realising sufficiently many instances of a formula) which are properties of n-tuples for n arbitrarily large.

Shelah's classification theory in model theory is concerned with classifying models of a first order theory up to isomorphism, and particularly with the number (cardinality) of different isomorphism types of models of a first order theory, A number of so called dividing lines of Shelah (properties of models and theories) can be reformulated as lifting properties ("Quillen negation of") certain "bad" infinite combinatorial structures using a simplicial category of generalised topological spaces. This makes formal a well-known model theoretic intuition that these properties of models and formulas are defined in terms of avoiding certain "bad" infinite combinatorial structures: the same diagram chasing "trick", the lifting property, applied to (a morphism associated with) a combinatorial structure defines the associated no-tree- or no-order- property of (objects associated with) models. The list of properties includes NOP, NTP, NATP, NTP_i, NSOP_i ($i \geqslant 1$) and NIP.

Another property NFCP(no finite cover property) of the same kind means being of finite dimension, in a certain precise sense.

QUESTION 3. *Is there a homotopy theoretic interpretation of the Shelah dividing lines in model theory [**Z1**, **Z2**] ?*

Say, what is the model theoretic meaning of the number of connected components of the space of maps from $T_\bullet^{\leqslant}$ to $M_\bullet$ defined in [**Z2**] where, perhaps, the space of maps is as defined in the $s\boldsymbol{\varphi}$ reformulation of geometric realisation [**GP**] ?

QUESTION 4. *Reformulate definable and Szemerédi regularity in terms of $s\boldsymbol{\varphi}$.*

Pillay-Starchenko [**PS**, Cor.1.2] uses φ-consistent tuples (take $\varphi(-,-) := E(-,-)$ there), i.e. data captured by $M_\bullet$, and so does [**Simon**, Def.1.1] to define generically stable measures, i.e. both use data captured by $M_\bullet$. [**Malliaris**] studies Szemerédi regularity of $M_\bullet$ viewed as a multigraph; can localisation and persistence of configurations there be described in $s\boldsymbol{\varphi}$?

4. *"Transcribing" the axioms of topology into simplicial language.*

> if a man bred to the seafaring life, and accustomed to think and talk only of matters relating to navigation, enters into discourse upon any other subject ; it is well known, that the language and the notions proper to his own profession are infused into every subject, and all things are measured by the rules of navigation : and if he should take it into his head to philosophize concerning the faculties of the mind, it cannot be doubted, but he would draw his notions from the fabric of his ship, and would find in the mind, sail, masts, rudder, and compass.
>
> Thomas Reid. An Inquiry into the Human Mind on the Principles of Common Sense. 1764.

Here our goal is to suggest a way to "extract" the category-theoretic language (reformulation) "implicit" in the text of the usual definitions and proofs.

Below we "transcribe" in simplicial language the text of the definition of uniform structure, of a characterisation of topological structure in terms of neighbourhoods of points, and of limit, in (Bourbaki, General Topology). A mathematically inclined reader might want to skip our verbose textual analysis and go directly to Definitions 1-5 motivated by it; the exposition there is self-contained. In §5 we demonstrate the same method on a simpler example of the definition of dense subspace and separation axiom T_0, by rewriting them in terms of diagram chasing maps of finite preorders (= finite topological spaces).

As in [**G**], the exposition is in the form of a story and aims to be self-contained and accessible to a first year student who has taken some first lectures in naive set theory, topology, and who has heard a definition of a simplicial set. A more sophisticated reader may find it more illuminating to recover our formulations herself by analysing the text of Bourbaki: Axioms $(\mathrm{V})_{\mathrm{I}}$-$(\mathrm{V})_{\mathrm{IV}}$ in [Bourbaki,I§1.2] and of Definition I in [ibid,II§1] and trying to rewrite it in the simplicial language. Rewriting in simplicial language the definition of uniform space is particularly straightforward, and we do recommend trying to do so yourself first. Rewriting the Bourbaki definition of a limit of a filter might be a fun exercise, either before or after reading our definition of a generalised topological space.

4.1. *"Transcribing" simplicially a definition of topological structure.* A topology is a collection of (filters of) neighbourhoods of points compatible in some sense. We now show that it is "compatible" in the sense that it is "functorial", i.e. defines a functor from Δ^{op} to a category of filters.

This is almost explicit in the axioms $(\mathrm{V_I})$-$(\mathrm{V_{IV}})$ of [Bourbaki,I§1.2] of topology in terms of neighbourhoods. We now quote:

Let us denote by $\mathfrak{B}(x)$ the set of all neighbourhoods of x. The sets $\mathfrak{B}(x)$ have the following properties :

$(\mathrm{V_I})$ *Every subset of* X *which contains a set belonging to* $\mathfrak{B}(x)$ *itself belongs to* $\mathfrak{B}(x)$.

$(\mathrm{V_{II}})$ *Every finite intersection of sets of* $\mathfrak{B}(x)$ *belongs to* $\mathfrak{B}(x)$.

$(\mathrm{V_{III}})$ *The element* x *is in every set of* $\mathfrak{B}(x)$.

Indeed, these three properties are immediate consequences of Definition 4 and axiom $(\mathrm{O_{II}})$.

$(\mathrm{V_{IV}})$ *If* V *belongs to* $\mathfrak{B}(x)$, *then there is a set* W *belonging to* $\mathfrak{B}(x)$ *such that, for each* $y \in$ W, V *belongs to* $\mathfrak{B}(y)$.

By Proposition 1, we may take W to be any open set which contains x and is contained in V.

This property may be expressed in the form that *a neighbourhood of* x *is also a neighbourhood of all points sufficiently near to* x.

What is "the set $\mathfrak{B}(x)$ of all neighbourhoods of x"? $\mathfrak{B}(x)$ is a set of subsets of X parametrised by $x \in X$, thus it is natural to view $\mathfrak{B}(x)$ as a set of subsets of $\{x\} \times X$, and then view "the sets $\mathfrak{B}(x)$", $x \in X$, as a filter on $X \times X = \bigsqcup_{x \in X} \{x\} \times X$ consisting of subsets of form

$$\bigsqcup_{x \in X, U_x \in \mathfrak{B}(x)} \{x\} \times U_x$$

Axioms $(\mathrm{V_I})$ and $(\mathrm{V_{II}})$ say exactly that it is indeed a filter on $X \times X$.

Axiom $(\mathrm{V_{III}})$ says that the filter induced on the diagonal $\{(x,x) : x \in X\} \subset X \times X$ is indiscreet, i.e. the only large subset is the whole set itself. To view this category-theoretically, first consider the inclusion as the diagonal map

$$X \longrightarrow X \times X, \quad x \longmapsto (x,x).$$

Axiom $(\mathrm{V_{III}})$ says that the preimage of any large subset contains the whole of X. To express "the whole of X", make it part of structure: equip X with the indiscreet filter. Then Axiom $(\mathrm{V_{III}})$ is expressed by saying that "the preimage of any large subset is large", which is a condition that makes sense for any map of sets equipped with filters.

This condition reminds us of the definition of a continuous map of topological spaces (the preimage of any open subset is open), and define a *continuous* map of filters to be a map such that the preimage of a large subset is large.

With these definitions, Axiom ($\mathrm{V_{III}}$) says precisely that the diagonal map $X \longrightarrow X \times X$, $x \longmapsto (x,x)$ is continuous.

At last, consider Axiom ($\mathrm{V_{IV}}$). The phrase "there is a set W belonging to $\mathfrak{B}(x)$ such that, for each $y \in W$, V belongs to $\mathfrak{B}(y)$" reads as a property of subsets of $X \times X$ or perhaps $\{x\} \times X \times X$: a subset $U \subset X \times X$ has this property iff there is a set W belonging to $\mathfrak{B}(x)$ such that, for each $y \in W$, the fibre $V_y := U \cap \{y\} \times X$ over y belongs to $\mathfrak{B}(y)$. This property depends on a parameter $x \in X$, and this leads us to define a filter on $X \times X \times X$: call a subset $U \subset X \times X \times X$ *large* iff

for all $x \in X$ there is a set W belonging to $\mathfrak{B}(x)$ such that, for each $y \in W$, the fibre $V_{(x,y)} := U \cap \{(x,y)\} \times X$ belongs to $\mathfrak{B}(y)$.

Equip $X \times X \times X$ with this filter. Then Axiom ($\mathrm{V_{IV}}$) says that the map $X \times X \times X \to X \times X, (x,y,z) \mapsto (x,z)$, is continuous.

These considerations are summed up in Definition 3.

4.2. *"Transcribing" simplicially a definition of uniform structure.* A uniform structure on a set X is a filter on $X \times X$ satisfying certain properties. We now see that properties mean it defines a functor from Δ^{op} to a category of filters which factors via $\Delta^{\mathrm{op}} \to$ FiniteNon-EmptySets.

This is almost explicit in Definition I of [Bourbaki,II§1.2] §2.1]. We now quote:

Definition I. ***A filter on a set* X *is a set* $\mathfrak{F}$ *of subsets of* X *which has the following properties:***

($\mathrm{F_I}$) ***Every subset of* X *which contains a set of* $\mathfrak{F}$ *belongs to* $\mathfrak{F}$.**

($\mathrm{F_{II}}$) ***Every finite intersection of sets of* $\mathfrak{F}$ *belongs to* $\mathfrak{F}$.**

($\mathrm{F_{III}}$) ***The empty set is not in* $\mathfrak{F}$.**

Definition 1. *A uniform structure* (or *uniformity*) *on a set* X *is a structure given by a set* $\mathfrak{U}$ *of subsets of* $X \times X$ *which satisfies axioms* (F_I) *and* (F_{II}) *of Chapter* I, § 6, *no.* 1 *and also satisfies the following axioms:*

(U_I) *Every set belonging to* $\mathfrak{U}$ *contains the diagonal* Δ.

(U_{II}) *If* $V \in \mathfrak{U}$ *then* $\overset{-1}{V} \in \mathfrak{U}$.

(U_{III}) *For each* $V \in \mathfrak{U}$ *there exists* $W \in \mathfrak{U}$ *such that* $W \circ W \subset V$.

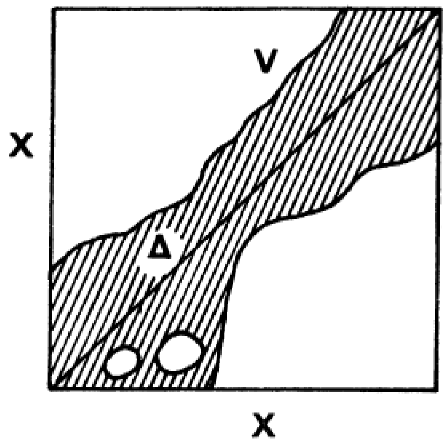

Figure 2.

The sets of $\mathfrak{U}$ *are called entourages of the uniformity defined on* X *by* $\mathfrak{U}$. *A set endowed with a uniformity is called a uniform space.*

(*) We recall (*Set Theory*, R, § 3, nos. 4 and 10) that if V and W are two subsets of $X \times X$, then the set of pairs $(x, y) \in X \times X$, such that $(x, z) \in W$ and $(z, y) \in V$ for some $z \in X$, is denoted by $V \circ W$ or VW, and that the set of pairs $(x, y) \in X \times X$ such that $(y, x) \in V$ is denoted by $\overset{-1}{V}$.

Axioms (F_I) and (F_{II}) say that $\mathfrak{U}$ is a filter on $X \times X$ (but allowing $\varnothing \in \mathfrak{U}$).

To rephrase Axiom (U_I) in the categorical language, first consider the diagonal map

$$X \longrightarrow X \times X, x \mapsto (x, x)$$

Axiom (U_I) says that the preimage of any set belonging to $\mathfrak{U}$ is the whole of X, i.e. in other words, belongs to the indiscreet filter on X. Thus, if we equip X with the indiscreet filter, Axiom (U_I) simply says that "the preimage of a large set is necessarily large". This remind us of the definition of continuity, and so we call a map of sets equipped with filters *continuous* iff the preimage of a large is necessarily large.

This definition of a *continuous map of filters* makes translation to categorical language straightforward. *Axiom (U_I) says that the diagonal map is continuous*, and *Axiom (U_{II}) says that the map permuting coordinates $X \times X \longrightarrow X \times X, (x, y) \mapsto (y, x)$, is continuous.*

In Axiom (U_{III}), first note that "$(x, z) \in W$ and $(z, y) \in V$ for some $y \in X$" describes

$$p_{12}^{-1}(W) \cap p_{23}^{-1}(V) \subset X \times X \times X = \{(x, z, y) : x, z, y \in X\}$$

and thus $W \circ W \subset V$ means that

$$p_{12}^{-1}(W) \cap p_{23}^{-1}(W) \subset p_{13}(V)$$

where $p_{ij} : X \times X \times X \to X \times X, (x_1, x_2, x_3) \mapsto (x_i, x_j)$ are coordinate projections. Thus, *Axiom (U_{III}) says that $p_{12} : X \times X \times X \to X \times X$ is continuous if $X \times X \times X$ is equipped with the pullback of the filter $\mathfrak{U}$ on $X \times X$ along p_{12} and p_{23}.*

These considerations are summed up in Definition 4.

4.3. *"Transcribing" simplicially the definition of limit.* Let us now express the Bourbaki definition of limit in terms of generalised topological spaces.

Definition 1. ***Let X be a topological space and $\mathfrak{F}$ a filter on X. A point $x \in$ X is said to be a limit point*** (or simply a ***limit***) ***of $\mathfrak{F}$, if $\mathfrak{F}$ is finer than the neighbourhood filter $\mathfrak{B}(x)$ of x; $\mathfrak{F}$ is also said to converge*** (or to be ***convergent***) ***to x.***

View "neighbourhood filter $\mathfrak{B}(x)$ as a filter on $\{x\} \times X$, to keep track of parameter "x". The phrase "$\mathfrak{F}$ is finer than the neighbourhood filter $\mathfrak{B}(x)$" means that the map $X \to \{x\} \times X, y \mapsto (x, y)$ is continuous when X is equipped with $\mathfrak{F}$ and $\{x\} \times X$ is equipped with $\mathfrak{B}(x)$. We would not want the target of a map depend on a parameter, and thus would rather consider the composition $X \to \{x\} \times X \to X \times X$. with $X \times X$ equipped with the finest filter such that the inclusion $\{x\} \times X \to X \times X$ is continuous for each parameter x. Explicitly, $X \times X$ is equipped with the filter of subsets of form

$$\bigsqcup_{x \in X, U_x \in \mathfrak{B}(x)} \{x\} \times U_x$$

appearing in our reformulation of the definition of topological spaces.

"$\mathfrak{F}$ a filter on X" suggests we consider an arrow $\mathfrak{F} \to X$ and then a diagram, whose meaning is yet unclear

$$\begin{tikzcd} & X \times X \arrow[d, "p_2 : (x,y) \mapsto y"] \\ \mathfrak{F} \arrow[ur, "y \mapsto (x,y)"] \arrow[r] & X \end{tikzcd}$$

The arrow $p_2 : X \times X \longrightarrow X$ suggests the map of "forgetting the first coordinate" if we view $X, X \times X, \ldots$ as part of the simplicial set $X_\bullet([n]) := \mathrm{Hom}([n], X)$, $n \geqslant 0$, represented by set X:

$$X_\bullet \circ [+1] \longrightarrow X_\bullet$$

where

$$[+1] : \Delta^{\mathrm{op}} \longrightarrow \Delta^{\mathrm{op}}$$

$$n \mapsto n+1, \quad f : n \to m \longmapsto f' : n+1 \to m+1, f'(1) := 1; f'(i+1) := f(i) + 1$$

The simplicial set $X_\bullet \circ [+1]$ is a disconnected union of copies of $X_\bullet$ parametrised by $x \in X$

$$X_\bullet \circ [+1] = \bigsqcup_{x \in X} X_\bullet \text{ (as simplicial sets)}$$

and the map $X_\bullet \circ [+1] \longrightarrow X_\bullet$ is identity on each connected component. Hence, if $\mathfrak{F}$ is connected in some appropriate sense, the diagonal map $\mathfrak{F} \longrightarrow X \times X$ is necessary of the form shown (i.e. $y \mapsto (x, y)$).

Thus, we would want $\mathfrak{F}$ to denote a connected simplicial set $\mathfrak{F}_\bullet$ such that $\mathfrak{F}_0$ is the set X equipped with filter $\mathfrak{F}$. A simple way to ensure that is to set $\mathfrak{F}_\bullet^{\mathrm{diag}}([n]) := \mathrm{Hom}([n], X)$ where each X^n is equipped with the finest filter such that the map $X \to X^n$ is continuous.

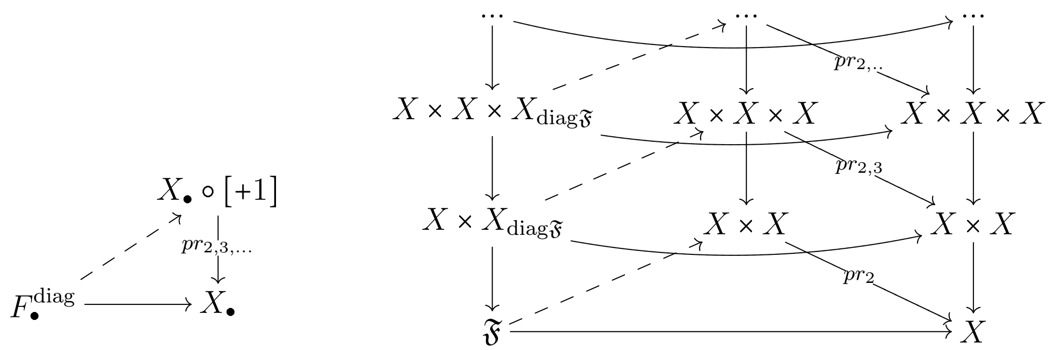

FIGURE 2. (a) The diagram in $s\boldsymbol{\varphi}$. (b) The same diagram in $s\boldsymbol{\varphi}$ expanded.

These considerations are summed up in Definition 5.

4.4. *A category theorist's view.* A category theorist will immediately find that the notions of limit and homotopy are defined by the same categorical construction, namely factoring through a simplicial path space which is defined precisely to be the simplicial object composed with a shift endofunctor of Δ, see §2.3 or [**Waldhausen**, §1.5].

Acknowledgments and historical remarks

It seems embarrassing to thank anyone for ideas so trivial, and we do that in the form of historical remarks. Ideas here have greatly influenced by extensive discussions with Grigori Mints, Martin Bays, and, later, with Alexander Luzgarev and Vladimir Sosnilo. At an early stage Xenia Kuznetsova helped to realise an earlier reformulation of compactness was inadequate and that labels on arrows are necessary to formalise topological arguments. "A category theorist" that interpreted our reformulation of convergence as homotopy is Vladimir Sosnilo. D.Rudskii explained that the metric used to define a geometric simplex as a space of functions, is well-known as the Levy-Prohorov or Skorokhod metric, used in Skorokhod spaces of upper semi-continuous functions. The exposition in §4 follows the style of the exposition in [**G**, §§1-3] which has been polished in the numerous conversations with students at St. Peterburg and Yaroslavl'2014 summer school.

Reformulations of surjectivity and injectivity, as well as connectedness and (not quite) compactness in terms of the lifting property, first appeared in early drafts of a paper [GH-I] with Assaf Hasson as trivial and somewhat curious examples of a lifting property but were removed during preparation for publication. After the reformulation of injectivity in terms the lifting property with respect to the simplest example of a non-injective map came up in a conversation with Misha Gromov the author decided to try to think seriously about such lifting properties, and in fact gave talks at logic seminars in 2012 at Lviv and in 2013 at Munster and Freiburg, and 2014 at St. Petersburg. At a certain point the author realised that possibly a number of simple arguments in point-set topology may become diagram chasing computations with finite topological spaces, and Grigori Mints insisted these obser-

vations be written. Later the author realised that several other simple definitions can also be expressed in terms of simplicial diagram chasing in the category of simplicial objects in the category of filters.

Ideas of [ErgB] influenced this paper (and [GH-I] as well), and particularly our computational approach to category theory. Alexandre Borovik suggested to write a note [**G**] for *The De Morgan Gazette* explaining the observation that 'some of human's "natural proofs" are expressions of lifting properties as applied to "simplest counterexample"'.

I thank Yuri Manin for several discussions motivated by [GH-I].

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I wish to express my deep thanks to Grigori Mints, to whose memory the initial version of [**G**] is dedicated . . .

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LP2 Lifting property. A list of examples of iterated lifting property. `ncatlab.com/show/lift`

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5. *Appendix: Transcribing "dense" and "T_0".*

We shall now transcribe the definitions of *dense* and *Kolmogoroff* T_0 spaces. An interested reader should read our exposition of compactness in [**mintsGE**, §2] from where this is taken.

5.1. *"A is a dense subset of X."*
By definition [Bourbaki, I§1.6, Def.12],

Definition 12. ***A subset* A *of a topological space* X *is said to be dense in* X (or simply *dense*, if there is no ambiguity about X) *if* $\overline{\mathbf{A}} = \mathbf{X}$, *i.e. if every non-empty open set* U *of* X *meets* A.**

Let us transcribe this by means of the language of arrows.

A subset A of a topological space X is an arrow $A \longrightarrow X$. (Note we are making a choice here: there is an alternative translation analogous to the one used in the next sentence). An open subset U of X is an arrow $X \longrightarrow \{U \searrow U'\}$; here $\{U \searrow U'\}$ denotes the topological space consisting of one open point U and one closed point U'; by the arrow $\searrow$ we mean that that $U' \in cl(U)$. Non-empty: a subset U of X is *empty* iff the arrow $X \longrightarrow \{U \searrow U'\}$ factors as $X \longrightarrow \{U'\} \longrightarrow \{U \searrow U'\}$; here the map $\{U'\} \longrightarrow \{U \searrow U'\}$ is the obvious map sending U' to U'. set U of X meets A: $U \cap A = \varnothing$ iff the arrow $A \longrightarrow X \longrightarrow \{U \searrow U'\}$ factors as $A \longrightarrow \{U'\} \longrightarrow \{U \searrow U'\}$.

Collecting above (Figure 1c), we see that a map $A \xrightarrow{f} X$ has dense image iff

$$A \xrightarrow{f} X \rightthreetimes \{U'\} \longrightarrow \{U \searrow U'\}$$

Note a little miracle: $\{U'\} \longrightarrow \{U \searrow U'\}$ is the simplest map whose image isn't dense. We'll see it happen again.

5.2. *Kolmogoroff spaces, axiom T_0.*
By definition [Bourbaki,I§1, Ex.2b; p.117/122],

***b*) A topological space is said to be a *Kolmogoroff space* if it satisfies the following condition : given any two distinct points *x*, *x'* of X, there is a neighbourhood of one of these points which does not contain the other. Show that an ordered set with the right topology is a Kolmogoroff space.**

Let us transcribe this. given any two ... points x, x' of X: given a map $\{x, x'\} \xrightarrow{f} X$. two *distinct* points: the map $\{x, x'\} \xrightarrow{f} X$ does not factor through a single point, i.e. $\{x, x'\} \longrightarrow X$ does not factor as $\{x, x'\} \longrightarrow \{x = x'\} \longrightarrow X$. The negation of the sentence there is a neighbourhood which does not contain the other defines a topology on the set $\{x, x'\}$: indeed, the antidiscrete topology on the set $\{x, x'\}$ is the only topology with the property that there is [no] neighbourhood of one of these points which does not contain the other. Let us denote by $\{x \leftrightarrow x'\}$ the antidiscrete

$$(a)\ \begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[r, "j"'] \arrow[ur, dashed, "\tilde{j}"] & Y \end{tikzcd} \quad (b)\ \begin{tikzcd} \{\} \arrow[r] \arrow[d] & X \arrow[d, "\therefore(surj)"] \\ \{\bullet\} \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \quad (c)\ \begin{tikzcd} A \arrow[r] \arrow[d, "\therefore(dense)"'] & \{U'\} \arrow[d] \\ B \arrow[r] \arrow[ur, dashed] & \{U \searrow U'\} \end{tikzcd} \quad (d)\ \begin{tikzcd} \{x \leftrightarrow x'\} \arrow[r] \arrow[d] & X \arrow[d, "\therefore(T_0)"] \\ \{x = x'\} \arrow[r] \arrow[ur, dashed] & \{x = x'\} \end{tikzcd}$$

Figure 1: Lifting properties. Dots $\therefore$ indicate free variables and what property of these variables is being defined; in a diagram chasing calculation, "$\therefore(dense)$" reads as: given a (valid) diagram, add label $(dense)$ to the corresponding arrow.
(a) The definition of a lifting property $f \rightthreetimes g$: for each $i : A \longrightarrow X$ and $j : B \longrightarrow Y$ making the square commutative, i.e. $f \circ j = i \circ g$, there is a diagonal arrow $\tilde{j} : B \longrightarrow X$ making the total diagram $A \xrightarrow{f} B \xrightarrow{\tilde{j}} X \xrightarrow{g} Y, A \xrightarrow{i} X, B \xrightarrow{j} Y$ commutative, i.e. $f \circ \tilde{j} = i$ and $\tilde{j} \circ g = j$. (b) $X \longrightarrow Y$ is surjective (c) the image of $A \longrightarrow B$ is dense in B (d) X is Kolmogoroff/T_0

space consisting of x and x'. Now we note that the text implicitly defines the space $\{x \leftrightarrow x'\}$, and the only way to use it is to consider a map $\{x \leftrightarrow x'\} \xrightarrow{f} X$ instead of the map $\{x, x'\} \xrightarrow{f} X$.

Collecting above (see Figure 1d), we see that *a topological space X is said to be a* Kolmogoroff *space iff any map $\{x \leftrightarrow x'\} \xrightarrow{f} X$ factors as $\{x \leftrightarrow x'\} \longrightarrow \{x = x'\} \longrightarrow X$.*

Note another little miracle: it also reduces to orthogonality of morphisms

$$\{x \leftrightarrow x'\} \longrightarrow \{x = x'\} \rightthreetimes X \longrightarrow \{x = x'\}$$

and $\{x \leftrightarrow x'\}$ is the simplest non-Kolmogoroff space.