REMARKS ON SHELAH'S CLASSIFICATION THEORY AND QUILLEN'S NEGATION

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Abstract. — We give category-theoretic reformulations of stability and NIP by observing that their characterisations in terms of indiscernible sequences are naturally expressed as Quillen lifting properties of certain morphisms associated with linear orders and with models, in a certain category of genenalised topological spaces extending the categories of topological spaces and of simplicial sets.

This suggests an approach to a homotopy theory for model theory.

Contents

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Then what do you gain by pretending so? 9.27/30,10.5/14,11.11,1.12 2020.

1. Introduction

The Stone space of types over a model is a topological space carrying important model-theoretic information, e.g. Cantor-Bendixon ranks, number of types. Unfortunately these spaces are "not nice" from the point of view of algebraic topology and the methods of homotopy theory do not apply to Stone spaces of models.

In this note we attempt to rectify this situation.

We define the notion of a generalised Stone space of a model living in a rather large category extending the category of topological spaces, simplicial sets, uniform structures (e.g., metric spaces with uniformly continuous maps), and of preorders (in several ways); a forgetful functor takes the generalised Stone space into the usual Stone space, or rather the space of elements of a model with the corresponding topology. These generalised Stone spaces and their morphisms carry information about indiscernible sequences: essentially, a morphism between the generalised Stone spaces of two models is a map between sets of elements preserving indiscernible sequences in the sense that the image of any indiscernible sequence is necessarily an indiscernible sequence. Our key observation is that an indiscernible sequence in a model is the same as a injective morphism from the linear order to the generalised Stone space of the model. This reformulations allows us to rewrite the characterisations of stability and NIP in terms of indiscernible sequences as examples of a standard trick from homotopy theory, the Quillen negation (orthogonality), with respect to certain explicitly given morphisms associated with linear orders.

Importantly, the definition of the generalised Stone space is easily obtained by "transcribing" the definition of an indiscernible sequence in a particularly mechanistic, oversimplified manner reminiscent of the "android" in [GH]. In a forthcoming paper[(1)] we observe that "transcribing" the tree property in the definition of a simple theory [Tent-Ziegler, Def.7.2.1] leads to a different object associated with a model, and also leads to a lifting property. This object is essentially the characteristic sequence of a first-order formula of [Malliaris].

The lifting property and Quillen negations (orthogonals) are a basic part of the language of a natural abstract setting for homotopy theory, the formalism of model categories introduced by Quillen [Quillen]; see [Wikipedia,Lifting_property] for details and examples of elementary properties defined as iterated Quillen negation. Note an analogy to the model theoretic intuition of the properties we reformulate: as their names suggest (not Independence Property, not Order Property), these properties are usually thought of as the negation of a corresponding property suggesting high combinatorial complexity.

The meaning of a morphism between two generalised Stone spaces is reminiscent of the notion of one structure *representing* another introduced by Shelah [CoSh:919] (we quote [Sh:1043]) 'try to formalise the intuition that "the class of models of a stable first order theory is not much more complicated than the class of models $M = (A, \dots, E_t, \dots)_{s\in I}$ where E_t^M is an equivalence relation on A refining E_s^M for $s < t$; and I is a linear order of cardinality $\leq |T|$".' We reformulate a corollary of a characterisation of stable theories in [CoSh:919] and give a more literal formalisation of this intuition: a theory is stable iff there is κ such that for each model of the theory there is a surjective morphism to its generalised Stone space from a structure whose language consists of at most κ equivalence relations and unary predicates (and nothing else). Based on this reformulation we suggest a

(1) A preliminary exposition is given in [8, Appendix 9], see §9.1.5 and §9.2.6.

conjecture with a category-theoretic characterisation of classes of models of stable theories.

The main construction. — A little of category theoretic terminology allows to explain our construction in a few words. Intuitively, our category is the category of simplicial sets equipped with a notion of smallness; formally, it is the category of simplicial objects in the category of filters in the sense of Bourbaki. The generalised Stone space of a model is the simplicial set *represented* by the set of elements of the model: an "n-simplex" is a tuple $(a_0, ..., a_n)$ of elements, and it is considered "very small" iff it is "very indiscernible", i.e. ϕ-indiscernible for "many" formulas ϕ (possibly with some elements repeated several times, so that $(a, a...a)$ is also "very small"). There is no single best definition of the generalised Stone space of a model: we may want consider slightly different notions of "smallness", e.g. "very indiscernible" may rather mean being part of an *infinite* ϕ-indiscernible sequence for "many" ϕ, cf. Definition 3.1.1.1 and Remark 3.1.2.

Stability and NIP as Quillen negation. — An indiscernible sequence is a map from a linear order to a model. Equivalently, it is a morphism of simplicial sets to the simplicial set represented by the set of elements of the model, from the simplicial set *represented by the linear order*: an n-simplex of that is a non-decreasing sequence $i_0 \le ... \le i_n$, and we always consider it "small". Then "the image of a small simplex is necessarily small" means exactly that the sequence is indiscernible, possibly with some elements repeated: for each "small (i.e. arbitary) n-simplex" $i_0 \le ... \le i_n$ the "n-simplex" $(a_{i_0}, ..., a_{i_n})$ is "small", i.e. indiscernible, though possibly with some elements repeated several times.

An indiscernible set is a map from a set to a model. In a similar way, it is equivalent to consider a morphism of simplicial sets to the simplicial set represented by the set of elements of the model, from the simplicial set represented by the set: an n-simplex of that is a tuple $(i_0, ..., i_n)$, and we always consider it "small". Then "the image of a small simplex is necessarily small" means exactly that the set is indiscernible: for each "small, i.e. arbitrary, n-simplex" $(i_0, ..., i_n)$ the "n-simplex $(a_{i_0}, ..., a_{i_n})$ is "small", i.e. indiscernible, though possibly with some elements repeated several times.

Hence, the definition of stability "each indiscernible sequence is an indiscernible set" says that, in the category of simplicial sets with a notion of smallness, a morphism to the model can be extended from one object to another, i.e. is a Quillen lifting property (Lemma 3.3.2.1,3.4.5.1).

In a similar manner we rewrite the characterisation of NIP "each eventually indiscernible sequence is eventually indiscernible over a parameter" (Lemma 3.4.3.1). To do this we use different notions of smallness: for the sequence, an n-simplex" $i_0 \le ... \le i_n$ is "small" if i_0 is large; and for the model, two notions: the "n-simplex $(a_{i_0}, ..., a_{i_n})$ is "small" iff the tuple is ϕ-indiscernible for "many" formulas ϕ with parameters, or for for "many" parameter-free formulas ϕ.

Forthcoming preliminary results. — In a forthcoming paper [8,Appendices §4-§9] we include results not ready for publication but which we hope might provide some context for the observations in this note. In [8,Appendix 5] we sketch preliminary characterisations of NIP, non-dividing, and NOP, which we think are not optimal. A Cauchy sequence and an indiscernible sequence are morphisms from the same object associated with the filter of final segments of a linear order [8,Lemma 5.1.4.1,5.1.3.1]. In topology the definition of a complete metric space "each Cauchy sequence has a limit" is expressed as a lifting property involving a endomorphism of our category "shifting dimension" [8,Lemma 5.2.5.1]; a modification of this lifting property defines NIP via existence of average/limits of types of indiscernible sequences [8,Lemma 5.2.4.1]; we do not pursue the analogy between Cauchy sequences and their limits in topology, and indiscernible sequences and their limit types in model theory. The same endomorphism is used in a reformulation of non-dividing [8,Lemma 5.3.2.1]. A reformulation [8,Lemma 5.4.2.1] of NOP is obtained by "transcribing" a definition in a particularly mechanistic way reminiscent of the "android" in [GH]. The process of transcribing is sensitive to the phrasing and details of the formulation being transcribed, and therefore we obtain a different lifting property, in fact it is a lifting property on the right and not on the left as for stability and NIP. In [8,§5.4.3] we consider a simplification of the reformulation of OP and discuss its relationship with NSOP. We also include in [8,Appendix 5] an exposition of stability very similar to Lemma 3.4.5.1 but using slightly simpler definition of the filters. In [8,Appendix 3] we sketch a number of examples of simplicial filters, in particular how to view a topological space as a simplicial filter. In [8,Appendix 6] we use category-theoretic language to sketch our construction of simplicial filters associated with models. In [8,Appendix 9] we sketch how to reformulate NTP appearing in the definition of a simple theory.

1.2. Further work. — These observations show that our generalised Stone space contain model theoretic information accessible by diagram chasing techniques, and, more generally, category theoretic methods, and may bring an additional geometric intuition and vision to model theory, particularly to the combinatorial methods of "negative" dividing lines and indiscernible sequences.

In [8,Appendix 7] we give a number of speculations elaborating our vision; here we mention general directions.

Our reformulations show that several properties (classes) of models can be defined very concisely in terms of iterated Quillen negation (orthogonals) starting from an explicitly given morphism. Can the diagram chasing technique arising from such reformulations be of use in model theory, say to shorten the exposition of some well-known arguments? Can one define interesting dividing lines by taking iterated Quillen negation of interesting examples or properties?

Our category(2) **ኒቀ** carries an intuition of point-set topology ([6,§3], [7]) which we hope might guide us to a definition of a notion of the space of indiscernible sequences in a model or the space of maps between two models, and their connected components.

Our category **ኒቀ** has two subcategories carrying a rich homotopy theory: topological spaces and simplicial sets. This raises a naive hope—or rather a grand project—that methods of homotopy theory can be developed for **ኒቀ** and will provide meaningful model theoretic information if applied to generalised Stone spaces of models.

2. The category

2.1. The category of filters. — We now introduce the main category.

(2) We suggest to pronounce **ኒቀ** as sF being visually similar to $s\Phi$ standing for "*s*implicial ϕilters", even though it is unrelated to the actual pronunciation of these symbols coming from the Amharic script.

2.1.1. The category of filters. — We slightly modify [Bourbaki, I§6.1, Def.I]:

DEFINITION I. ***A filter on a set* X *is a set* $\mathfrak{F}$ *of subsets of* X *which has the following properties:***

(F_I) ***Every subset of* X *which contains a set of* $\mathfrak{F}$ *belongs to* $\mathfrak{F}$.**

($F_{II\,a}$) The intersection of two sets of $\mathfrak{F}$ belongs to $\mathfrak{F}$.

($F_{II\,b}$) X belongs to $\mathfrak{F}$.

Subsets in $\mathfrak{F}$ are called *neighbourhoods* or $\mathfrak{F}$*-big*. We call X *the underlying set* of filter $\mathfrak{F}$ and denote it by $X = |\mathfrak{F}|$.

By abuse of language, often by a filter we mean a set together with a filter on it, and by X mean $\mathfrak{F}$.

A *morphism of filters* is a mapping of underlying sets such that the preimage of a neighbourhood is necessarily a neighbourhood; we call such maps of filters *continuous*. Note that $\mathfrak{F} \cup \{\varnothing\}$ is a topology on X, but this notion of continuity is stronger: the preimage of an open set is not allowed to be empty (unless $\varnothing$ is a neighbourhood).

Let $\boldsymbol{\Phi}$ denote the category of filters.

Unlike the definition of filter in [Bourbaki, I§6.1, Def.I],(3) we do not require that $\varnothing \notin \mathfrak{F}$, i.e. for us the set of all subsets of X is a filter. In particular it is possible that $X = \varnothing$ and $\mathfrak{F} = \{\varnothing\}$. We do so in order for the category of filters to have limits.

2.1.2. Filters: notation and intuition. — We shall denote filters by $X, Y, \ldots$, and neighbourhoods by $\varepsilon, \delta, ..$, as this enables us to write in analogy with analysis that a map $f : |X| \longrightarrow |Y|$ of filters is continuous iff for each neighbourhood $\varepsilon \subset |Y|$ there is neighbourhood $\delta \subset |X|$ such that $f(\delta) \subset \varepsilon$.

With the the notion of a filter precision can be given to the concept of sufficiently small error and it enables one to give an exact meaning to the phrase "whenever x is sufficiently small, x has the property $P\{x\}$", by saying that there is a neighbourhood $\varepsilon \subset |X|$ such that x has the property $P\{x\}$ whenever $x \in \varepsilon$. We may also *define* a neighbourhood to mean a set of points having some property we are interested in, and using the word "neighbourhood" in this sense brings about the intuition and terminology of the mathematical idea of neighbourhood in topology. For example, it makes the language more expressive if we say "$x \in |X|$ is ε-small" to mean $x \in \varepsilon$, i.e. that x has the property we are interested in.

This notation and intuition is demonstrated by the following example: a map $f : |M'| \longrightarrow |M''|$ of metric spaces M', M'' is uniformly continuous if it induces a continuous map from the filter of ϵ-neighbourhoods of the main diagonal on $M' \times M'$ to that on $M'' \times M''$. The *filter of ϵ-neighbourhoods of the main diagonal* on a metric space M is defined as being generated by

$$\{(x, y) \in |M| \times |M| : \mathrm{dist}(x, y) < \epsilon\}$$

Indeed, being continuous means that for every $\varepsilon := \{(x,y) \in |M'| \times |M'| : \mathrm{dist}(x,y) < \epsilon\}$ there exists $\delta := \{(x,y) \in |M''| \times |M''| : \mathrm{dist}(x,y) < \delta\}$ such that $f(\delta) \subset \varepsilon$. In more expressive terms we may say that f is uniformly continuous if the "error" $(f(x), f(y))$ is as small as we please whenever the "error" (x, y) is small enough,

(3)Our modification of the notion of filter is used in the literature on formalisation [HIH13], [Formalising perfectoid spaces], where instead of $f : X \longrightarrow Y$ they say "f tends to filter Y with respect to filter X".

or, more vaguely, the "homotopy" $f(x) \rightsquigarrow f(y)$ is as short as we please whenever the "homotopy" $x \rightsquigarrow y$ is short enough.

Similarly, a map $f : |X'| \longrightarrow |X''|$ of topological spaces X', X'' is continuous if it induces a continuous map from the filter of coverings on $X' \times X'$ to that on $X'' \times X''$. The *filter of coverings* on $|X| \times |X|$ consists of subsets of form

$$\bigsqcup_{x \in X} \{x\} \times U_x$$

where $U_x \ni x$ is a (not necessarily open) neighbourhood of x, i.e. $(U_x)_{x \in X}$ is a covering of X. [(4)]

2.1.3. The category of simplicial filters. — Now we introduce the main object of concern of this paper.

Let $n_\leq$ for $n > 0$ denote the finite linear order $1 < 2 < ... < n$ on the first n natural numbers. Let Δ be the category of finite linear orders whose objects are finite linear orders,and morphisms are non-decreasing maps, and Δ^{op} denote the opposite category of Δ.

Let ${}_{\wr}\boldsymbol{\varphi} = Func(\Delta^{\mathrm{op}}, \boldsymbol{\varphi})$ be the category of functors from Δ^{op} to the category $\boldsymbol{\varphi}$ of filters; morphisms are natural transformations between functors. One may refer to an object of ${}_{\wr}\boldsymbol{\varphi}$ as either *a simplicial filter* or *a situs*, if one prefers a short name.

2.1.4. Simplicial notation. — We usually let $X_\bullet, Y_\bullet$ denote simplicial filters, the subscript $\bullet$ indicating it is a functor. We may write $X_\bullet : {}_{\wr}\boldsymbol{\varphi}$ to indicate that X is an object of ${}_{\wr}\boldsymbol{\varphi}$. $X_\bullet(n_\leq) \in \mathrm{Ob}\, \boldsymbol{\varphi}$ shall denote the value of $X_\bullet$ at $n_\leq \in \mathrm{Ob}\, \Delta^{\mathrm{op}}$.

If we defined a construction associating an object of ${}_{\wr}\boldsymbol{\varphi}$ with a mathematical object of certain type (a linear order, a model, a topological or metric space, ...), we usually let $X_\bullet, Y_\bullet, ...$ denote the object of ${}_{\wr}\boldsymbol{\varphi}$ associated with $X, Y, ..$; a superscript may indicate the nature of the construction.

We view a weakly increasing sequence $1 \leq i_1 \leq ... \leq i_n \leq m$ as a monotone map $n_\leq \longrightarrow m_\leq$, and denote by $[i_1 \leq ... \leq i_n] : m_\leq \longrightarrow n_\leq$ the corresponding morphism of Δ^{op}. Because $X_\bullet$ is a functor, $[i_1 \leq ... \leq i_n]$ induces a morphism $[i_1 \leq ... \leq i_n]$:

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(4) This intuition is the usual intuition of topology as described in [Bourbaki, Introduction], though use of simplicial techniques makes our formalism somewhat more flexible then that of [Bourbaki], as we will see later. For example, it allows to treat uniformly the topological and uniform structures: The filter of coverings gives the topological structure on a set, and the filter of ε-neighbourhoods of the main diagonal gives the uniform structure. We quote [Bourbaki]:

– (Introduction) To formulate the idea of neighbourhood we started from the vague concept of an element "sufficiently near" another element. Conversely, a topological structure now enables us to give precise meaning to the phrase "such and such a property holds for all points sufficiently near a": by definition this means that the set of points which have this property is a neighbourhood of a for the topological structure in question. .... a topological structure on a set enables one to give an exact meaning to the phrase "whenever x is sufficiently near a, x has the property $P(x)$". But, apart from the situation in which a "distance" has been defined, it is not clear what meaning ought to be given to the phrase "every pair of points x, y which are sufficiently near each other has the property $P(x, y)$", since a priori we have no means of comparing the neighbourhoods of two different points.
– (I§1.2) The everyday sense of the word "neighbourhood" is such that many of the properties which involve the mathematical idea of neighbourhood appear as the mathematical expression of intuitive properties; the choice of this term thus has the advantage of making the language more expressive.
– (II§2.1) In more expressive terms we may say that f is uniformly continuous if $f(x)$ and $f(y)$ are as close to each other as we please whenever x and y are close enough.

$X_\bullet(m_\leq) \longrightarrow X_\bullet(n_\leq)$ which we may also denote by $[i_1 \leq \ldots \leq i_n]$. For $x \in X_\bullet(m_\leq)$, we denote the image of x under this morphism by $x[i_1 \leq \ldots \leq i_n] \in X_\bullet(n_\leq)$.

For $X_\bullet : s\boldsymbol{\varphi}$, elements of $X_\bullet(n_\leq)$ are called $(n-1)$-simplicies where $n-1$ is the *dimension*; to avoid confusion we may sometimes write $n_\leq$-simplex instead. An element of the form $x[i_1 \leq \ldots \leq i_n] \in X_\bullet(n_\leq)$ is called a *face* of simplex x.

We denote by $\left\{X \underset{C}{\rightarrow} Y\right\} := \mathrm{Hom}_C(X, Y)$ the set of morphisms from $X : C$ to $Y : C$ in a category C; we may omit C when it is clear. By $\left\{- \rightarrow Y\right\}$ or $\left\{- \underset{C}{\rightarrow} Y\right\}$ we denote the functor $C^{\mathrm{op}} \longrightarrow \mathrm{Sets}$, $X \longmapsto \left\{X \underset{C}{\rightarrow} Y\right\}$

2.1.5. Simplicial filters: intuition. — Recall that the category of simplicial sets is the category of functors $\mathrm{sSets} := Func(\Delta^{\mathrm{op}}, \mathrm{Sets})$. The forgetful functor $|\cdot| : \boldsymbol{\varphi} \longrightarrow \mathrm{Sets}$ taking a filter to its underlying set induces a forgetful functor $|\cdot| : s\boldsymbol{\varphi} \longrightarrow \mathrm{sSets}$, sending a simplicial filter to its *underlying simplicial set.*

A simplicial filter is a simplicial set equipped with filters, in the following precise sense. To give a simplicial filter $s\boldsymbol{\varphi}$ is to give a simplicial set $X_\bullet : Func(\Delta^{\mathrm{op}}, \mathrm{Sets})$ and a filter $\mathfrak{F}_n$ on set $X(n_\leq)$ for each $n > 0$ such that all the face maps $X(m_\leq) \longrightarrow X(n_\leq)$ are continuous with respect to these filters. The continuity condition means explicitly that for each $m, n > 0$, for each weakly increasing sequence $1 \leq i_1 \leq \ldots \leq i_n \leq m$ for each neighbourhood $\varepsilon \in \mathfrak{F}_n$

$$\{\, x : x[i_1 \leq \ldots \leq i_n] \in \varepsilon \} \in \mathfrak{F}_m$$

or, using notation analogous to mathematical analysis, for each neighbourhood $\varepsilon \in \mathfrak{F}_n$ there is a neighbourhood $\delta \in \mathfrak{F}_m$ such that for each $x \in \delta$ $x[i_1 \leq \ldots \leq i_n] \in \varepsilon$.

Recall our intuition in §2.1.2 that a filter is a notion of smallness. Hence, intuitively we think of a simplicial filter as a *simplicial set equipped with a notion of smallness.*

2.2. Examples of simplicial filters and their morphisms.

— We give examples of simplicial filters we use later.

In most of our examples of simplicial filters the underlying simplicial set is represented; here we explain what this means. We also sketch how to embed into $s\boldsymbol{\varphi}$ the category of simplicial sets and that of uniform structures; we hope these examples will aid the intuition.

More examples are sketched in [8,Appendix 4], notably the full subcategory of $s\boldsymbol{\varphi}$ of topological spaces. The reader may find it helpful to browse through to gain intuition and the the wider context.

2.2.1. Discrete and indiscrete. — The set of subsets consisting of X alone is a filter on X called the *indiscrete* filter, and there is a functor $\cdot^{\mathrm{indiscrete}} : \mathrm{Sets} \longrightarrow \boldsymbol{\varphi}$, $X \longmapsto X^{\mathrm{indiscrete}}$, sending a set to itself equipped with indiscrete filter $\{X\}$. For us the set of all subsets of X is also a filter which we call *discrete.*

The functor $\cdot^{\mathrm{indiscrete}} : \mathrm{Sets} \longrightarrow \boldsymbol{\varphi}$ induces a fully faithful embedding $\cdot^{\mathrm{indiscrete}} : \mathrm{sSets} \hookrightarrow s\boldsymbol{\varphi}$, and in this way any simplicial set is also a simplicial filter.

2.2.2. Represented simplicial sets. — The underlying simplicial sets of most of the examples will be variations of the following well-known construction in category theory. A reader less familiar with category theory may wish to read first §2.2.4 where we give this construction in the set-theoretic language.

Let C be a category. To each object $Y \in \mathrm{Ob}\, C$ corresponds a functor $h_Y : X \longmapsto \left\{ X \underset{C}{\to} Y \right\}$ sending each object $X \in \mathrm{Ob}\, C$ into the set of morphisms from X to Y. A functor $h_Y : C \longrightarrow \mathrm{Sets}$ of this form is called *represented by* Y. Yoneda Lemma implies that this correspondence $Y \longmapsto h_Y$ defines a fully faithful embedding $C \longrightarrow Func(C^{\mathrm{op}}, \mathrm{Sets})$. Indeed, a natural transformation $\eta : h_{Y'} \longrightarrow h_{Y''}$ is fully determined by morphism $\eta_{Y'}(id_{Y'}) \in h_{Y''} = \left\{ Y' \to Y'' \right\}$: for arbitrary X, $\eta_X : \left\{ X \to Y' \right\} \longrightarrow \left\{ X \to Y'' \right\}$ is given by $f \longmapsto f \circ \eta_{Y'}(id_{Y'})$.

A *simplicial set* $I^{\leq}_{\bullet} : \Delta^{\mathrm{op}} \longrightarrow \mathrm{Sets}$ *co-represented by* a preorder $I^{\leq}$ is the functor sending each finite linear order $n_{\leq}$ into the set of monotone maps from $n_{\leq}$ to $I^{\leq}$:

$$n_{\leq} \longmapsto \left\{ n_{\leq} \xrightarrow[\text{preorders}]{} I^{\leq} \right\} = \{\, (t_1, ..., t_n) \in I^n \,:\, t_1 \leq ... \leq t_n \,\}$$

A map $[i_1 \leq ... \leq i_n] : n_{\leq} \longrightarrow m_{\leq}$ induces by composition a map

$$\left\{ m_{\leq} \xrightarrow[\text{preorders}]{} I^{\leq} \right\} \longrightarrow \left\{ n_{\leq} \xrightarrow[\text{preorders}]{} I^{\leq} \right\}$$

$$(t_1 \leq ... \leq t_m) \longmapsto (t_{i_1} \leq ... \leq t_{i_n})$$

A monotone map $f : I^{\leq} \longrightarrow J^{\leq}$ of preorders induces a natural transformation of functors $f_{\bullet} : I^{\leq}_{\bullet} \longrightarrow J^{\leq}_{\bullet}$: for each $n > 0$, a weakly increasing sequence $(x_1 \leq ... \leq x_n) \in I^{\leq}(n_{\leq})$ goes into a weakly increasing sequence $(f(x_1) \leq ... \leq f(x_n)) \in J^{\leq}_{\bullet}(n_{\leq})$. Moreover, every natural transformation $f_{\bullet} : I^{\leq}_{\bullet} \longrightarrow J^{\leq}_{\bullet}$ is necessarily of this form, as the following easy argument shows. Let $(y_1, ..., y_n) = f_n(x_1, .., x_n)$; by functoriality using maps $[i] : 1 \longrightarrow n, 1 \mapsto i$ we know that $y_i = (y_1, .., y_n)[i] = f_n(x_1, .., x_n)[i] = f_1(x_i)$. In a more geometric language, we may say that we used that each "simplex" $(y_1, .., y_n) \in J^{\leq}(n_{\leq})$ is uniquely determined by its "0-dimensional faces" $y_1, ..., y_n \in J^{\leq}$.

In the category-theoretic language, the facts above are expressed by saying that preorders with monotone maps form a full subcategory of sSets and therefore of ɀ$\boldsymbol{\varphi}$.

An important special case is when the preorder is an equivalence relation with one equivalence class, i.e. a set with no additional structure. In that case we call the functor just defined *represented by the set* I, denote it by $|I|_{\bullet} : \Delta^{\mathrm{op}} \longrightarrow \mathrm{Sets}$. This gives a fully faithful embedding Sets $\longrightarrow$ sSets.

2.2.3. Metric spaces and the filter of uniform neighbourhoods of the main diagonal. — Let M be a metric space. Consider the *simplicial set* $|M|_{\bullet} : \Delta^{\mathrm{op}} \longrightarrow \mathrm{Sets}$ *represented by* the set $|M|$ of points of M defined above, i.e. the functor

$$n_{\leq} \longmapsto \left\{ n \xrightarrow[\text{Sets}]{} |M| \right\} = \{\, (t_1, ..., t_n) \in |M|^n \} = |M|^n$$

Now equip $|M|_{\bullet}(n_{\leq}) = |M|^n$ with the "filter of uniform neighbourhoods of the main diagonal" generated by

$$\{(x_1, .., x_n) \in |M|^n \,:\, \mathrm{dist}(x_i, x_j) < \epsilon \ \forall 1 \leq i, j \leq n\,\}$$

as ϵ ranges over $\mathbb{R}_{>0}$. A map $f : |M'| \longrightarrow |M''|$ is uniformly continuous iff it induces a natural transformation $f_{\bullet} : M'_{\bullet} \longrightarrow M''_{\bullet}$ of functors $\Delta^{\mathrm{op}} \longrightarrow \boldsymbol{\varphi}$. For $n = 2$ this is checked in §2.1.2, and for $n > 2$ the argument is the same.

Thus we see that the category of metric spaces and uniformly continuous maps is a fully faithful subcategory of ɀ$\boldsymbol{\varphi}$.

Though not used, we present the following reformulation of the definition of uniform structure [Bourbaki, II§I.1,Def.I] to illuminate how translation to the language of $\wr\varphi$ works. Below we put in "()" the properties as formulated in [Bourbaki]; the notation is explained in the proof.

Lemma 2.2.3.1 (**[Bourbaki, II§I.1,Def.I]**). — *A uniform structure (or uniformity) on a set X is a structure given by a filter $\mathfrak{U}$ of subsets of $X \times X$ such that there is an object $X_\bullet : \Delta^{\mathrm{op}} \longrightarrow \varphi$ of $\wr\varphi$ satisfying*

(U_0) *Its underlying simplicial set $|X_\bullet| = |X|_\bullet$ is represented by the set $|X|$ of points of X.*
(U_I) *("Every set belonging to $\mathfrak{U}$ contains the diagonal Δ.")*
The filter on $X_\bullet(1_\leq)$ is indiscrete, i.e. is $\{|X_\bullet(1_\leq)|\}$.
(U_{II}) *("If $V \in \mathfrak{U}$ then $V^{-1} \in \mathfrak{U}$.")*
The functor $X_\bullet$ factors as

$$X_\bullet : \Delta^{\mathrm{op}} \longrightarrow \mathit{FiniteSets}^{op} \longrightarrow \varphi$$

(U_{III}) *("For each $V \in \mathfrak{U}$ there exists $W \in \mathfrak{U}$ such that $W \circ W \subset V$.")*
for $n > 2$ $|X|_\bullet(n_\leq) = |X|^n$ is equipped with the coarsest filter such that the maps $X^n \longrightarrow X \times X, (x_1, .., x_n) \mapsto (x_i, x_{i+1})$, $0 < i < n$, of filters are continuous.

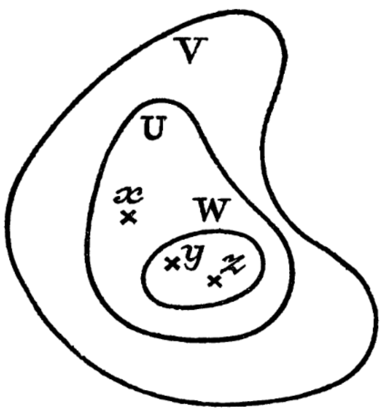

Figure 1.

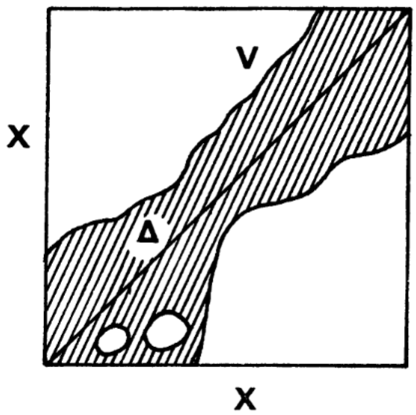

Figure 2.

Figure 1 [Bourbaki, I§I.1,Def.I] illustrates either of the following equivalent statements:

(V_{IV}) *If* V *belongs to* $\mathfrak{B}(x)$*, then there is a set* W *belonging to* $\mathfrak{B}(x)$ *such that, for each* $y \in$ W, V *belongs to* $\mathfrak{B}(y)$.

where $\mathfrak{B}(x)$ denotes the set of neighbourhoods of point $x \in X$.

(V_{IV}) The map $[1,3] : X \times X \times X \longrightarrow X \times X$ is continuous when $X \times X$ is equipped with the filter of coverings and $X \times X \times X$ is equipped with the coarsest filter such that both maps $[1,2],[2,3] : X \times X \times X \longrightarrow X \times X$ are continuous. To see the equivalence to the previous item, consider that the preimage of the set $|V| \times |V| \subset |X| \times |X|$ contains $\{x\} \times W \times V \subset |X| \times |X| \times |X|$.

Figure 2 [Bourbaki, II§I.1,Def.I] illustrates the filter $\mathfrak{U}$ of subsets of $X \times X$ above.

Proof(sketch). — (U_I): Consider the map $|X| \longrightarrow |X| \times |X|$, $x \longmapsto (x,x)$ induced by the map $[1,1] : 2_\leq \longrightarrow 1_\leq$. Being continuous with respect to the indiscrete filter $\{|X|\}$ and the filter $\mathfrak{U}$ on $|X| \times |X|$ means exactly every set belonging to $\mathfrak{U}$ contains the diagonal $d := \{(x,x) : x \in X\}$.

(U_{II}): As the functor factors via $\mathrm{FiniteSets}^{\mathrm{op}}$, the permutation $|X|\times|X| \longrightarrow |X|\times|X|$, $(x,y)\longmapsto(y,x)$ induces a continuous map of $\mathfrak{U}$ into itself. This means exactly that if $V\in\mathfrak{U}$ then $V^{-1}\in\mathfrak{U}$ where $V^{-1}:=\{(y,x) : (x,y)\in U\}$. ($U_{III}$): Recall $W'\circ W'' := \{(x_1,x_3) :$ exists x_2 such that $(x_1,x_2)\in W'$ and $(x_2,x_3)\in W''\}$. The coarsest filter on $|X|_\bullet(3_\le) = |X|^3$ such that the two maps $X^3\longrightarrow X\times X, (x_1,x_2,x_3)\mapsto(x_1,x_2), (x_1,x_2,x_3)\mapsto(x_2,x_3)$ of filters are continuous, is explicitly described as being generated by

$$\{(x_1,x_2,x_3) : (x_1,x_2)\in W' \text{ and } (x_2,x_3)\in W''\}, \quad \text{for } W',W''\in\mathfrak{U}$$

Continuity of the map $[1,3]:|X|^3\longrightarrow|X|^2$ means exactly that for any $V\in\mathfrak{U}$ there exists $W',W''\in\mathfrak{U}$ such that $W'\circ W''\subset V$. Now take $W := W'\cap W''$. □

2.2.4. An explicit set-theoretic description of a simplicial filter on a simplicial set represented by a preorder. — A class of examples of objects of ɀ$\boldsymbol{\varphi}$ associated with preorders can be described explicitly as follows.

Let $(I,\le)$ be a preorder, i.e. $\le$ is a binary transitive relation on I. We will be interested in the examples where it is a linear order and where it is an equivalence relation with only one equivalence class, i.e. a set with no further structure.

For each $0<n\in\omega$, let $\mathfrak{F}_n$ be a filter on

$$I_n := \{(t_1,...,t_n)\in|I|^n : t_1\le...\le t_n\}$$

such that for each $m,n>0$, weakly increasing sequence $1\le i_1\le...\le i_n\le m$, for each neighbourhood $\varepsilon\in\mathfrak{F}_n$

$$\{(t_1,...,t_m)\in|I|^m : (t_{i_1},...,t_{i_n})\in\varepsilon\}\in\mathfrak{F}_m$$

Such a sequence of filters gives rise to a simplicial filter $I^{\le}_\bullet:\Delta^{\mathrm{op}}\longrightarrow\boldsymbol{\varphi}$ defined as follows:

- $I^{\le}_\bullet(n_\le) := \mathfrak{F}_n$
- for each weakly increasing sequence $1\le i_1\le...\le i_n\le m$, the continuous map of filters $[i_1\le...\le i_n]: I^{\le}_\bullet(m_\le)\longrightarrow I^{\le}_\bullet(n_\le)$ is given by

$$(t_1,...,t_m)\longmapsto(t_{i_1},...,t_{i_n})$$

The condition on the filters $\mathfrak{F}_n$ means exactly that these maps are continuous.

A verification shows that these maps commute as required by functoriality: explicitly, this requirement states that

- the composition $I_l\xrightarrow{[j_1\le...\le j_m]}I_m\xrightarrow{[i_1\le...\le i_n]}I_n$ is equal to $I_l\xrightarrow{[j_{i_1}\le...\le j_{i_n}]}I_n$ as shown

$$I_l\xrightarrow{[j_1\le...\le j_m]}I_m\xrightarrow{[i_1\le...\le i_n]}I_n, \qquad I_l\xrightarrow{[j_{i_1}\le...\le j_{i_n}]}I_n$$

  for each $l,m,n>0$, each weakly increasing sequences $1\le j_1\le...\le j_m\le l$, $1\le i_1\le...\le i_n\le m$,

Let $I'^{\le}_\bullet:\Delta^{\mathrm{op}}\longrightarrow\boldsymbol{\varphi}$ be a simplicial filter associated with another preorder $(I',\le)$ and a sequence $\mathfrak{F}'_n$, $n>0$ of filters.

A verification shows that a monotone map $f: I\longrightarrow I'$ induces a morphism $I^{\le}_\bullet\longrightarrow I'^{\le}_\bullet$ iff for every n, every $\varepsilon\in\mathfrak{F}_n$ there is $\delta\in\mathfrak{F}'_n$ such that $f(\delta)\subset\varepsilon$, and that each such ɀ$\boldsymbol{\varphi}$-morphism is induced by a unique such map.

2.2.5. The coarsest and the finest simplicial filter induced by a filter. — Let $\mathfrak{F}$ be a filter on I. We can define the coarsest and the finest such sequence such that $\mathfrak{F}_1 = \mathfrak{F}$ as follows:

(C_1) $\mathfrak{F}_n$ is the coarsest filter such that all maps $[i] : \mathfrak{F}_n \longrightarrow \mathfrak{F}_1$, $1 \le i \le n$ are continuous. Explicitly, $\mathfrak{F}_n$ is generated by $\{I_n \cap \varepsilon^n : \varepsilon \in \mathfrak{F}_1\}$, i.e. by

$$\{\,\{(t_1, ..., t_n) \in I_n : t_1, ..., t_n \in \varepsilon\}\ :\ \varepsilon \in \mathfrak{F}_1\}$$

(F_1) $\mathfrak{F}_n$ is the finest filter such that the map $[1 \le ... \le 1] : \mathfrak{F}_1 \longrightarrow \mathfrak{F}_n$ is continuous, i.e. is generated by

$$\{\,\{(t, ..., t) \in I_n : t \in \varepsilon\}\ :\ \varepsilon \in \mathfrak{F}_1\}$$

Let $I_\bullet^{\le,\mathfrak{F}} : \wr\boldsymbol{\varphi}$ be the object of $\wr\boldsymbol{\varphi}$ defined in (C_1). Later we will use it when I is a linear order and $\mathfrak{F}$ is the filter of final segments.

Let I' be another preorder, and let $\mathfrak{F}'$ be a filter on I'. A verification shows that a monotone map $I \longrightarrow I'$ continuous with respect to filters $\mathfrak{F}$ and $\mathfrak{F}'$ induces an $\wr\boldsymbol{\varphi}$-morphism $I_\bullet^{\le,\mathfrak{F}} \longrightarrow I'^{\le,\mathfrak{F}'}_\bullet$, and each such $\wr\boldsymbol{\varphi}$-morphism is induced by a unique such map.

Let $|I|_\bullet^{\mathfrak{F}} : \wr\boldsymbol{\varphi}$ denote the object of $\wr\boldsymbol{\varphi}$ defined in (C_1) when the preorder on I is the equivalence relation with only one equivalence class, and $\mathfrak{F}$ is a filter on I. Note that this defines a functor $\boldsymbol{\varphi} \longrightarrow \wr\boldsymbol{\varphi}$.

2.2.6. Topological and uniform structures as simplicial filters. — More generally, let $\mathfrak{F}$ be a filter on I_m for some $m > 0$. We can define the coarsest and the finest such sequence such that $\mathfrak{F}_m = \mathfrak{F}$ as follows:

- $\mathfrak{F}_n$ is the coarsest filter such that all the face maps $[i_1 \le ... \le i_m] : \mathfrak{F}_n \longrightarrow \mathfrak{F}_m$, $1 \le i_1 \le ... \le i_m \le n$ are continuous. Explicitly, $\mathfrak{F}_n$ is generated by
$$\{\,\{x \in I_n : x[i_1 \le ... \le i_m] \in \varepsilon\} : \varepsilon \in \mathfrak{F}_m,\ 1 \le i_1 \le ... \le i_m \le n\}$$
- $\mathfrak{F}_n$ is the finest filter such that the map $[i_1 \le ... \le i_n] : \mathfrak{F}_m \longrightarrow \mathfrak{F}_n$, $1 \le i_1 \le ... \le i_n \le m$ is continuous, i.e. is generated by
$$\{\,\{(t_{i_1}, ..., t_{i_n}) \in I_n : (t_1, .., t_m) \in \varepsilon\}\ :\ \varepsilon \in \mathfrak{F}_m,\ 1 \le i_1 \le ... \le i_n \le m\}$$

Let $\mathfrak{F}'_n$, $n > 0$ be the sequence of coarsest filters associated with a filter $\mathfrak{F}'$ on I'_m for some preorder I'. A monotone map $f : I \longrightarrow I'$ induces a map $f_n : I_n \longrightarrow I'_n$, and this map is continuous with respect to $\mathfrak{F}_n$ and $\mathfrak{F}'_n$ iff $f_m : I_m \longrightarrow I'_m$ is continuous with respect to $\mathfrak{F}_m = \mathfrak{F}$ and $\mathfrak{F}'_m = \mathfrak{F}'$. Indeed, as $\mathfrak{F}'_n$ is the coarsest filter, we only need to check that the preimages in I_n of the preimages in I'_n of neighbourhoods in I'_m are neighbourhoods, and this follows from commutativity as they contain the preimages in I_n of the preimages under f of the neighbourhoods in I'_m.

Take the preorder I be the set of points of a topological space X equipped with the equivalence relation with only one equivalence class, and take $\mathfrak{F}$ to be the filter of coverings on $|X| \times |X|$. This defines an object of $\wr\boldsymbol{\varphi}$ corresponding to a topological space, which we will denote $X_\bullet$. In §2.1.2 we observed a map $f : |X'| \longrightarrow |X''|$ of topological spaces X', X'' is continuous iff it induces a continuous map from the filter of coverings on $X' \times X'$ to that on $X'' \times X''$. A verification shows this implies that $f : |X'| \longrightarrow |X''|$ is continuous iff it induces an $\wr\boldsymbol{\varphi}$-morphism $X_\bullet \longrightarrow X'_\bullet$, and each such morphism is induced by a unique continuous map. In the language of category theory, this is expressed by saying that we just constructed a fully faithful embedding of the category Top of topological spaces into the category of $\wr\boldsymbol{\varphi}$.

Similarly, we can take $\mathfrak{F}$ to be the filter of ϵ-neighbourhoods of the main diagonal on $|M| \times |M|$, for a metric space M. This defines an object of $\wr\boldsymbol{\varphi}$ corresponding to a metric space, which we will denote $M_\bullet$, a fully faithful embedding of the category of metric spaces and uniformly continuous maps into the category of $\wr\boldsymbol{\varphi}$.

3. Model theory

We now proceed to reformulate several notions in model theory in the language of $\wr\boldsymbol{\varphi}$.

3.1. Generalised Stone spaces. — Now we define a notion of generalised Stone space of a model in $\wr\boldsymbol{\varphi}$, and give some examples. We also point out that the generalised Stone space of a model carries information of the usual Stone space of 1-types, or, rather more precisely, the topology on the set of elements of the model induced by formulas of one variable as clopen subsets.

To define NTP in Appendix 9, we use another way to associate an object of $\wr\boldsymbol{\varphi}$ with a model.

3.1.1. Generalised Stone spaces associated with structures. — Call a sequence *(totally) ϕ-indiscernible with repetitions* iff each subsequence with *distinct* elements is necessarily (totally) ϕ-indiscernible. Recall that an totally indiscernible sequence is the same as an indiscernible set. Note that we allow that there are only finitely many distinct elements in a ϕ-indiscernible sequence with repetitions, e.g. $(a,b,a,b,a,b,...)$ is ϕ-indiscernible with repetitions for $\{a,b\}$ an indiscernible set.

Definition 3.1.1.1 ($M_\bullet^{\{\phi\}}, M_\bullet^\Sigma, M_\bullet^\Sigma/A : \wr\boldsymbol{\varphi}$). — Let M be a model, and let Σ be a set of formulas in the language of M.

Let $M_\bullet^\Sigma : \Delta^{\mathrm{op}} \longrightarrow \boldsymbol{\varphi}$ be the simplicial filter whose underlying simplicial set is $|M|_\bullet$ represented by the set of elements of M, i.e. the functor $\left\{- \xrightarrow[\text{Sets}]{} |M|\right\} : \Delta^{\mathrm{op}} \longrightarrow \text{Sets}$. A subset

$$\varepsilon \subseteq \left\{n_\leq \xrightarrow[\text{Sets}]{} |M|\right\} = \{(x_1,...,x_n) \in |M|^n\} = |M|^n$$

is a neighbourhood iff there is a finite subset $\Sigma' \subset \Sigma$ and $N > 0$ such that it contains each sequence which can be extended to a Σ'-indiscernible sequence with repetitions with at least N distinct elements. In notation:

A subset

$$\varepsilon \subseteq \left\{n_\leq \xrightarrow[\text{Sets}]{} |M|\right\} = \{(x_1,...,x_n) \in |M|^n\} = |M|^n$$

is a neighbourhood iff the following holds for some finite subset $\Sigma' \subset \Sigma$ and $N > 0$

- $(a_1,...,a_n) \in \varepsilon$ whenever there exist distinct $a_{n+1},...,a_N$ such that for any $m > 0$ for any $1 \leq i_1 < ... < i_m \leq N$ any subsequence $a_{i_1},..,a_{i_m}$ with distinct elements $a_{i_k} \neq a_{i_l}$, $1 \leq k \neq l \leq N$, is Σ'-indiscernible.

For a subset $A \subset |M|$ of parameters, let $M_\bullet/A$ be the quotient of $M_\bullet^\Sigma$ by the equivalence relation of having the same type over A, i.e. $M_\bullet^\Sigma/A(n_\leq)$ is the set $S_n^M(A)$ of n-types over A realised in M equipped with the filter induced from $M_\bullet(n_\leq)$.

Let $M_\bullet$ denote $M_\bullet^\Sigma$ for Σ the set of all parameter-free formulas of the language of M, and let $M_\bullet^{L(A)}$ denote $M_\bullet^\Sigma$ for Σ the set of all formulas of the language of M with parameters in A. Let $M_\bullet^{\mathrm{qf}}$ denote $M_\bullet^\Sigma$ for Σ the set of all quantifier-free formulas of the language of M.

To verify that $M_\bullet^\Sigma : \Delta^{\mathrm{op}} \longrightarrow \mathcal{P}$ is indeed a functor to $\mathcal{P}$ rather than just Sets, it is enough to verify that for each $\phi \in \Sigma$ $M_\bullet^{\{\phi\}} : \Delta^{\mathrm{op}} \longrightarrow \mathcal{P}$ is indeed a functor to $\mathcal{P}$ rather than just Sets. We only need to check that maps

$$[i_1 \leq ... \leq i_n] : M^m \longrightarrow M^n, \quad (x_1, .., x_m) \longmapsto (x_{i_1}, ..., x_{i_n})$$

are continuous for any $1 \leq i_1 \leq ... \leq i_n \leq m$. This says that for each neighbourhood $\varepsilon \subseteq |M|^n$ there is a neighbourhood $\delta \subset |M|^m$ such that $f(\delta) \subseteq \varepsilon$. That is, for some N, a tuple $(a_{i_1}, ..., a_{i_n})$ can be can be extended to a ϕ-indiscernible sequence with repetitions with at least N distinct elements, whenever, for some N' not depending on N, tuple $(a_1, ..., a_n)$ can be extended to a ϕ-indiscernible sequence with repetitions with at least N' distinct elements. It is enough to take $N' = N+n$.

Remark 3.1.1.2 **(Generalised Stone spaces: variants)**
In the definition above one may drop N and define the filter on $|M|^n$ as:

– A subset

$$\varepsilon \subseteq \left\{ n_\leq \underset{\text{Sets}}{\longrightarrow} |M| \right\} = \{(x_1, ..., x_n) \in |M|^n\} = |M|^n$$

is a neighbourhood iff there is a finite subset $\Sigma' \subset \Sigma$ such that ε contains each Σ'-indiscernible sequence $(a_1, ..., a_n)$ with repetitions

This is simpler but arguably less model-theoretically natural. Either variant of the definition can be used in the reformulations below involving the filter of tails and eventually indiscernible sequences, but the verification of "degenerate" cases, i.e. of diagrams involving non-injective maps, will be different. We shall denote these spaces by $\tilde{M}_\bullet^{\{\phi\}}$ and $\tilde{M}_\bullet^\Sigma$.

With this version of the definition the filter on $|M|$ is necessarily indiscrete, i.e. $\{|M|\}$; the filter on $|M \times M|$ is generated by sets $\phi(x) \leftrightarrow \phi(y)$, $\phi \in \Sigma$ unary. and, more generally, the filter on $|M^n|$ is generated by sets of $\phi()$-indiscernible sequences with repetitions where $\phi \in \Sigma$.

3.1.2. Generalised Stone spaces and the usual Stone spaces. — It is easy to see how to recover the usual Stone space of 1-types from the filter on $|M \times M|$ as defined in Remark 3.1.1.2:

– a subset U of $|M|$ is open, i.e. of form $\phi(x)$ for some unary $\phi(-)$, iff $U_2 := U \times U \cup (|M| \setminus U) \times (|M| \setminus U)\}$ is a neighbourhood.

Indeed, if $U = \{x : M \vDash \phi(x)\}$ then $U_2 = \{(x, y) : M \vDash \phi(x) \leftrightarrow \phi(y)\}$. Converse is a compactness argument: if U_2 is a neighbourhood, then there are finitely many formulas $\phi_1(x), ..., \phi_l(x)$ such that $U_2 \supset \{(x, y) : M \vDash \bigwedge_i \phi_i(x) \leftrightarrow \phi_i(y)\}$. By the definition of U_2, this means that that for any x it holds that

$$M \vDash \forall x \forall y (\bigwedge_i \phi_i(x) \leftrightarrow \phi_i(y) \implies x \in U \leftrightarrow y \in U)$$

Thus $U = \bigwedge_{x \in U} \{y : M \vDash \phi(x) \leftrightarrow \phi(y)\}$ and $|M| \setminus U = \bigwedge_{x \in |M| \setminus U} \{y : M \vDash \phi(x) \leftrightarrow \phi(y)\}$. By compactness it is enough to take finitely many formulas, and hence U is defined by a formula, i.e. $U = \{x : M \vDash \phi(x)\}$ for some formula $\phi(-)$.

In fact the forgetful functor $\wr\mathcal{P} \longrightarrow \mathrm{Top}$ (see [7,§2.6.3], also [6,§2.2.4]) takes $\tilde{M}_\bullet^\Sigma / A$ so defined into the Stone space of 1-types over A, and takes $\tilde{M}_\bullet$ into the set of elements of M with Stone topology, where we consider $\tilde{M}_\bullet^\Sigma / A$ and $\tilde{M}_\bullet$ as defined in Remark 3.1.1.2.

Analogously, the the forgetful functor $\wr\mathcal{P} \longrightarrow \mathrm{Top}$ takes $M_\bullet^\Sigma/A$ as defined in Definition 3.1.1.1 into the subspace of the Stone space of 1-types $p(-)$ consisting of types which admit infinite indiscernible sequences with elements of type $p(-)$.

3.1.3. Examples of generalised Stone spaces of a model with a formula. — Generalised Stone spaces as defined in Remark 3.1.1.2 are somewhat easier to describe, as we do not have to deal with finiteness. We start with them.

Example 3.1.3.1 ($\tilde{M}_\bullet^{\{\phi\}}$). — Let M be a model, and let ϕ be a formula in the language of M.

- Let $\phi(-)$ be a unary formula. Then the filter on $|\tilde{M}_\bullet^{\{\phi\}}(1)| = |M|$ is indiscrete, the filter on $|\tilde{M}_\bullet^{\{\phi\}}(n)| = |M|^n$ is generated by the set
$$\{(x_1, .., x_n) : \phi(x_1) \leftrightarrow \phi(x_1) \leftrightarrow ... \leftrightarrow \phi(x_n)\}$$
- Let $\phi(x, y)$ be a binary formula. Then the filters on $|M| = |\tilde{M}_\bullet^{\{\phi\}}(1)|$ and $|M \times M| = |\tilde{M}_\bullet^{\{\phi\}}(2)|$ are indiscrete.
- Let binary formula $\phi(x, y)$ be either $x \le y$ or $x < y$ where $<$ is a linear order on M. Then the filters on $|M| = |\tilde{M}_\bullet^{\{\phi\}}(1)|$ and $|M \times M| = |\tilde{M}_\bullet^{\{\phi\}}(2)|$ are indiscrete. The filter on $|M|^n = |\tilde{M}_\bullet^{\{\phi\}}(n)|$ is generated by the set of monotone sequences
$$\{(x_1, ..., x_n) : x_1 \le ... \le x_n \text{ or } x_1 \ge ... \ge x_n\}$$
- Let binary formula $\phi(x, y)$ be an equivalence relation. Then the filters on $|M| = |\tilde{M}_\bullet^{\{\phi\}}(1)|$ and $|M \times M| = |\tilde{M}_\bullet^{\{\phi\}}(2)|$ are indiscrete. The filter on $|M|^n = |\tilde{M}_\bullet^{\{\phi\}}(n)|$ is generated by the set which contains sequences whose elements all represent the same equivalence class, or all represent different equivalence classes:
$$\{(x_1, ..., x_n) : x_1 \approx ... \approx x_n \text{ or } \forall 1 \le i < j \le n (x_i \not\approx x_j)\}$$

In the generalised Stone spaces as defined in Definition 3.1.1.1, issues of being finite are important: whether elements are algebraic, the number of equivalence classes is finite, etc.

Example 3.1.3.2 ($M_\bullet^{\{\phi\}}$). — As before, let M be a infinite model, and let ϕ be a formula in the language of M.

- Let $\phi(-)$ be a unary formula. If both $\phi(x)$ and $\neg\phi(x)$ are infinite, the filter on $|\tilde{M}_\bullet^{\{\phi\}}(1)| = |M|$ is indiscrete. If say $\phi(x)$ is infinite and $\neg\phi(x)$ is finite, then the filter on $|\tilde{M}_\bullet^{\{\phi\}}(1)| = |M|$ generated by the set $\{\phi(x) : M \vDash \phi(x)\}$. The filter on $|\tilde{M}_\bullet^{\{\phi\}}(n)| = |M|^n$ is generated by
$$\{(x_1, .., x_n) : \phi(x_1) \leftrightarrow \phi(x_2) \leftrightarrow ... \leftrightarrow \phi(x_n) \text{ and } \exists^\infty y\, (\phi(y) \leftrightarrow \phi(x_1))\}$$
- Let ϕ be a formula. Then the filter on $|M| = |M_\bullet^{\{\phi\}}(1)|$ is generated by the set of elements which belong to an infinite ϕ-indiscernible sequence. The filter on $|M|^n = |M_\bullet^{\{\phi\}}(n)|$ is generated by the set of ϕ-indiscernible sequence with repetitions which can be extended to an infinite ϕ-indiscernible sequence.
- Let binary formula $\phi(x, y)$ be either $x \le y$ or $x < y$ where $<$ is a linear order on M. The filter on $|M|^n = |\tilde{M}_\bullet^{\{\phi\}}(n)|$ is generated by the set of monotone sequences

$$\{(x_1, ..., x_n) : \forall N \exists x_{n+1}..x_N (x_1 \le ... \le x_n < x_{n+1} < ... < x_N \bigvee x_1 \ge ... \ge x_n > x_{n+1} > .. > x_N)\}$$

Thus, for example, if $a < b \in M$ and b is a maximal element of M, then $(a, b) \in |M \times M|$ lies outside of this neighbourhood in $M \times M$.

– Let binary formula $\phi(x, y)$ be an equivalence relation. Then the filter on $|M| = |M_\bullet^{\{\phi\}}(1)|$ is generated by the union of *infinite* equivalence classes. If the number of equivalence classes is finite, the filter on $|M \times M| = |M_\bullet^{\{\phi\}}(2)|$ is generated by the pairs of elements of infinite equivalence classes. If the number of equivalence classes is infinite, then the filter on $|M \times M| = |M_\bullet^{\{\phi\}}(2)|$ is generated by the set of the pairs of elements of infinite equivalence classes, as before, and of pairs of non-equivalent elements. In notation, this is described as follows. The filter on $|M|^n = |M_\bullet^{\{\phi\}}(n)|$ is generated by the set which contains sequences whose elements all represent the same *infinite* equivalence class, or all represent different equivalence classes *if there are infinitely many of them*:

$$\{(x_1, ..., x_n) : x_1 \approx ... \approx x_n \text{ and the equivalence class of } x_1, x_2, ..., x_n \text{ is infinite,}$$
$$\text{or there are infinitely many equivalence classes and } \forall 1 \le i < j \le n (x_i \not\approx x_j)\}$$

3.1.4. Generalised Stone spaces of the dense linear order and of an equivalence relation. — Now let us describe the generalised Stone spaces associated with the theory of one or many equivalence relations, and the theory of the dense linear order.

Again we start with $\tilde{M}_\bullet$ so that we do not have to deal with issues of finiteness.

***Example 3.1.4.1* (Dense linear order DLO).** — Let $(M; <)$ be a model of DLO, the theory of dense linear order. Quantifier elimination implies that any $<$-indiscernible sequence is also ϕ-indiscernible for an arbitrary parameter-free formula ϕ of DLO. Thus $\tilde{M}_\bullet = \tilde{M}_\bullet^{\{<\}}$, and thus the filters on $|M| = |\tilde{M}_\bullet(1)|$ and $|M \times M| = |\tilde{M}_\bullet(2)|$ are indiscrete,. and the filter on $|M|^n = \tilde{M}_\bullet(n)$ is generated by the set of monotone sequences

$$\{(x_1, ..., x_n) : x_1 \le ... \le x_n \text{ or } x_1 \ge ... \ge x_n\}$$

Now consider $\tilde{M}_\bullet^{L(A)}$ for $A \subset |M|$, the Stone space with parameters in $A \subset |M|$. Quantifier elimination implies that for an arbitrary parameter-free formula ϕ of DLO with parameters in A there are finitely many 1-ary formulas of form $x < a$, $a < x$, $a < x < b$, such that a sequence is ϕ-indiscernible iff it is indiscernible with respect to $<$ and each of these formulas. Hence, the filter on $\tilde{M}_\bullet^{L(A)}$ is generated by the filters $\tilde{M}_\bullet^{\{<\}}$, and $\tilde{M}_\bullet^{\{x<a\}}$, and $\tilde{M}_\bullet^{\{a<x<b\}}$, $a, b \in A$.

Explicitly, the filter on $|M| = |\tilde{M}_\bullet(1)|$ is indiscrete; the filter on $|M \times M| = |\tilde{M}_\bullet(2)|$ is generated by sets $\{x : x < a\} \times \{x : x < a\} \cup \{x : x \ge a\} \times \{x : x \ge a\}$, and $\{x : x <\le a\} \times \{x : x \le a\} \cup \{x : x > a\} \times \{x : x > a\}$.

Remark 3.1.4.2. — A model M is o-minimal iff there is a model $(Q; <)$ of DLO such that there are isomorphisms $\tilde{M}_\bullet^{L(M)}(1) \xrightarrow{(iso)} \tilde{Q}_\bullet^{L(Q)}(1)$ and $\tilde{M}_\bullet^{L(M)}(2) \xrightarrow{(iso)} \tilde{Q}_\bullet^{L(Q)}(2)$ commuting with the simplicial maps.

Equivalently, there is a bijection $|M| \xrightarrow{(iso)} |Q|$ of sets inducing isomorphisms of filters $\tilde{M}_\bullet^{L(M)}(1) \xrightarrow{(iso)} \tilde{Q}_\bullet^{L(Q)}(1)$ and $\tilde{M}_\bullet^{L(M)}(2) \xrightarrow{(iso)} \tilde{Q}_\bullet^{L(Q)}(2)$.

Indeed, the map $\tilde{M}_\bullet^{L(M)}(2) \xrightarrow{(iso)} \tilde{Q}_\bullet^{L(Q)}(2)$ being an isomorphism of filters means exactly that each 1-ary formula of M corresponds to a 1-ary formula of Q, and hence is a union of intervals.

***Example 3.1.4.3* (The theory of an equivalence relation)**
Let $(M;\approx)$ be a model of a theory with a single equivalence relation with infinitely many classes, all of which are infinite. Quantifier elimination implies that any $\approx$-indiscernible sequence is also ϕ-indiscernible for an arbitrary parameter-free formula ϕ of the theory. Hence, $\tilde{M}_\bullet = \tilde{M}_\bullet^{\{\approx\}}$, and, by assumption that the classes are infinite and there are infinitely many of them, both versions of the Stone space coincide $M_\bullet = \tilde{M}_\bullet$.

Now consider $\tilde{M}_\bullet^{L(A)}$ for $A \subset |M|$, the Stone space with parameters in $A \subset |M|$. Quantifier elimination implies that for an arbitrary parameter-free formula ϕ of the theory with parameters in A there are finitely many 1-ary formulas of form $x \approx a$ such that a sequence is ϕ-indiscernible iff it is indiscernible with respect to $\approx$ and each of these formulas. Hence, the filter on $\tilde{M}_\bullet^{L(A)}$ is generated by the filters $\tilde{M}_\bullet^{\{\approx\}}$, and $\tilde{M}_\bullet^{\{x\approx a\}}$, $a \in A$, and also we have that both versions of the Stone space coincide $M_\bullet^{L(A)} = \tilde{M}_\bullet^{L(A)}$.

Explicitly, the filter on $|M| = |\tilde{M}_\bullet^{L(A)}(1)|$ is indiscrete; the filter on $|M \times M| = |\tilde{M}_\bullet^{L(A)}(2)|$ is generated by sets $\{x : x \approx a\} \times \{x : x \approx a\} \cup \{x : x \not\approx a\} \times \{x : x \not\approx a\}$, $a \in A$. The filter on $|M \times M \times M| = |\tilde{M}_\bullet(3)|$ is generated by, for $a \in A$,

$$\{(x_1, x_2, x_3) :: x_1 \approx ... \approx x_3 \text{ or } \forall 1 \leq i < j \leq 3(x_i \not\approx x_j)\}$$

and

$$\{(x_1, x_2, x_3) :: a \approx x_1 \approx x_2 \approx x_3 \text{ or } \forall 1 \leq i < j \leq 3(x_i \not\approx a)\}$$

The latter sets are in fact the pullbacks of filters on $|M \times M| = |\tilde{M}_\bullet^{L(A)}(2)|$ along simplicial maps $M^n \longrightarrow M^2$, $(x_1, ..., x_n) \longmapsto (x_i, x_j)$, $1 \leq i \leq j \leq n$.

The filter on $|M^n| = |\tilde{M}_\bullet(n)|$ is generated by

$$\{(x_1, ..., x_n) : x_1 \approx ... \approx x_n \text{ or } \forall 1 \leq i < j \leq n(x_i \not\approx x_j)\}$$

and

$$\{(x_1, x_2, x_3) : a \approx x_1 \approx ... \approx x_n \text{ or } \forall 1 \leq i < j \leq n(x_i \not\approx a)\}$$

The latter sets are in fact the pullbacks of filters on $|M \times M| = |\tilde{M}_\bullet^{L(A)}(2)|$ along simplicial maps The latter sets are the pullbacks of filters on $|M \times M| = |\tilde{M}_\bullet^{L(A)}(2)|$ along simplicial maps $M^n \longrightarrow M^2$, $(x_1, ..., x_n) \longmapsto (x_i, x_j)$, $1 \leq i \leq j \leq n$.

3.2. Shelah representation of stable theories by equivalence relations. — We start with an elementary Proposition 3.2.1.1 which says that if models of a theory are "approximated" in ጌፐ by structures with boundedly many equivalence relations, then the theory is stable. "Approximated" here means there is a surjection from the generalised Stone space of such a structure with equivalence relations, to that of the model.

We then show that a theorem of Shelah on representations of stable theories [CoSh:919, Theorem 3.1(7)] gives a characterisation of stable theories in these terms, though the statement is slightly more technical.

Finally, we remark that Proposition 3.2.2.1 formalises the intuition behind [CoSh:919] rather more literally then the paper itself.

We end by suggesting a characterisation Corollary 3.2.4.2 of stable theories in terms of category theory.

3.2.1. Surjective images of structures with boundedly many equivalence relations are stable. — The following is elementary.

Proposition 3.2.1.1. — *A theory T is stable if there is κ such that for each model M of T there is a structure $\mathbf{I}$ on the same domain, $|\mathbf{I}| = |M|$, with at most κ equivalence relations $\approx_\alpha, \alpha < \kappa$ (and nothing else), such that there is a $\rightthreetimes$-surjection $\mathbf{I}_\bullet^{\{\approx_\alpha:\alpha<\kappa\}} \longrightarrow M_\bullet$.*

Proof (sketch). — Let M be a large enough model of T and let $\mathbf{I}_\bullet^{\{\approx_\alpha:\alpha<\kappa\}} \longrightarrow M_\bullet$ be an $\rightthreetimes$-surjection. A long enough sequence indiscernible in a model M of T has an infinite subsequence quantifier-free indiscernible in $\mathbf{I}$, as the number of quantifier-free types in $\mathbf{I}$ is bounded. In $\mathbf{I}$, a quantifier-free indiscernible sequence is necessarily quantifier-free order-indiscernible, and therefore order-indiscernible in M, because $\rightthreetimes$-morphisms preserve indiscernible sequences. Hence, in M every long enough indiscernible sequence has an infinite order-indiscernible subsequence, and hence is order-indiscernible itself. Hence, any large enough and saturated enough model of T is stable, and therefore T is stable. □

3.2.2. Shelah representation as a characterisation of stable theories as surjective images of structures with boundedly many equivalence relations. — Now we use a result of Shelah to prove a characterisation of stable theories in terms of $\rightthreetimes$. Essentially, in the statement above we replace $\mathbf{I}_\bullet^{\{\approx_\alpha:\alpha<\kappa\}} \longrightarrow M_\bullet$ by a finer structure $\mathbf{I}_\bullet^{\Sigma}$ where Σ is the set of all quantifier-free types in $\mathbf{I}$. Note that the size of Σ is still bounded.

Proposition 3.2.2.1. — *A theory T is stable iff there is κ such that for each model M of T there is a structure $\mathbf{I}$ on the same domain, $|\mathbf{I}| = |M|$, with at most κ equivalence relations $\approx_\alpha, \alpha < \kappa$ (and nothing else), such that there is a $\rightthreetimes$-surjection $\mathbf{I}_\bullet^{\{\Sigma\}} \longrightarrow M_\bullet$ where Σ is the set of quantifier-free types in $\mathbf{I}$.*

Proof (sketch). — $\Leftarrow$: The same argument applies, as the cardinality of Σ is bounded.

$\Rightarrow$: We now use [CoSh:919, Theorem 3.1(7)]:

> A theory T is stable iff there is κ such that for each model M of T there is a structure $\mathbf{I}$ with at most κ unary functions (and equality, and nothing else) which represents M.

Our argument is based on the following easy lemma.

Lemma 3.2.2.2. — *In a theory in a language consisting only of equality and unary functions, which we assume closed under composition, the quantifier-free type of an indiscernible sequence of $n \geq 3$ elements is isolated, among types of indiscernible sequences, by a formula of the form*

$$\bigwedge_{\substack{1\leq i\leq n\\(f,g)\in F_1}} f(x_i) = g(x_i) \;\;\&\; \bigwedge_{\substack{1\leq i\leq n\\(f,g)\in F_2}} f(x_i) \neq g(x_i) \;\;\&\; \bigwedge_{\substack{i<j\\f\in F_3}} f(x_i) = f(x_j) \;\;\&\; \bigwedge_{\substack{i<j\\f\in F_4}} f(x_i) \neq f(x_j)$$

for some sets F_1, F_2 of pairs of unary functions, and some sets F_3, F_4 of unary functions.

Proof. — Indeed, let $f(x_1) = g(x_2)$ be in the quantifier-free type of an indiscernible sequence (a_1, a_2, a_3). Then so are $f(x_1) = g(x_3)$, $f(x_2) = g(x_3)$, and therefore $f(x_1) = f(x_2) = g(x_2) = g(x_3)$, which is equivalent to the conjunction of $f(x_1) = f(x_2)$, $f(x_2) = g(x_2)$, and $g(x_2) = g(x_3)$ of the required form, and implies the formula $f(x_1) = g(x_2)$ we started with. □

Note that in the formula in the statement above we omit atomic formulas $f(x_i) = g(x_j)$ for $i \neq j$, $f \neq g$, that are not equivalence relations.

Let M be a model of T, and let $\mathbf{I}'$ represent M as in Theorem 3.1(7). By definition of representation [CoSh:919, Def. 2.1], a quantifier-free indiscernible sequence in $\mathbf{I}'$ is necessarily indiscernible in M, hence the identity map $|\mathbf{I}'| \longrightarrow |M|$ induces an $\wr\mathcal{P}$-morphism $\mathbf{I}'^{\Sigma'}_\bullet \longrightarrow M_\bullet$ where Σ' is the set of quantifier-free types of indiscernible sequences in the language of $\mathbf{I}'$. This morphism is obviously surjective. (The requirement in the definition of representation talks about arbitrary types, not only types of indiscernible sequences.) We may assume that the unary functions of $\mathbf{I}'$ are closed under composition. Consider the reduct $\mathbf{I}$ of $\mathbf{I}'$ in the language containing the equivalence relations $f(x) = f(y)$ and unary predicates $f(x) = g(y)$ where f, g are functions in the language of $\mathbf{I}'$. Lemma 3.2.2.2 implies that there is an $\wr\mathcal{P}$-surjection $\mathbf{I}^\Sigma_\bullet \longrightarrow \mathbf{I}'^{\Sigma'}_\bullet$ where Σ is the set of quantifier-free types of $\mathbf{I}$. To finish the proof, consider the composition $\mathbf{I}^\Sigma_\bullet \longrightarrow \mathbf{I}'^{\Sigma'}_\bullet \longrightarrow M_\bullet$. □

Remark 3.2.2.3. — [Sh:1043, Theorem 2.1(6)] gives a similar characterisation of superstable theories in terms of representation by the class $Ex^{0,\mathrm{lf}}_{\mu,\kappa}$ of "locally finite" structures with unary functions. It suggests there might exist a similar reformulation in terms of $\wr\mathcal{P}$-surjections from quantifier-free generalised Stone spaces of models with boundedly many equivalence relations and nothing else.

3.2.3. Shelah's intuition of representability. — Thus we saw that $\wr\mathcal{P}$ can express formally in a very literal manner the following intuition expressed by [Sh:1043]:

> Here we deal with another external property, *representability*. This notion was a try to formalize the intuition that "the class of models of a stable first order theory is not much more complicated than the class of models $M = (A, \dots, E_t, \dots)_{s \in I}$ where E^M_t is an equivalence relation on A refining E^M_s for $s < t$; and I is a linear order of cardinality $\leq |T|$. It was first defined in Cohen-Shelah [SC16], where it was shown that one may characterize stability and $\aleph_0$-stability by means of representability.

Note that $\wr\mathcal{P}$-reformulation explicitly talks about equivalence relations, unlike [CoSh:919, Theorem 3.1(7)] or [Sh:1043, Theorem 2.1(6)].

3.2.4. A category-theoretic characterisation of classes of stable models. —

Remark 3.2.4.1. — A structure $\mathbf{I}$ with equivalence relations $\approx_\alpha$, $\alpha < \kappa$,(and nothing else) gives rise to a functor $\mathbf{I}^{\{\approx_\alpha:\alpha<\kappa\}}_\bullet : \Delta^{\mathrm{op}} \longrightarrow \mathcal{P}$ which factors as

$$\mathbf{I}^{\{\approx_\alpha:\alpha<\kappa\}}_\bullet : \Delta^{\mathrm{op}} \longrightarrow \mathrm{FiniteSets}^{\mathrm{op}} \longrightarrow \mathcal{P}$$

which could be then be called of a "2-dimensional" "symmetric" simplicial filter. Here by "symmetric" we mean that the simplicial filter $\mathbf{I}^{\{\approx_\alpha:\alpha<\kappa\}}_\bullet$ factors as shown, i.e. via the inclusion of categories $\Delta^{\mathrm{op}} \longrightarrow \mathrm{FiniteSets}^{\mathrm{op}}$. By "2-dimensional" we mean that for each $n > 3$ the filter on $|\mathbf{I}^n| = |\mathbf{I}^{\{\approx_\alpha:\alpha<\kappa\}}(n_\leq)|$ is induced from that on $|\mathbf{I}^3| = |\mathbf{I}^{\{\approx_\alpha:\alpha<\kappa\}}_\bullet(3_\leq)|$, i.e. is the coarsest filter such that all face maps $[i \leq j \leq k] : \mathbf{I}^{\{\approx_\alpha:\alpha<\kappa\}}_\bullet(n_\leq) \longrightarrow \mathbf{I}^{\{\approx_\alpha:\alpha<\kappa\}}_\bullet(3_\leq)$ are continuous. In fact, this functor is probably the free $\wr\mathcal{P}$-object "started by" (i.e. 3-coskeleton generated by) $(\mathbf{I}, \mathbf{I} \times \mathbf{I}, \mathbf{I} \times \mathbf{I} \times \mathbf{I})$ equipped with appropriate filters.

Note that a similar reformulation can be given to the axioms of uniform structure §2.2.3 or topological space §4.2.4, also cf. [Bourbaki,II§1.1,Def.1], for details see §2.2.3 or [6,Exercise 4.2.1.5]: a uniform structure is a 1-dimensional symmetric simplicial set such that the filter of 0-simplicies is indiscrete.

Corollary 3.2.4.2. — *A theory T is stable iff there is a cardinal κ such that for each model $M \models T$ there is a surjective $\wr\varphi$-morphism $I_\bullet \longrightarrow M_\bullet$ from a "2-dimensional" "symmetric" simplicial filter $I_\bullet : \Delta^{\mathrm{op}} \longrightarrow FiniteSets^{op} \longrightarrow \varphi$ with at most κ neighbourhoods, or, equivalently, such that its filter structure is pulled back from at most κ morphisms to filters of form*

$$J_\bullet : \Delta^{\mathrm{op}} \longrightarrow FiniteSets^{op} \longrightarrow \varphi$$

where for each $n > 0$ $|J_\bullet(n_\leq)|$ is a finite set.

Proof. — By Proposition 3.2.2.1 and Remark 3.2.4.1 this does hold for a stable theory. The same proof as in Proposition 3.2.2.1 shows that in a saturated enough models indiscernible sequences are in fact indiscernible sets. □

It will be interesting to compare this to [Boney, Erdos-Rado Classes, Thm 6.8].

3.3. Stability as a Quillen negation analogous to a path lifting property. — Now we use the definitions and intuitions introduced in §2.1.4 to reformulate the characterisation of stability that "each infinite indiscernible sequence is necessarily an indiscernible set" as a Quillen lifting property/negation.(5) Surprisingly(6) such a naive, oversimplified, and mechanistic way of "transcribing" this characterisation in terms of $\wr\varphi$ produces a correct conjecture; being oversimplified is an essential feature. We explain the process in a verbose way in §3.3.3.

3.3.1. Indiscernible sequences with repetitions. — The following lemma is the key observation which started this paper.

Let $|I|_\bullet$ and $I^\leq_\bullet$ denote the simplicial sets represented by the set $|I|$, resp. the linear order $I^\leq$, equipped with the indiscrete filters (see Example 2.2.2 for details). Recall that being equipped with indiscrete filters means that for each $n > 0$, the only neighbourhood on $|I|_\bullet(n_\leq) = |I|^n$ is the whole set $\{||I|(n_\leq)|\}$, and the only neighbourhood on $I^\leq_\bullet(n_\leq) = \{(i_1, ..., i_n) \in |I|^n : i_1 \leq ... \leq i_n\}$ is the whole set $\{|I^\leq_\bullet(n_\leq)|\}$.

Call an (totally) ϕ-indiscernible sequence with repetitions *infinitely extendable* iff it is a subsequence of a an (totally) ϕ-indiscernible sequence with repetitions with infinitely many distinct elements. This notion is only non-trivial for sequences with finitely many distinct elements.

Lemma 3.3.1.1. — *For any infinite linear order I and any structure M the following holds.*

- *An infinitely extendable ϕ-indiscernible sequence $(a_i)_{i \in I}$ in M with repetitions induces an $\wr\varphi$-morphism $a_\bullet : I^\leq_\bullet \longrightarrow M^{\{\phi\}}_\bullet$, and, conversely, each morphism $I^\leq_\bullet \longrightarrow M^{\{\phi\}}_\bullet$ is induced by a unique such sequence.*

(5) A morphism $i : A \to B$ in a category has *the left lifting property* with respect to a morphism $p : X \to Y$, and $p : X \to Y$ also has *the right lifting property* with respect to $i : A \to B$, denoted $i \rightthreetimes p$, iff for each $f : A \to X$ and $g : B \to Y$ such that $p \circ f = g \circ i$ there exists $h : B \to X$ such that $h \circ i = f$ and $p \circ h = g$. This notion is used to define properties of morphisms starting from an explicitly given class of morphisms, often a list of (counter)examples, and a useful intuition is to think that the property of left-lifting against a class C is a kind of negation of the property of being in C, and that right-lifting is also a kind of negation. See [Wikipedia,Lifting_property] for details and examples.

(6) In fact, for the author this is the most curious observation of the paper which calls for an explanation—why (and whether!) does this kind of naive, oversimplified, and mechanistic way of "transcribing" produces a correct conjecture so often. A temptation is to connect this with the speculative notion of ergologic by Gromov, by saying it is a rule of ergologic.

– *An infinitely extendable order ϕ-indiscernible sequence $(a_i)_{i\in I}$ with repetitions induces an $\wr\!\!\mathcal{P}$-morphism $a_\bullet : |I|_\bullet \longrightarrow M_\bullet^{\{\phi\}}$, and, conversely, each morphism $|I|_\bullet \longrightarrow M_\bullet^{\{\phi\}}$ is induced by a unique such sequence.*

Proof. — First note that by Example 2.2.2 maps $|I| \longrightarrow |M|$ of sets are into one-to-one correspondence with morphisms $|I|_\bullet \longrightarrow |M|_\bullet$, i.e. the morphisms of the underlying simplicial sets $|I|_\bullet = ||I|_\bullet|$ and $|M|_\bullet = |M_\bullet^{\{\phi\}}|$, and the same is true for $|I_\bullet^\leq| = |I_\bullet^\leq| \longrightarrow |M|_\bullet = |M_\bullet^{\{\phi\}}|$. Hence, at the level of the underlying simplicial sets, any sequence $(a_i)_{i\in I}$ gives rise to morphisms $|I_\bullet^\leq| \longrightarrow |M_\bullet^{\{\phi\}}|$ and $||I|_\bullet| \longrightarrow |M_\bullet^{\{\phi\}}|$ of the underlying simplicial sets. Therefore, we only need to check what it means to be continuous for the following induced maps of filters for each $n > 0$:

$$|I_\bullet^\leq(n_\leq)| = \{(i_1, ..., i_n) \in |I|^n : i_1 \leq ... \leq i_n\} \longrightarrow M^n$$

$$||I|_\bullet(n_\leq)| = |I|^n \longrightarrow M^n$$

The sets $|I|^n$ and $\{(i_1, ..., i_n) \in |I|^n : i_1 \leq ... \leq i_n\}$ are equipped with the indiscrete filter, hence continuity means that the image of the map in contained in any neighbourhood. For the first map it means that for any weakly increasing sequence $i_1 \leq ... \leq i_n$ for any N the sequence $(a_{i_1}, ..., a_{i_n})$ can be extended a ϕ-indiscernible sequence with repetitions with arbitrarily many distinct elements. This exactly means that the sequence $(a_i)_{i\in I}$ in M is an infinitely extendable ϕ-indiscernible sequence with repetitions.

For the second map it means that for any sequence $i_1, ..., i_n$ (not necessarily ordered, and with repetitions) the sequence $(a_{i_1}, ..., a_{i_n})$ can be extended to a ϕ-indiscernible sequence with repetitions with arbitrarily many distinct elements. This means that the sequence $(a_i)_{i\in I}$ is totally indiscernible, and is also infinitely extendable ϕ-indiscernible, as required. □

Remark 3.3.1.2. — Using $M_\bullet/A$ instead of $M_\bullet$ in the lemma above would give us a bit of flexibility at the cost of some simplicial combinatorics. For a structure I, I-indiscernibles induce a map of simplicial sets

$$|I_\bullet^{\mathrm{qf}}/\approx| \longrightarrow |M_\bullet/A|$$

and each such map of simplicial sets is induced by I-indiscernibles. For example, this allows one to talk about mutually indiscernible sequences $(I_t)_t$ over A, though in this case it is probably better to consider a different simplicial set which "remembers" the order: instead of $|\cup_t I_t|_\bullet(n_\leq) = |\cup_t I_t|^n$ take its subset consisting of tuples where elements of each I_t occur in an weakly increasing order.

3.3.2. Stability as Quillen negation of indiscernible sets (fixme: better subtitle..)— Let $\top = |\{pt\}|_\bullet$ denote the terminal object of the category $\wr\!\!\mathcal{P}$, i.e. "the simplicial set represented by a singleton equipped with indiscrete filters": for any $n > 0$ $\top(n_\leq) := \{pt\}$, and the only big subset is $\{pt\}$ itself.

***Proposition 3.3.2.1* (Stability as Quillen negation)**

Let M be a model, and let ϕ be a formula in the language of M. The following are equivalent:

(i) *in the model M, each infinite ϕ-indiscernible sequence is necessarily a ϕ-indiscernible set*
(ii) *in the model M, each ϕ-indiscernible sequence with repetitions and with infinitely many distinct elements is necessarily order ϕ-indiscernible with repetitions*

(iii) *in* ɀ𝝋 *the following lifting property holds for each infinite linear order* I*:*

$$I^{\leq}_{\bullet} \longrightarrow |I|_{\bullet} \;\rightthreetimes\; M^{\{\phi\}}_{\bullet} \longrightarrow \top$$

i.e. the following diagram in ɀ𝝋 *holds:*

$$\begin{array}{ccc} I^{\leq}_{\bullet} & \longrightarrow & M^{\{\phi\}}_{\bullet} \\ \downarrow & \nearrow & \downarrow \\ |I|_{\bullet} & \longrightarrow & \top \end{array}$$

(iv) *in* ɀ𝝋 *the following lifting property holds*

$$\omega^{\leq}_{\bullet} \longrightarrow |\omega|_{\bullet} \;\rightthreetimes\; M^{\{\phi\}}_{\bullet} \longrightarrow \top$$

i.e. the following diagram in ɀ𝝋 *holds:*

$$\begin{array}{ccc} \omega^{\leq}_{\bullet} & \longrightarrow & M^{\{\phi\}}_{\bullet} \\ \downarrow & \nearrow & \downarrow \\ |\omega|_{\bullet} & \longrightarrow & \top \end{array}$$

Proof. — (i)⇔(ii) is obvious.

(iii) $\Longrightarrow$ (i): Let $(a_i)_{i\in I}$ be an infinite ϕ-indiscernible sequence. By Lemma 3.3.1.1 it induces an ɀ𝝋-morphism $a_\bullet : I^{\leq}_{\bullet} \longrightarrow M^{\{\phi\}}_{\bullet}$. It fits into the commutative square, as any square with top vertex $\top$ commutes. By the lifting property it lifts to a ɀ𝝋-morphism $a'_\bullet : |I|_\bullet \longrightarrow M^{\{\phi\}}_{\bullet}$. By commutativity of the upper triangle both morphisms correspond to the same map of the underlying simplicial sets, i.e. to the same sequence $(a_i)_{i\in I}$. Again by Lemma 3.3.1.1, this sequence is order ϕ-indiscernible with repetitions, and, in this case, just order ϕ-indiscernible as its elements are all distinct.

(ii) $\Longrightarrow$ (iii): Let $a_\bullet : I^{\leq}_{\bullet} \longrightarrow M^{\{\phi\}}_{\bullet}$ be the ɀ𝝋-morphism corresponding to the lower horizontal arrow. By Lemma 3.3.1.1 it corresponds to an infinitely extendable ϕ-indiscernible sequence $(a_i)_{i\in I}$ with repetitions. Pick such an infinite extension. It is ϕ-indiscernible with repetitions, and by (ii) it is order ϕ-indiscernible with repetitions. Hence, so is its subsequence $(a_i)_{i\in I}$, and by Lemma 3.3.1.1 it corresponds to an ɀ𝝋-morphism $|I|_\bullet \longrightarrow M^{\{\phi\}}_{\bullet}$, as required.

(i)⇔(iv): by compactness (i) holds if it holds for sequences $(a_n)_{n\in\omega}$ indexed by ω, which is what the proof of the equivalence (i)⇔(iii) gives us in the case $I = \omega$. □

3.3.3. An informal explanation. — Surprisingly, the following naive, oversimplified, and mechanistic way of thinking produces a correct conjecture. Let us try "transcribe" in terms of ɀ𝝋 the characterisation "each infinite indiscernible sequence is necessarily an indiscernible set" we used; we explain the process in a verbose way; being oversimplified is an essential feature.

An indiscernible *sequence* is indexed by a linear order and we would like to think of it as a map from *a linear order* to a model. An indiscernible *set* is indexed by a set rather that a linear order, and we would like to think of it as a map from a *set* to a model.

In ɀ𝝋 or indeed in sSets, a straightforward way to interpret "a map from a linear order $I^{\leq}$ to a model M" is to consider a morphism $I^{\leq}_{\bullet} \longrightarrow |M|_\bullet$ from the simplicial set $I^{\leq}_{\bullet}$ represented by the linear order, to the simplicial set $|M|_{\bullet}$ represented by the set of elements of the model.

Note an oversimplification: we say nothing on the crucial property assuring that this map encodes an indiscernible sequence.

Similarly, a straightforward way to interpret "a map from a set $|I|$ to a model M" is to consider a morphism $|I|_{\bullet} \longrightarrow |M|_{\bullet}$ from the simplicial set $|I|_{\bullet}$ represented by the set $|I|$ of elements of the linear order, to the simplicial set $|M|_{\bullet}$ represented by the set of elements of the model.

Note that in this simplicial formalisation, a map from an ordered set is not automatically a map from its underlying set, even though it is more natural when thinking of homomorphisms. And indeed, if in our formalism a map from an ordered set were automatically a map from its underlying set, we would not be able to reformulate the characterisation of stability (it would hold trivially).

The characterisation says that each infinite indiscernible sequence gives rise to an indiscernible set; accordingly, we'd like to say that in ɀ𝝋 and sSets, a map $I^{\leq}_{\bullet} \longrightarrow |M|_{\bullet}$ gives rise to a map $|I|_{\bullet} \longrightarrow |M|_{\bullet}$. Let us continue to ignore the meaning of being indiscernible.

The characterisation says that each infinite indiscernible sequence *is* necessarily an indiscernible set; accordingly, in ɀ𝝋 and sSets, there is a morphism $I^{\leq}_{\bullet} \longrightarrow |I|_{\bullet}$.

This enables us to rephrase the characterisation by saying that each map $I^{\leq}_{\bullet} \longrightarrow |M|_{\bullet}$ *extends to* to a map $|I|_{\bullet} \longrightarrow |M|_{\bullet}$, which is written as a lifting property

$$I^{\leq}_{\bullet} \longrightarrow |I|_{\bullet} \rightthreetimes M_{\bullet} \longrightarrow \top$$

where $\top$ is the terminal object of ɀ𝝋 and thus can be ignored.

So far we did not discuss "indiscernability". A sequence is indiscernible iff for each n "each n-tuple in increasing order is". A straightforward way to talk in ɀ𝝋 about "indiscernible n-tuples" is rather tautological: to consider the filter on $|M|^n = |M_{\bullet}(n_{\leq})|$ generated by the set of indiscernible n-tuples, or say the sets of ϕ-indiscernible tuples for various ϕ. The phrase "*each* n-tuple" suggests that we consider *each* tuple in $I^{\leq}_{\bullet}$ and $|I|_{\bullet}$ to be "small", i.e. equip $I^{\leq}_{\bullet}$ and $|I|_{\bullet}$ with the indiscrete filters. A verification now shows that, with these definitions, an injective (continuous) morphism $I^{\leq}_{\bullet} \longrightarrow |M|_{\bullet}$ is the same as an indiscernible sequence in M indexed by I, and an injective (continuous) morphism $|I|_{\bullet} \longrightarrow |M|_{\bullet}$ is the same as an indiscernible set in M indexed by I.

Now we see that the preceding fulfils details of this little program and gives a precise definition of $M_{\bullet}$ so that this lifting property does indeed say that M is stable. We'd like to stress again the particularly mechanistic and oversimplified nature of the considerations above. Indeed, in the big picture, perhaps the observation of most consequence is that such mechanistic and oversimplified consideration are useful.

3.4. NIP and eventually indiscernible sequences. — The notion of an eventually indiscernible sequence needed for NIP involves the filter of final segments of a linear order.

First we associate ɀ𝝋 objects with the filter of final segments of a linear order, so that an eventually indiscernible sequence is a morphism from that object to the model. Then we rewrite the characterisation of NIP "each indiscernible sequence is eventually indiscernible over a parameter" as a diagram which is almost, but not quite, a lifting property. We then modify slightly the definition of $M_{\bullet}$ so that it becomes a lifting property.

We note that there is a bit of flexibility in choosing details of the construction of $M_\bullet$, which sometimes matter: the filters we use for NIP do not fit for stability because they lack symmetry.

3.4.1. Simplicial filters associated with filters on linear orders. —

Definition 3.4.1.1 **($|I|_\bullet^{\mathfrak{F}}, I_\bullet^{\le\text{tails}}, |I|_\bullet^{\text{tails}}$: ɀ𝝋).** — Let I be a linear order, and let $\mathfrak{F}$ be a filter on $|I|$.

Let $|I|_\bullet^{\mathfrak{F}}$ be the simplicial filter whose underlying simplicial set is represented by the set $|I|$, i.e. the functor $\left\{- \underset{\text{Sets}}{\longrightarrow} |I|\right\} : \Delta^{\text{op}} \longrightarrow \text{Sets}$. The set

$$\left\{n_\le \underset{\text{Sets}}{\longrightarrow} |I|\right\} = \{(t_1, ..., t_n) \in |I|^n\} = |I|^n$$

is equipped with the filter generated by sets of the form ε^n, $\varepsilon \in \mathfrak{F}$.

Let $I_\bullet^{\le,\mathfrak{F}}$ be the simplicial filter whose underlying simplicial set is represented by the linear order I, i.e. the functor $\left\{- \underset{preorders}{\longrightarrow} I^\le\right\} : \Delta^{\text{op}} \longrightarrow \text{Sets}$. The set

$$\left\{n_\le \underset{preorders}{\longrightarrow} I^\le\right\} = \{(t_1, ..., t_n) \in |I|^n \,:\, t_1 \le ... \le t_n\}$$

is equipped with the filter generated by sets of the form

$$\{(t_1, ..., t_n) \in \varepsilon^n \,:\, t_1 \le ... \le t_n\}, \quad \varepsilon \in \mathfrak{F}$$

Let $I_\bullet^{\le\text{tails}} := I_\bullet^{\le,\mathfrak{F}}$, $|I|_\bullet^{\text{tails}} := |I|_\bullet^{\mathfrak{F}}$ for $\mathfrak{F} := \{\{x : x \ge i\} : i \in I\}$ the filter generated by non-empty final segments of I.

3.4.2. NIP as almost a lifting property. — Let $\bot = |\{\}|_\bullet = |\varnothing|_\bullet$ denote the initial object of the category ɀ𝝋, i.e. "the simplicial set represented by the empty set: for any $n > 0$ $\bot(n_\le) := \varnothing$, and the only big subset is $\varnothing$ itself.

Lemma 3.4.2.1 **(NIP as almost a lifting property)**

Let M be a structure. Let I be an infinite linear order. The following are equivalent:

(i) *in the model M, for each $b \in M$, each formula $\phi(-, b)$ each eventually indiscernible I-sequence (over $\varnothing$) is eventually $\phi(-, b)$-indiscernible*

(ii) *in ɀ𝝋 each injective morphism $I_\bullet^{\le tails} \longrightarrow M_\bullet$ factors as $I_\bullet^{\le tails} \longrightarrow M_\bullet^{L(M)} \longrightarrow M_\bullet$, i.e. the following diagram holds:*

$$\begin{array}{ccc} \bot & \longrightarrow & M_\bullet^{L(M)} \\ \downarrow & \nearrow & \downarrow \\ I_\bullet^{\le tails} & \text{-}(inj)\longrightarrow & M_\bullet \end{array}$$

(iii) *in ɀ𝝋 each injective morphism $\omega_\bullet^{\le tails} \longrightarrow M_\bullet$ factors as $\omega_\bullet^{\le tails} \longrightarrow M_\bullet^{L(M)} \longrightarrow M_\bullet$, i.e. the following diagram holds:*

$$\begin{array}{ccc} \bot & \longrightarrow & M_\bullet^{L(M)} \\ \downarrow & \nearrow & \downarrow \\ \omega_\bullet^{\le tails} & \text{-}(inj)\longrightarrow & M_\bullet \end{array}$$

Proof. — (ii) $\Longrightarrow$ (i): Let $(a_i)_{i\in I}$ be an indiscernible sequence. It induces a morphism $I_\bullet^{\leq\text{tails}} \longrightarrow M_\bullet$ (the bottom arrow), which by the diagram in (ii) lifts to a diagonal arrow $I_\bullet^{\leq\text{tails}} \longrightarrow M_\bullet^{L(M)}$. By commutativity of the triangle it corresponds to the same sequence, and continuity means this sequence is indiscernible.

(i) $\Longrightarrow$ (ii): Consider the bottom arrow $I_\bullet^{\leq\text{tails}} \longrightarrow M_\bullet$. It corresponds to a sequence $(a_i)_{i\in I}$. Injectivity means its elements are all distinct. For such a sequence continuity of the morphism then means the sequence is ϕ-indiscernible, hence is eventually indiscernible over any parameter. Hence, the diagonal map $I_\bullet^{\leq\text{tails}} \longrightarrow M_\bullet^{L(M)}$ is continuous.

(i)$\leftrightarrow$(ii): by compactness. □

However, note the diagram (ii) may fail for a non-injective morphism. Indeed, consider a sequence $(a,b,a,b,...)$ where a,b are elements of an indiscernible set, is not indiscernible over b. It represents a continuous map $I_\bullet^{\leq\text{tails}} \longrightarrow M_\bullet$, but it is not continuous as a map to $M_\bullet^{L(M)}$.

In the next subsection we slightly modify the definition of the filters on $M_\bullet$ to take care of this "degenerate" case, and define NIP as a lifting property

$$\bot \longrightarrow I_\bullet^{\leq\text{tails}} \rightthreetimes M_\bullet^{L(M)} \longrightarrow M_\bullet$$

In [8, Appendix §5.2] we also reformulate as a lifting property the characterisation of NIP using average/limit types as a lifting property reminiscent of the lifting property defining completeness of metric spaces. They both use an endomorphism "shifting dimension" of Δ and $s\mathcal{P}$ which is also used in topology to define limits (and being locally trivial).

3.4.3. NIP as a lifting property. — Call a sequence (totally) ϕ-indiscernible with *consecutive* repetitions iff each subsequence with *distinct consecutive* elements is necessarily (totally) ϕ-indiscernible. The sequence $(a,b,a,b,a,b,...)$ where $\{a,b\}$ is an indiscernible set, is an example of a sequence which is indiscernible with repetitions but not indiscernible with consecutive repetitions. A sequence $(a,a,b,...,b,c,..,c,..,)$ is ϕ-indiscernible with consecutive repetitions. Note that an infinite indiscernible sequence with consecutive repetitions is necessarily either eventually constant or has infinitely many distinct elements.

Let $M_\bullet'^{\Sigma}$ denote the simplicial set $|M|_\bullet$ equipped with filters defined as in Definition 3.1.1.1 where everywhere words "with repetition" are replaced by "with consecutive repetitions".

Note that for an injective map $a:|I| \longrightarrow |M|$, the induced map $a_\bullet : I_\bullet^{\leq\text{tails}} \longrightarrow M_\bullet'$ is continuous if and only if the induced map $a_\bullet : I_\bullet^{\leq\text{tails}} \longrightarrow M_\bullet$ is continuous.

***Lemma 3.4.3.1* (NIP as a lifting property).** — *Let M be a structure. Let I be an infinite linear order. The following are equivalent:*

(i) *in the model M, for each $b \in M$, each formula $\phi(-,b)$ each eventually indiscernible I-sequence (over $\varnothing$) is eventually $\phi(-,b)$-indiscernible*

(ii) *in $s\mathcal{P}$ each morphism $I_\bullet^{\leq tails} \longrightarrow M_\bullet$ factors as $I_\bullet^{\leq tails} \longrightarrow M_\bullet'^{L(M)} \longrightarrow M_\bullet'$, i.e. the following lifting property holds:*

$$\bot \longrightarrow I_\bullet^{\leq tails} \rightthreetimes M_\bullet'^{L(M)} \longrightarrow M_\bullet'$$

i.e. in $\wr\mathcal{P}$ *the following diagram holds:*

$$\begin{array}{ccc} \perp & \longrightarrow & M_\bullet'^{L(M)} \\ \downarrow & \nearrow & \downarrow \\ I_\bullet^{\leq tails} & \longrightarrow & M'_\bullet \end{array}$$

(iii) *in* $\wr\mathcal{P}$ *each morphism* $\omega_\bullet^{\leq tails} \longrightarrow M_\bullet$ *factors as* $\omega_\bullet^{\leq tails} \longrightarrow M_\bullet'^{L(M)} \longrightarrow M'_\bullet$, *i.e. the following lifting property holds:*

$$\perp \longrightarrow \omega_\bullet^{\leq tails} \rightthreetimes M_\bullet'^{L(M)} \longrightarrow M'_\bullet$$

i.e. in $\wr\mathcal{P}$ *the following diagram holds:*

$$\begin{array}{ccc} \perp & \longrightarrow & M_\bullet'^{L(M)} \\ \downarrow & \nearrow & \downarrow \\ \omega_\bullet^{\leq tails} & \longrightarrow & M'_\bullet \end{array}$$

Proof. — (ii) $\Longrightarrow$ (i): same as before because here we need to consider only injective maps.

(i) $\Longrightarrow$ (ii): Consider the bottom arrow $I_\bullet^{\leq \text{tails}} \longrightarrow M'_\bullet$. It corresponds to a sequence $(a_i)_{i\in I}$. For such a sequence continuity of the morphism then means for each ϕ some final segment $(a_i)_{i>i_0}$ is ϕ-indiscernible with consecutive repetitions. If this final segment has only finitely many distinct elements, it is eventually constant, hence eventually indiscernible over any parameter. If it has infinitely many distinct elements, we can use (i) to conclude it is eventually indiscernible over any parameter. Hence, the diagonal map $I_\bullet^{\leq \text{tails}} \longrightarrow M_\bullet'^{L(M)}$ is continuous.

(ii)$\leftrightarrow$(iii): by compactness. □

3.4.4. Cauchy sequences: a formal analogy to indiscernible sequences. — This is formally unnecessary but might help the reader's intuition.

Recall that with a metric space M we associate its simplicial filter $M_\bullet$ whose underlying simplicial set $|M|_\bullet$ is represented by the set of elements of M, and where we equip $|M|^n$ with *the filter of* ε*-neighbourhoods of the main diagonal* generated by the subsets, for $\epsilon > 0$

$$\{ (x_1, ..., x_n) \,:\, \text{dist}(x_i, x_j) < \epsilon \text{ for } 1 \leq i < j \leq n \}$$

The lemma below establishes a formal analogy between Cauchy sequences and indiscernible sequences with repetitions: they both are defined as in $\wr\mathcal{P}$ as morphisms from the same object associated with a linear order.

Lemma 3.4.4.1. — *For any linear order* I *and any metric space* M *the following holds.*

(i) *A sequence* $(a_i)_{i\in I}$ *in* M *induces an morphism* $a_\bullet : |I|_\bullet \longrightarrow |M|_\bullet$ *of* sSets, *and, conversely, each morphism* $|I|_\bullet \longrightarrow |M|_\bullet$ *is induced by a unique such sequence*

(ii) *A Cauchy sequence* $(a_i)_{i\in I}$ *in* M *induces an* $\wr\mathcal{P}$*-morphism* $a_\bullet : I_\bullet^{\leq tails} \longrightarrow M_\bullet$, *and, conversely, each morphism* $I_\bullet^{\leq tails} \longrightarrow M_\bullet$ *is induced by to a unique such sequence.*

(iii) *A Cauchy sequence* $(a_i)_{i\in I}$ *in* M *induces an* $\wr\mathcal{P}$*-morphism* $a_\bullet : |I|_\bullet^{tails} \longrightarrow M_\bullet$, *and, conversely, each morphism* $|I|_\bullet^{tails} \longrightarrow M_\bullet$ *induced by a unique such sequence.*

Proof. — (i): A map $a : |I| \longrightarrow |M|$ of sets induces a natural transformation of functors $f_\bullet : I^{\leq}_\bullet \longrightarrow |M|_\bullet$: for each $n > 0$, a tuple $(i_1 \leq .., \leq i_n) \in I^{\leq}_\bullet(n_\leq)$ goes into tuple $(a_{i_1}, , ..., a_{i_n}) \in |M|^n = |M|_\bullet(n_\leq)$. Now let us show that every natural transformation $a_\bullet : I^{\leq}_\bullet \longrightarrow |M|_\bullet$ is necessarily of this form, by the following easy argument. Let $(y_1, .., y_n) = f_n(x_1, .., x_n)$; by functoriality using maps $[i] : 1 \longrightarrow n, 1 \mapsto i$ we know that $y_i = (y_1, .., y_n)[i] = f_n(x_1, .., x_n)[i] = f_1(x_i)$. Finally, use that for any $y_1, .., y_n \in |M|_\bullet(1_\leq) = |M|$ there is a unique element $\tilde{y} \in |M|_\bullet(n_\leq) = |M|^n$ such that $y_i = \tilde{y}[i]$. In a more geometric language, we may say that we used the following property of the simplicial set $|M|_\bullet$: that each "(n-1)-simplex" $(y_1, .., y_n)$ is uniquely determined by its "0-dimensional faces" $y_1, ..., y_n \in M$.

(ii): We need to check what it means to for the map $f_n : I^{\leq}_\bullet(n_\leq) \longrightarrow M^n$ of filters to be continuous. Consider $n = 2$; for $n > 2$ the argument is the same. Continuity means that for each $\epsilon > 0$ and thereby "ϵ-neighbourhood $\varepsilon := \{(x, y) : x, y \in M, \mathrm{dist}(x, y) < \epsilon\}$ of the main diagonal" there is $N > 0$ and thereby a "N-tail" neighbourhood $\delta := \{(i, j) \in \omega \times \omega : N \leq i \leq j\}$ such that $a(\delta) \subset \varepsilon$, i.e. for each $j \geq i > N$ it holds $\mathrm{dist}(a_i, a_j) < \epsilon$.

(iii): We need to check what it means to for the map $f_n : |I|^n \longrightarrow M^n$ of filters to be continuous. Consider $n = 2$; for $n > 2$ the argument is the same. Continuity means that for each $\epsilon > 0$ and thereby "ϵ-neighbourhood $\varepsilon := \{(x, y) : x, y \in M, \mathrm{dist}(x, y) < \epsilon\}$ of the main diagonal" there is $N > 0$ and thereby a "N-tail" neighbourhood $\delta := \{(i, j) \in \omega \times \omega : i, j > N\}$ such that $a(\delta) \subset \varepsilon$, i.e. for each $i, j > N$ it holds $\mathrm{dist}(a_i, a_j) < \epsilon$. □

3.4.5. Stability as Quillen negation of eventually (totally) indiscernible sequences. — The characterisation of stability "each eventually ϕ-indiscernible sequence is necessarily an eventually order ϕ-indiscernible" and is a lifting property with respect to $I^{\leq \mathrm{tails}}_\bullet$.

***Proposition 3.4.5.1* (Stability as Quillen negation)**

Let M be a model, and let ϕ be a formula in the language of M. Let I be a linear order. The following are equivalent:

(i) *in the model M, each eventually ϕ-indiscernible sequence is necessarily eventually order ϕ-indiscernible*

(ii) *in the model M, each eventually ϕ-indiscernible sequence with repetitions is necessarily eventually order ϕ-indiscernible with repetitions*

(iii) *the following lifting property holds in the category ¿Ƥ:*

$$I^{\leq tails}_\bullet \longrightarrow |I|^{tails}_\bullet \;\rightthreetimes\; M^{\{\phi\}}_\bullet \longrightarrow \top$$

i.e. the following diagram in ¿Ƥ holds:

$$\begin{array}{ccc} I^{\leq tails}_\bullet & \longrightarrow & M^{\{\phi\}}_\bullet \\ \downarrow & \nearrow & \downarrow \\ |I|^{tails}_\bullet & \longrightarrow & \top \end{array}$$

Proof. — (ii) $\Longrightarrow$ (i) is trivial so we only need to prove (i) $\Longrightarrow$ (ii): Consider an eventually ϕ-indiscernible sequence with repetitions. Take a maximal subsequence with distinct elements. First assume it is infinite. Then it is eventually ϕ-indiscernible, hence eventually order ϕ-indiscernible by (ii), hence the original sequence is eventually order ϕ-indiscernible with repetitions.

So what happens if there are only finitely many distinct elements?(7) Call this sequence $(a_i)_{i\in I}$. Pick an initial segment $(a_i)_{i\leq i_0}$ of the sequence which contains all the elements of $(a_i)_{i\in I}$ which occur only finitely many times; then in the corresponding final segment $(a_i)_{i>i_0}$ each element occurs infinitely many times whenever it occurs there at all. Therefore any finite subsequence of that final segment occurs there in an arbitrary order, hence $(a_i)_{i>i_0}$ is ϕ-indiscernible with repetitions iff it is order ϕ-indiscernible with repetitions.

(iii) $\implies$ (ii): Let $(a_i)_{i\in I}$ be an eventually ϕ-indiscernible sequence with repetitions. By Lemma 3.3.1.1 it induces an $s\mathcal{P}$-morphism $a_\bullet : I_\bullet^{\leq\text{tails}} \longrightarrow M_\bullet^{\{\phi\}}$. By the lifting property it lifts to a $s\mathcal{P}$-morphism $a'_\bullet : |I|_\bullet^{\text{tails}} \longrightarrow M_\bullet^{\{\phi\}}$. By commutativity they both correspond to the same map of the underlying simplicial sets, i.e. to the same sequence $(a_i)_{i\in I}$. Again by Lemma 3.3.1.1, this sequence is eventually order ϕ-indiscernible with repetitions.

(ii) $\implies$ (iii): Let $a_\bullet : I_\bullet^{\leq\text{tails}} \longrightarrow M_\bullet^{\{\phi\}}$ be the $s\mathcal{P}$-morphism corresponding to the lower horizontal arrow. By Lemma 3.3.1.1 it corresponds to an eventually ϕ-indiscernible sequence $(a_i)_{i\in I}$ with repetitions. By (ii') this sequence is also eventually order ϕ-indiscernible with repetitions. By Lemma 3.3.1.1 it corresponds to an $s\mathcal{P}$-morphism $|I|_\bullet^{\text{tails}} \longrightarrow M_\bullet^{\{\phi\}}$, as required. □

3.5. Questions. — Let us now formulate several questions the technique of Quillen negation allows us to formulate. Note it would be rather cumbersome to reformulate these questions back into model theoretic language.

3.5.1. Double Quillen negation/orthogonal of a model. — In the notation of Quillen negation,(8) Proposition 3.3.2.1(iii) can be stated concisely as:

– a model M is stable iff $M_\bullet \longrightarrow \top \in \{\omega_\bullet \longrightarrow |\omega|_\bullet\}^{\mathrm{r}}$

(7) We remark that in category theory it is often important that things work in a "degenerate" case, such as, here, the case of a ϕ-indiscernible sequence which has only finitely many distinct elements. Note that we would not have been able to write the lifting property if not for this set-theoretic argument.

(8) For a class C of morphisms in a category, its *left orthogonal* C^{l} with respect to the lifting property, respectively its right orthogonal C^{r}, is the class of all morphisms which have the left, respectively right, lifting property with respect to each morphism in the class C. In notation,

$$C^{\mathrm{l}} := \{f \ : \ \forall g \in C,\ f \rightthreetimes g\} \qquad C^{\mathrm{r}} := \{g \ : \ \forall f \in C,\ f \rightthreetimes g\}$$

Taking the orthogonal of a class C is a simple way to define a class of morphisms excluding non-isomorphisms from C, in a way which is useful in a diagram chasing computation. Thus, in the category Set of sets, the right orthogonal $\{\varnothing \to \{*\}\}^{\mathrm{r}}$ of the simplest non-surjection $\varnothing \to \{*\}$, is the class of surjections. The left and right orthogonals of $\{x_1, x_2\} \to \{*\}$, the simplest non-injection, are both precisely the class of injections,

$$\{\{x_1, x_2\} \to \{*\}\}^{\mathrm{l}} = \{\{x_1, x_2\} \to \{*\}\}^{\mathrm{r}} = \{f \ : \ f \text{ is an injection }\}.$$

A number of notions can be defined by passing to the left or right orthogonal several times starting from a list of explicit examples, i.e. as $C^{\mathrm{l}}, C^{\mathrm{r}}, C^{\mathrm{lr}}, C^{\mathrm{ll}}$, where C is a class consisting of several explicitly given morphisms. A useful intuition is to think that the property of left-lifting against a class C is a kind of negation of the property of being in C, and that right-lifting is also a kind of negation. Hence the classes obtained from C by taking orthogonals an odd number of times, such as $C^{\mathrm{l}}, C^{\mathrm{r}}, C^{\mathrm{rrr}}, C^{\mathrm{lll}}$ etc., represent various kinds of negation of C, so $C^{\mathrm{l}}, C^{\mathrm{r}}, C^{\mathrm{rrr}}, C^{\mathrm{lll}}$ each consists of morphisms which are far from having property C.

It is convenient to refer to C^{l} and C^{r} as the property of *left, resp. right, Quillen negation of the property of being in the class* C, and $C^{\mathrm{rl}} := (C^{\mathrm{r}})^{\mathrm{l}} \supset C$ and $C^{\mathrm{lr}} := (C^{\mathrm{l}})^{\mathrm{r}} \supset C$ as *Quillen generalisation of property* C.

Purely by a standard elementary diagram chasing calculation the Proposition 3.3.2.1 above implies that if a model M is stable and N is an arbitrary model,

- $N_\bullet \longrightarrow \top \in \{M_\bullet \longrightarrow \top\}^{\mathrm{lr}}$ implies N is stable
- N is stable iff $N_\bullet \longrightarrow \top \in \{M_\bullet \longrightarrow \top \,:\, M \text{ a stable model}\}^{\mathrm{lr}}$

Indeed, Proposition 3.3.2.1 states that for a stable M $\{M_\bullet \longrightarrow \top\}^{\mathrm{l}}$ contains a particular morphism, and thus every morphism in $\{M_\bullet \longrightarrow \top\}^{\mathrm{lr}}$ lifts with respect to that morphism. This allows us to formulate the following amusingly concise questions:

Question 3.5.1.1. — *Is the following true?*(9)

- *A model M is stable iff $M_\bullet \longrightarrow \top \in \{(\mathbb{C};+,*)_\bullet \longrightarrow \top\}^{lr}$.*
- *A model M is distal iff $M_\bullet \longrightarrow \top \in \{(\mathbb{Q}_p;+,*)_\bullet \longrightarrow \top\}^{lr}$.*
- *For an algebraically closed field $\mathbb{C}_p$ of prime characteristic p,*

$$\{(\mathbb{C};+,*)_\bullet \longrightarrow \top\}^{lr} = \{(\mathbb{C}_p;+,*)_\bullet \longrightarrow \top\}^{lr}$$

- $\{(\mathbb{Q}_p;+,*)_\bullet \longrightarrow \top\}^{lr} = \{(\mathbb{R};+,*)_\bullet \longrightarrow \top\}^{lr}$

Answering these questions, perhaps negatively, will represent a grasp of ɀ𝒫 and its expressive power at an elementary level.

3.5.2. ACF_0- and stable replacement of a model. — And, in a similarly concise manner, our category theoretic technique(10)(11)(12) leads us to define an "ACF_0-replacement" of a model N as the decomposition

$$N_\bullet \xrightarrow{\{\mathbb{C}_\bullet \longrightarrow \top\}^{\mathrm{l}}} \bullet \xrightarrow{\{\mathbb{C}_\bullet \longrightarrow \top\}^{\mathrm{lr}}} \top$$

or a "stable replacement" as

$$N_\bullet \xrightarrow{\{M_\bullet \longrightarrow \top \,:\, M \text{ a stable model}\}^{\mathrm{l}}} \bullet \xrightarrow{\{M_\bullet \longrightarrow \top \,:\, M \text{ a stable model}\}^{\mathrm{lr}}} \top$$

(9) In set theoretic notation the classes of models defined by these double Quillen negations are explained in Lemma 8.1.2.2. This explantation is preliminary.

(10) It is convenient to say that a morphism $A \longrightarrow B$ has a property P by writing it above the arrow as $A \xrightarrow{(P)} B$.

(11) Given a property P of morphisms in a category, it is desirable that each morphism decomposes as $\bullet \xrightarrow{(P)^{\mathrm{l}}} \bullet \xrightarrow{(P)^{\mathrm{lr}}} \bullet$ and as $\bullet \xrightarrow{(P)^{\mathrm{rl}}} \bullet \xrightarrow{(P)^{\mathrm{r}}} \bullet$. The axiom $M2$ of closed model categories is of this form where P is the class of fibrations or cofibrations, and, for example, the connected components of a topological space X fit into decomposition

$$X \xrightarrow{(\{0,1\}\longrightarrow\{0=1\})^{\mathrm{l}}} \pi_0(X) \xrightarrow{(\{0,1\}\longrightarrow\{0=1\})^{\mathrm{lr}}} \{0=1\}$$

where $(\{0,1\} \longrightarrow \{0=1\})$ denotes the (class consisting of the single) morphism gluing together the two points of a discrete space of two points; a similar definition can be made in ɀ𝒫 as well. Some details appear in [6,§4.7].

(12) Such decomposition may fail to exist, particularly for set theoretic reasons if the property involved is a class, not a set, see the next footnote.

or "indiscernible sequence" replacement[13]

$$N_\bullet \xrightarrow{\{I_\bullet^{\leq\text{tails}}\longrightarrow|I|_\bullet^{\text{tails}}\}^{\text{rl}}} \bullet \xrightarrow{\{|I|_\bullet^{\leq\text{tails}}\longrightarrow|I|_\bullet^{\leq\text{tails}}\}^{\text{r}}} \top$$

Question 3.5.2.1. — Is it true that the stable part St_M of a model M can be defined in terms of one of the M2-decompositions above, i.e. by one of the equations

$$M_\bullet \xrightarrow{(P)^{\text{rl}}} (St_M)_\bullet \xrightarrow{(P)^{\text{r}}} \bot$$

or

$$M_\bullet \xrightarrow{(P)^{\text{l}}} (St_M)_\bullet \xrightarrow{(P)^{\text{lr}}} \bot$$

for some nice class P of morphisms ?

Note that if this holds for any class, it does for $(P) := \{M_\bullet \to (St_M)_\bullet : M \text{ is a model}\}$ (top) or $(P) := \{(St_M)_\bullet \longrightarrow \top : M \text{ is a model}\}$ (bottom), and this constitutes a precise question.

3.5.3. Levels of stability as iterated Quillen negations. — More generally, we may think of the class

$$\left\{M_\bullet : (M_\bullet \longrightarrow \bot) \in \{N_\bullet \longrightarrow \top\}^{\text{lr}}\right\}$$

is a class of models "at the same level of stability as N or lower", i.e. having "better" properties of indiscernible sequences than N. We may be more explicit about these properties and define instead the class

$$\{M_\bullet : (M_\bullet \longrightarrow \bot) \in (I)^{\text{r}}\}$$

where (I) is a class of morphisms associated with partial orders. Of course, these is a lot of flexibility in this, e.g. we may consider not $M_\bullet \longrightarrow \top$ but some other morphism associated with a model, as we do in NIP; or Quillen negations $^{\text{lrr}}, ^{\text{rr}}, \ldots$ iterated several times.

This is just one of the possible questions of this type.

Question 3.5.3.1. — *Can the definition of distality be expressed in this way as Quillen negation, perhaps as a lifting property with respect to a class of morphisms associated with linear orders, say for example as*

$$(I_1 + a + I_2 + I_3)_\bullet^{\leq} \bigcup_{(I_1+I_2+I_3)_\bullet^{\leq}} (I_1 + I_2 + b + I_3)_\bullet^{\leq} \longrightarrow (I_1 + a + I_2 + b + I_3)_\bullet^{\leq} \;\rightthreetimes\; M_\bullet \longrightarrow \top$$

in the notation of the definition of distality in [Simon, Oleron]? Can the notion of domination of indiscernible sequences [Simon, Type decomposition in NIP theories] or collapse of indiscernibles in [Scow,Characterization of NIP theories by ordered graph-indiscernibles] and [GHS,Characterizing Model-Theoretic Dividing Lines via

[13]Some explicit remarks can be made about this decomposition, which perhaps show it is of little interest. Typically a decomposition of form $\bullet_1 \xrightarrow{(P)^{\text{lr}}} \bullet_2 \xrightarrow{(P)^{\text{r}}} \bullet_3$ is constructed by a Quillen small object argument, which is a careful induction taking pushback of the $(P)^{\text{r}}$ maps in the decompositions $\bullet_1 \xrightarrow{(arbitrary)} \bullet \xrightarrow{(P)^{\text{r}}} \bullet_3$; such a pullback necessarily has property $(P)^{\text{r}}$. In this case the "indiscernability" decomposition of $M_\bullet \longrightarrow \top$ can probably be done explicitly, by weakening the filter structure on $M_\bullet$ such that the map

$$|I|_\bullet^{\leq\text{tails}} \longrightarrow M'_\bullet$$

is $\wr\!\mathcal{P}$-continuous (i.e. is an $\wr\!\mathcal{P}$-morphism) for I an indiscernible sequence with repetitions in M, i.e. an $\wr\!\mathcal{P}$-morphism

$$I_\bullet^{\leq\text{tails}} \longrightarrow M_\bullet$$

Collapse of Generalized Indiscernibles]) be expressed in terms of Quillen lifting properties? Distality is thought of as a notion of pure instability, hence we may ask whether the stable replacement of a distal model is necessarily trivial?

Are p-adics or say CODF "distal-NIP-complete" in the sense that the following is the class of all distal models:

$$\left\{ M : M_\bullet \longrightarrow \bot \in \{(\mathbb{Q}_p; +, *)_\bullet \longrightarrow \top\}^{lr} \right\}$$

$$\left\{ M : M_\bullet \longrightarrow \bot \in \{N_\bullet \longrightarrow \top : N \vDash CODF\}^{lr} \right\}$$

3.5.4. Simple theories and tree properties. — We think answering the following question is important as it might give a technical clue towards homotopy theory for model theory.

Question 3.5.4.1. — *Define the notion of a simple theory as an iterated Quillen negation.*(14)

The motivation for singling out simplicity is that its characterisation uses indiscernible trees: we think of such a tree as a *family of indiscernible sequences (its branches) compatible in some way*, and thus analogous to a compatible family of paths (or its negation(!)), i.e. a homotopy between paths. Another coincidence is that a homotopy $I \times I \longrightarrow X$ or $I \longrightarrow (I \longrightarrow X)$ between paths involves two linear orders, and so does an indiscernible tree in a tree property $\mathrm{NTP}_{1,2}$. We hope that an answer to this question might help give a clue towards how to define a notion of the space of indiscernible sequences in a model, a start of homotopy theory.

3.5.5. Axiomatize non-abelian homotopy theory. — We take the liberty to state a wild speculation that understanding dividing lines of Shelah in terms of diagram chasing may suggest an approach towards axioms of "non-abelian" homotopy theory. Our motivation in stating that is not any positive evidence but rather that the technique of classification theory is so alien and unknown in homotopy theory that any ideas there would be fresh and new in homotopy theory. Our motivation of thinking of a homotopy theory based on ɀφ being non-abelian is elementary and not very convincing either: the definition of a natural notion of homotopy in ɀφ is not symmetric. Our motivation of mentioning the endomorphism $[+1] :$ ɀφ $\longrightarrow$ ɀφ is that it appears in reformulations of the notions of being locally trivial as a base change diagram (see [7,§3.4] for a sketch and for details [6,§2.2.5,§4.8]), of limit and compactness [6,§2.1.4,§4.11].

Question 3.5.5.1. — *Formulate axioms of homotopy theory in terms of iterated Quillen lifting properties/negation and the "shift" endomorphism* $[+1] :$ ɀφ $\longrightarrow$ ɀφ.

(14) An answer is suggested in Appendix 9, or rather a straightforward way to "read it off" the definition of the tree property, and involves a slightly different definition of Stone space of a model; the filters on $|M|^n = |M|_\bullet(n_\leq)$ are defined to consist of tuples $(a_1, ..., a_n)$ such that the set $\{\phi(x, a_1), ..., \phi(x, a_n)\}$ is consistent. This suggests a variation of Question 3.5.1.1: is a (saturated enough) model M simple iff $M_\bullet \longrightarrow \top \in \left\{(\text{Random graph})_\bullet \longrightarrow \top\right\}^{\text{lr}}$. The same trick probably gives NTP_1 and $NSOP_1$ as well. However, it still remains to intrepret these reformulations as a clue towards homotopy theory.

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APPENDICES (PRELIMINARY)

The Appendices contain an assorie of examples and constructions not quite ready for publication, and are intended to provide some context and suggestions for future research.

Appendix 4 contains an assortie of constructions of objects of ʐ𝝋, notably a sketch of a fully faithful inclusion Top ⊂ ʐ𝝋.

Appendix 5 contains reformulations of NIP, NOP, non-dividing, and considerations of NSOP.

Appendix 6 sketches in category-theoretic language a constructions of ʐ𝝋 objects motivated by Ramsey theory.

Appendix 7 presents speculations about a homotopy theory for ʐ𝝋 non-trivial for the subcategory of models.

Appendix 8 sketches in set-theoretic notation the definition of the class of models $\{(\mathbb{C};+,*)_\bullet \longrightarrow \top\}^{\mathrm{lr}}$, and, more genererally, $\{M_\bullet \longrightarrow \top\}^{\mathrm{lr}}$ for a model M, appearing in Question 3.5.1.1.

Appendix 9 is a preliminary exposition of the reformulation of the tree property solving Question 3.5.4.1.

4. Appendix. Examples of simplicial filters.

We sketch a number of constructions of simplicial filters, is a somewhat informal style. The aim is to provide context and intuitions for our constructions.

Importantly, we explain how to view the topological spaces as objects of ʐ𝝋.

4.1. Examples of filters and their morphisms.— We list a number of examples of filters in the hope that some may aid the reader's intuition. The reader should skip examples they find unhelpful or uneasy to follow.

4.1.1. Discrete and indiscrete. — The set of subsets consisting of X alone is a filter on X called the *indiscrete* filter, and there is a functor $\cdot^{\text{indiscrete}} : \text{Sets} \longrightarrow$ 𝝋, $X \longmapsto X^{\text{indiscrete}}$, sending a set to itself equipped with indiscrete filter $\{X\}$. For us the set of all subsets of X is also a filter which we call *discrete*.

The functor Sets $\longrightarrow$ 𝝋 induces a fully faithful embedding $\cdot^{\text{indiscrete}} : \text{sSets} \longrightarrow$ ʐ𝝋, thus any simplicial set is also a simplicial filter.

4.1.2. Neighbourhood filter. — In a topological space X, the *set of all neighbourhoods* of an arbitrary subset A of X (and in particular the set of all neighbourhoods of a point of X) is a filter, called the *neighbourhood filter* of A. The *filter of coverings* on $X \times X$ consists of subsets of form

$$\sqcup_{x \in X} \{x\} \times U_x$$

where $U_x \ni x$ is a (not necessarily open) neighbourhood of x, i.e. $(U_x)_{x \in X}$ is a covering of X.

A map $f : X \longrightarrow Y$ of topological spaces is continuous iff for every point $x \in X$ it induces a continuous map of filters from the neighbourhood filter of x to the neighbourhood filter of $f(x)$, or, equivalently, iff it induces a continuous map from the filter of coverings of X on $|X| \times |X|$ to that of Y on $|Y| \times |Y|$.

4.1.3. Cofinite filter. — If X is an set, the *complements of the finite subsets* of X are the elements of a filter. The filter of complements of finite subsets of the set $\mathbb{N}$ of integers ≥ 0 is called the *Fréchet filter.*

4.1.4. Filter of tails of a preorder. — In a partial preorder $I^{\leq}$, the *filter of tails* consists of sets $\varepsilon \subset I$ with the following property: for every $i \in I$ there is $j \geq i$ such that for every $k \geq j$ it holds $k \in \varepsilon$.

4.1.5. Uniformly continuous maps. — Let M be a metric space, and $n \geq 0$. A *uniform neighbourhood of the main diagonal* on M^n is a subset which contain all tuples of diameter $< \epsilon$ for some $\epsilon > 0$; equivalently, a subset $\varepsilon \subset M^n$ with the following property: there is $\epsilon > 0$ such that for arbitrary tuple $(x_1, .., x_n) \in M^n$, it holds that tuple $(x_1, .., x_n) \in \varepsilon$ whenever for each $1 \leq i, j \leq n$ $\mathrm{dist}(x_i, x_j) < \epsilon$. Thus, the *filter of uniform neighbourhoods of the main diagonal* on M^n is generated by the subsets

$$\{ (x_1, .., x_n) \in M^n : \mathrm{dist}(x_i, x_{i+1}) < \epsilon, 0 < i < n\}$$

For $n = 2$ we may rewrite this filter as being generated by subsets

$$\bigsqcup_{x \in X} \{x\} \times B_{r<\epsilon}(x)$$

where $B_{r<\epsilon}(x) \ni x$ is the ball around x of diameter ϵ, We may call it the *filter of uniform coverings* on $M \times M$, to emphasize similarity to topological spaces.

A map $f : M' \longrightarrow M''$ of metric spaces is uniformly continuous iff it induces a continuous map from the filter of uniform neighbourhoods of the main diagonal of $M' \times M'$ into that of $M'' \times M''$. Indeed, continuity just says that for each $\epsilon > 0$ and thereby "ϵ-neighbourhood $\varepsilon := \{(x, y) : x, y \in M'', \mathrm{dist}(x, y) < \epsilon\}$ of the main diagonal" there is $\delta > 0$ and thereby a "δ-neighbourhood $\mathfrak{d} := \{(x, y) : x, y \in M', \mathrm{dist}(x, y) < \delta\} \subset M' \times M'$ of the main diagonal" such that $f(\mathfrak{d}) \subset \varepsilon$. In fact, the same holds for M^n for any $n > 1$ instead of $M \times M$.

4.1.6. A Cauchy sequence. — A Cauchy sequence is a map $a : \omega \longrightarrow |M|$ such that it induces a continuous map $(\omega^{\leq} \times \omega^{\leq})^{\mathrm{tails}} \longrightarrow M \times M$ from the linear order $\omega^{\leq} \times \omega^{\leq}$ equipped with the filter of tails to $M' \times M'$ equipped with the filter of uniform neighbourhoods of the main diagonal. Indeed, continuity of the induced map $\bar{a} : (\omega^{\leq} \times \omega^{\leq})^{\mathrm{tails}} \longrightarrow M \times M$ just means that for each $\epsilon > 0$ and thereby "ϵ-neighbourhood $\varepsilon := \{(x, y) : x, y \in M, \mathrm{dist}(x, y) < \epsilon\}$ of the main diagonal" there is $N > 0$ and thereby a "N-tail" neighbourhood $\mathfrak{d} := \{(i, j) \in \omega \times \omega : i, j > N\}$ such that $a(\mathfrak{d}) \subset \varepsilon$, i.e. for each $i, j > N$ it holds $\mathrm{dist}(a_i, a_j) < \epsilon$.

In fact, the same holds for M^n for any $n > 1$ instead of $M \times M$, i.e. for each $n > 1$ it holds that a map $a : \omega \longrightarrow |M|$ represents a Cauchy sequence iff the induced map $((\omega^{\leq})^n)^{\mathrm{tails}} \longrightarrow M^n$ is continuous, where M^n is equipped with the filter of uniform neighbourhoods of the main diagonal.

Later we shall see that the fact this holds for each n means that a Cauchy sequence gives rise a morphism in $\wr\!\mathfrak{P}$ from a certain object associated with the linear order ω to a certain object associated with metric space M. ...

4.1.7. EM-filters and indiscernible sequences: our main example. — This is a sketch of our main model-theoretic construction in $\wr\!\mathfrak{P}$. We will give details in Definition 3.1.1.1.

A *EM-formula* is a formula of form

$$\bigwedge_{1 \leq s < t \leq r} x_{i_s} \neq x_{i_t} \& x_{j_s} \neq x_{j_t} \implies (\phi(x_{i_1}, .., x_{i_r}) \leftrightarrow \phi(x_{j_1}, .., x_{j_r}))$$

for some formula $\phi(x_1, ..., x_r)$, $1 \leq i_1 < ... < i_r \leq n$, $1 \leq j_1 < ... < j_r \leq n$.

For an r-ary formula ϕ, let $\phi^{n\text{-EM}}$ denote

$$\bigwedge_{1\le i_1<\ldots<i_r\le n, 1\le j_1<\ldots<j_r\le n} \left(\bigwedge_{1\le s<t\le r} x_{i_s} \ne x_{i_t} \& x_{j_s} \ne x_{j_t} \implies (\phi(x_{i_1},..,x_{i_r}) \leftrightarrow \phi(x_{j_1},..,x_{j_r})) \right)$$

Note that a sequence of distinct elements is indiscernible iff its EM-type contains all the EM-formulas.

The *ϕ-EM-filter on M^n* is the filter generated by the subset $\phi^{n\text{-EM}}(M^n)$; for a set of formulas Σ, the *Σ-EM-filter on M^n* is generated by the subsets $\phi^{n\text{-EM}}(M^n)$, $\phi \in \Sigma$. The *Σ-EM$^\infty$-filter on M^n* is the filter generated by arbitrary intersections of sets of form $\{\underline{x} : M \models \phi^{n\text{-EM}}(\underline{x})\}$, $\phi \in \Sigma$. We omit Σ when Σ is the set of all formulas in the language of M.

The EM-filter on M is indiscrete: the only neighbourhood is the whole set. On $M \times M$ it is generated by equivalence relations $\psi(x_1) \leftrightarrow \psi(x_2)$, for ψ an arbitrary 1-ary formula, and thus (together with the two coordinate projections $|M| \times |M| \longrightarrow |M|$) carries the same information as the usual topological Stone space of 1-types. The EM-filter on $M \times M \times M$ is the first non-trivial filter. The EM-filters on $M^n, n > 0$ capture the notion of an indiscernible sequence in the following way: a sequence (a_i) of distinct elements is indiscernible iff every finite subsequence $a_{i_1}, ..., a_{i_n}, i_1 < ... < i_n$ belongs to every EM-neighbourhood in M^n. In terms of the category $\mathfrak{F}$ of filters this is rewritten as follows:

– an injective map $I^{\le} \longrightarrow M$ is an indiscernible sequence iff for each $n > 0$ it induces a continuous map

$$\left\{ n_{\le} \xrightarrow[\text{preorders}]{} I \right\}^{\text{indiscrete}} \longrightarrow M^n$$

For $n = 3$ this means there is a continuous map

$$\{(i,j,k) \in I \times I \times I : i \le j \le k\}^{\text{indiscrete}} \longrightarrow M \times M \times M$$

4.2. Examples of simplicial filters.— We list a number of examples of simplicial filters in the hope that some may aid the reader's intuition. Some are indented for a category theoretically minded reader. The reader should skip examples they find unhelpful or uneasy to follow.

4.2.1. Discrete, indiscrete and the filter of main diagonals on a simplicial set. — The functor $\cdot^{\text{indiscrete}} : \text{Sets} \longrightarrow \mathfrak{F}$ induces a fully faithful embedding $\cdot^{\text{indiscrete}} : \text{sSets} \longrightarrow {}_{\wr}\mathfrak{F}$, thus any simplicial set is also a simplicial filter.

Let $X_\bullet : \Delta^{\text{op}} \longrightarrow \text{Sets}$ be a simplicial set. For each $n > 0$, equip $X_\bullet(n_\le)$ with the filter generated by the image of the map $X_\bullet(1_\le) \longrightarrow X_\bullet(n_\le)$ corresponding to the unique morphism $n_\le \longrightarrow 1_\le$ in Δ. A verification shows that by functoriality all the maps $X_\bullet(m_\le) \longrightarrow X_\bullet(n_\le)$, $n_\le \longrightarrow m_\le$, are continuous, and that this construction defines a functor $\Delta^{\text{op}} \longrightarrow \mathfrak{F}$, and, in fact, a fully faithful embedding $\cdot^{\text{diag}} : \text{sSets} \longrightarrow {}_{\wr}\mathfrak{F}$.

4.2.2. Represented simplicial sets. — The underlying simplicial sets of most of the examples will be variations of the following well-known construction.

Let C be a category. To each object $Y \in \text{Ob}\, C$ there correspond a functor $h_Y : X \longmapsto \left\{ X \underset{C}{\to} Y \right\}$ sending each object $X \in \text{Ob}\, C$ into the set of morphisms from X to Y, and it can be checked that this defines a fully faithful embedding $C \longrightarrow Func(C^{\text{op}}, \text{Sets})$. A functor $h_Y : C \longrightarrow \text{Sets}$ of this form is called *represented by Y*.

A *simplicial set* $M_\bullet : \Delta^{\mathrm{op}} \longrightarrow \mathrm{Sets}$ *co-represented by* a set M is the functor sending each finite linear order $n_\leq$ into the set of maps from n to M:

$$n_\leq \longmapsto \left\{ n \underset{\mathrm{Sets}}{\longrightarrow} M \right\} = \{\,(x_1, ..., x_n) \in M^n\} = M^n$$

A map $[i_1 \leq ... \leq i_n] : n_\leq \longrightarrow m_\leq$ by composition induces a map

$$\left\{ m \underset{\mathrm{Sets}}{\longrightarrow} M \right\} \longrightarrow \left\{ n \underset{\mathrm{Sets}}{\longrightarrow} M \right\}$$

$$(x_1, ..., x_n) \longmapsto (x_{i_1}, ..., x_{i_n})$$

A map $f : M' \longrightarrow M''$ of sets induces a natural transformation of functors $f_\bullet : M'_\bullet \longrightarrow M''_\bullet$: for each $n > 0$, a tuple $(x_1, .., x_n) \in M'^n = M'_\bullet(n_\leq)$ goes into tuple $(f(x_1), ..., f(x_n)) \in M''^n = M''_\bullet(n_\leq)$. Moreover, every natural transformation $f_\bullet : M'_\bullet \longrightarrow M''_\bullet$ is necessarily of this form, as the following easy argument shows. Let $(y_1, .., y_n) = f_n(x_1, .., x_n)$; by functoriality using maps $[i] : 1 \longrightarrow n, 1 \mapsto i$ we know that $y_i = (y_1, .., y_n)[i] = f_n(x_1, .., x_n)[i] = f_1(x_i)$. In a more geometric language, we may say that we used that each "simplex" $(y_1, .., y_n) \in M''^n$ is uniquely determined by its "0-dimensional faces" $y_1, ..., y_n \in M'$.

In the category-theoretic language, the facts above are expressed by saying that we get a fully faithful embedding $\cdot_\bullet : \mathrm{Sets} \longrightarrow \mathrm{sSets}$.

4.2.3. Metric spaces and the filter of uniform neighbourhoods of the main diagonal. — Let M be a metric space. Consider the *simplicial set* $|M|_\bullet : \Delta^{\mathrm{op}} \longrightarrow \mathrm{Sets}$ *represented by* the set $|M|$ of points of M defined above, i.e. the functor

$$n_\leq \longmapsto \left\{ n \underset{\mathrm{Sets}}{\longrightarrow} M \right\} = \{\,(t_1, ..., t_n) \in M^n\} = M^n$$

Now equip $|M|_\bullet(n_\leq) = |M|^n$ with the filter of uniform neighbourhoods of the main diagonal. Remarks in §def:filt:metr above about uniform continuity imply that a map $f : |M'| \longrightarrow |M''|$ is uniformly continuous iff it induces a natural transformation $f_\bullet : M'_\bullet \longrightarrow M''_\bullet$ of functors $\Delta^{\mathrm{op}} \longrightarrow \mathfrak{F}$.

Thus we see that the category of metric spaces and uniformly continuous maps is a fully faithful subcategory of $\wr\mathfrak{F}$.

In fact, the definition [Bourbaki, II§I.1,Def.I] of a uniform structure in can be phrased in the language of $\wr\mathfrak{F}$ as follows:

Lemma 4.2.3.1 (**[Bourbaki, II§I.1,Def.I]**). — *A uniform structure (or uniformity) on a set X is a structure given by a filter $\mathfrak{U}$ of subsets of $X \times X$ such that there is an object $X_\bullet : \Delta^{\mathrm{op}} \longrightarrow \mathfrak{F}$ of $\wr\mathfrak{F}$ which satisfies the following properties:*

(V_{I}) *("Every set belonging to U contains the diagonal Δ.")*
The filter on $X_\bullet(1_\leq)$ is indiscrete, i.e. is $\{|X_\bullet(1_\leq)|\}$.

(V_{II}) *("If $V \in \mathfrak{U}$ then $V^{-1} \in \mathfrak{U}$.")*
The functor $X_\bullet$ factors as

$$X_\bullet : \Delta^{\mathrm{op}} \longrightarrow \mathit{FiniteSets}^{op} \longrightarrow \mathfrak{F}$$

(V_{III}) *("For each $V \in \mathfrak{U}$ there exists $W \in \mathfrak{U}$ such that $W \circ W \subset V$.")*
for $n > 2$ $|X|_\bullet(n_\leq) = |X|^n$ is equipped with the coarsest filter such that the maps $X^n \longrightarrow X \times X, (x_1, .., x_n) \mapsto (x_i, x_{i+1})$, $0 < i < n$, of filters are continuous

4.2.4. Topological spaces and the filter of coverings. — This example is slightly technically complicated and should be skimmed at first reading. Let X be a topological space. Consider the *simplicial set* $|X|_\bullet : \Delta^{\mathrm{op}} \longrightarrow \mathrm{Sets}$ *represented by* the set $|X|$ of points of X. Now equip $|X|_\bullet(1_\leq) = |X|$ with the indiscrete filter $\{X\}$; equip $|X|_\bullet(2_\leq) = |X|^2$ with the filter of coverings; for $n > 2$ equip $|X|_\bullet(n_\leq) = |X|^n$ with the coarsest filter such that the map $X^n \longrightarrow X \times X, (x_1, .., x_n) \mapsto (x_i, x_{i+1})$, $0 < i < n$, of filters are continuous. A verification shows that this construction does indeed define a simplicial filter $X_\bullet :$ ɀ𝝋.

In fact, axioms of topology in terms of neighbourhoods [Bourbaki,I§1.2: $(\mathrm{V_I})$-$(\mathrm{V_{IV}})$, Proposition 2] may be interpreted as saying this construction does define an object of ɀ𝝋, see [6,3.1.2] for details. We paraphrase [Bourbaki, I§1.2, Proposition 2] giving the axioms of topology in terms of neighbourhoods:

Lemma 4.2.4.1 **([Bourbaki, I§1.2, Proposition 2])**. — *If to each element x of a set X there corresponds a set $\mathfrak{N}(x)$ of subsets of X, and there is an object $X_\bullet$ of* ɀ𝝋 *such that*

– *its underlying simplicial set is $|X|_\bullet$*

$(\mathrm{V_{III}})$ *("The element x is in every set of $\mathfrak{N}(x)$.")*
$|X|_\bullet(1_\leq) = |X|$ *is equipped with the indiscrete filter* $\{X\}$

$(\mathrm{V_I})$-$(\mathrm{V_{II}})$ *((V_I) "Every subset of X which contains a set belonging to $\mathfrak{N}(x)$ itself belongs to $\mathfrak{N}(x)$.*
(V_{II}) "Every finite intersection of sets of $\mathfrak{N}(x)$ belongs to $\mathfrak{N}(x)$.")
$\{\bigcup_{x\in X}\{x\} \times U_x : U_x \in \mathfrak{N}(x)\}$ *is a filter on* $X \times X$

– $|X|_\bullet(2_\leq) = |X| \times |X|$ *is equipped with this filter*

$(\mathrm{V_{IV}})$ *("If V belongs to $\mathfrak{N}(x)$, then there is a set W belonging to $\mathfrak{N}(x)$ such that, for each $y \in W$, V belongs to $\mathfrak{N}(y)$.")*
for $n > 2$ $|X|_\bullet(n_\leq) = |X|^n$ is equipped with the coarsest filter such that the maps $X^n \longrightarrow X \times X, (x_1, .., x_n) \mapsto (x_i, x_{i+1})$, $0 < i < n$, of filters are continuous

then there is a unique topological structure on X such that, for each $x \in X$, $\mathfrak{N}(x)$ is the set of neighbourhoods of x in this topology.

This defines a fully faithful embedding $\mathrm{Top} \longrightarrow$ ɀ𝝋; in fact it has an inverse ɀ𝝋 $\longrightarrow \mathrm{Top}$. [7,§2.6.1].

4.2.5. Linear order and the filter of tails. — Let $I^\leq$ be a linear order. Consider the *simplicial set* $I^\leq_\bullet : \Delta^{\mathrm{op}} \longrightarrow \mathrm{Sets}$ *represented by* the linear order $I^\leq$ defined as the functor

$$n_\leq \longmapsto \left\{ n_\leq \xrightarrow[\text{preorders}]{} I^\leq \right\} = \{\, (t_1, ..., t_n) \in I^n : t_1 \leq ... \leq t_n \}$$

Morphisms are defined similarly to the above. Now equip $I^\leq_\bullet(n_\leq)$ with the filter of tails, where we consider $I^\leq_\bullet(n_\leq)$ equipped with tuples ordered element-wise: $(t'_1, ..., t'_n) \leq (t''_1, ..., t''_n)$ if for each i $t'_i \leq t''_i$.

This defines a functor $I^{\leq \mathrm{tails}}_\bullet : \Delta^{\mathrm{op}} \longrightarrow$ 𝝋, i.e. an object of ɀ𝝋.

In the same way we may define the filter of tails on the simplicial filter represented by $|I|$ as a set. In more details, let $|I|^{\leq \mathrm{tails}}_\bullet : \Delta^{\mathrm{op}} \longrightarrow$ 𝝋 denote the simplicial filter whose underlying sset

$$n_\leq \longmapsto \left\{ n_\leq \xrightarrow[\text{Sets}]{} I^\leq \right\} = |I|^n$$

is equipped with the *filter of tails* generated by, the subsets $\{(i_1, ..., i_n) : i_1 \geq i_0, ..., i_n \geq i_0\} \subset |I|^n$, for $i_0 \in |I|$.

The identity map defines an inclusion $I_\bullet^{\leq\text{tails}} \hookrightarrow |I|_\bullet^{\leq\text{tails}}$ in $s\boldsymbol{\varphi}$.

4.2.6. Cauchy sequences and indiscernible sequences. — In §4.1.6 we saw that a Cauchy sequence is a map $a : \omega \longrightarrow |M|$ such that it induces a continuous map $((\omega^{\leq})^n)^{\text{tails}} \longrightarrow M^n$ from the linear order $(\omega^{\leq})^{n\leq}$ equipped with the filter of tails to M^n equipped with the filter of uniform neighbourhoods of the main diagonal. In the notation of the previous example, it says that a Cauchy sequence is a map $a : \omega \longrightarrow |M|$ such that it induces an $s\boldsymbol{\varphi}$-morphism $|I|_\bullet^{\leq\text{tails}} \longrightarrow M_\bullet$, and therefore also a map $I_\bullet^{\leq\text{tails}} \longrightarrow M_\bullet$. In fact each morphism of the underlying simplicial sets of these sfilters is induced by a map $\omega \longrightarrow |M|$ of sets, and hence we get a definition of a Cauchy sequence in $s\boldsymbol{\varphi}$:

- a Cauchy sequence is a morphism $I_\bullet^{\leq\text{tails}} \longrightarrow M_\bullet$
- or, equivalently,
- a Cauchy sequence is a morphism $|I|_\bullet^{\leq\text{tails}} \longrightarrow M_\bullet$

We shall later see an *eventually indiscernible sequence* in a model M is an injective morphism $I_\bullet^{\leq\text{tails}} \longrightarrow M_\bullet$ to a certain object of $s\boldsymbol{\varphi}$ associated with model M. This is the observation which started these notes.

4.2.7. A glossary of our notation. — For a natural number $k \in \mathbb{N}$, by $k_\leq$ we denote the linearly ordered set $1 < 2 < .. < k$ with k elements viewed as an object of Δ^{op}; a morphism $\theta : k_\leq \longrightarrow l_\leq$ in Δ we denote by $[i_1 \leq ... \leq i_k]$ where $1 \leq i_1 = \theta(1) \leq ... i_k = \theta(k) \leq l$, and $[i..j]$ is short for $[i \leq i+1 \leq .. \leq j]$.

By $X_\bullet, Y_\bullet, ..$ we shall usually denote objects of $s\boldsymbol{\varphi}$ or, rarely, another category of functors; often X,Y,.. is some mathematical object (a linear order, a model, a topological or metric space, ...) and $X_\bullet, Y_\bullet, ...$, possibly with a superscript, denotes the object of $s\boldsymbol{\varphi}$ corresponding to $X, Y, ..$; the superscript may indicate the nature of the correspondence. Sometimes we write $X_\bullet : s\boldsymbol{\varphi}$ to indicate that $X_\bullet$ is an object of $s\boldsymbol{\varphi}$.

For $X_\bullet : s\boldsymbol{\varphi}$, by $X_\bullet(n_\leq) = X(n_\leq) = X_{n-1}$ we denote the functor $X_\bullet$ evaluated at $n_\leq$; elements of $X_\bullet(n_\leq) = X(n_\leq) = X_{n-1}$ are called $(n-1)$*-simplicies.*

By functoriality a non-decreasing map $[i_1 \leq ... \leq i_k] : k_\leq \longrightarrow l_\leq$ and a $(l-1)$-simplex $x \in X_\bullet(l_\leq) = X_{l-1}$ determine a $(k-1)$-simplex $x[i_1 \leq ... \leq i_k] \in X_\bullet(l_\leq) = X_{k-1}$ called a $(k-1)$*-dimensional face of* x, and, sometimes by abuse of language, the $(k-1)$*-dimensional face of* x *with vertices or coordinates* $i_1 \leq ... \leq i_k$.

A k-simplex $x \in X_k$ is *degenerate* iff it is a face of some l-simplex of smaller dimension $l < k$.

A k-simplex of form $x[i_1 \leq ... \leq i_r \leq i_r \leq ... \leq i_k] \in X_k$ is necessarily degenerate, as it is a face of $(k-1)$-simplex $x[i_1 \leq ... \leq i_k] \in X_{k-1}$, as seen by equality $x[i_1 \leq ... \leq i_r \leq i_r \leq ... \leq i_k] = (x[i_1 \leq ... \leq i_r \leq ... \leq i_k])[1 \leq .. \leq r \leq r \leq ... \leq k]$.

By $\left\{X \rightarrow Y\right\} := \text{Hom}\,(X, Y)$ or $\left\{X \underset{C}{\rightarrow} Y\right\} := \text{Hom}_C\,(X, Y)$ we denote the set of morphisms from $X : C$ to $Y : C$ in a category C. By $\left\{- \rightarrow Y\right\}$ or $\left\{- \underset{C}{\rightarrow} Y\right\}$ we denote the functor $C^{\text{op}} \longrightarrow \text{Sets}$, $X \longmapsto \left\{X \underset{C}{\rightarrow} Y\right\}$ hence Δ_{N-1}, $\left\{- \rightarrow N_\leq\right\}$ and $\left\{- \xrightarrow[\text{preorders}]{} N_\leq\right\}$ denote the $(N-1)$-dimensional simplex (as a simplicial set). By $\left\{X \Rightarrow Y\right\} := \underline{\text{Hom}}\,(X, Y)$ or $\left\{X \underset{C}{\Rightarrow} Y\right\} := \underline{\text{Hom}}_C\,(X, Y)$ we denote the inner hom from $X : C$ to $Y : C$ in a category C whenever it is defined.

Thus $\left\{X \underset{\text{sSets}}{\Longrightarrow} Y\right\}$ denotes the inner hom in the category of simplicial sets, and thereby

$$\left\{X \underset{\text{sSets}}{\Longrightarrow} Y\right\}(1_{\le}) = \left\{X \underset{\text{sSets}}{\Longrightarrow} Y\right\}_0 = \left\{X \underset{\text{sSets}}{\longrightarrow} Y\right\} = \mathrm{Hom}_{\text{sSets}}(X,Y)$$

denotes the set of morphism from X to Y.

We put in quotation marks words intended to aid intuition but formally unnecessary; thus formally $x \in \delta$ and "δ-small" $x \in \delta$ mean the same.

5. Appendix. NIP, NOP, and non-dividing.

We sketch definitions of NIP and OP by lifting properties, and discuss NSOP. For completeness we also include a somewhat different exposition of stability. The exposition here is sketchy and preliminary.

The category Δ of finite linear orders has an endofunctor which "shifts dimension" $n \longmapsto n+1$. The category $\wr\mathcal{P} = Funct(\Delta^{\mathrm{op}}, \mathcal{P})$ is a category of functors on Δ^{op}, and therefore an endofunctor $\Delta \longrightarrow \Delta$ of category Δ of finite linear orders induces an endofunctor $\wr\mathcal{P} \longrightarrow \wr\mathcal{P}$. In topology [6,§2.1.3-4,§4.8-10] the "local" notions of limit and local triviality are expressed of an endfunctor of Δ "shifting dimension" sending $n_{\le} \longmapsto (n+1)_{\le}$.

We use this endofunctor of Δ and somewhat cumbersome modifications of the topological definitions to define NIP and non-dividing. The lifting property defining NOP is analogous to the finite cover property defining compactness.

5.1. Stability as a Quillen negation analogous to a path lifting property.

— Now we may reformulate the characterisation of stability that "each infinite indiscernible sequence is necessarily an indiscernible set" as a Quillen lifting property/negation [15]

5.1.1. Simplicial filters associated with linear orders and filters. —

Definition 5.1.1.1 **($|I|_\bullet^{\mathfrak{F}}, I_\bullet^{\le\text{tails}}, |I|_\bullet^{\text{tails}} : \wr\mathcal{P}$).** — Let I be a linear order, and let $\mathfrak{F}$ be a filter on $|I|$.

Let $|I|_\bullet^{\mathfrak{F}}$ be the simplicial filter whose underlying simplicial set is represented by the set $|I|$, i.e. the functor $\left\{- \underset{\text{Sets}}{\longrightarrow} |I|\right\} : \Delta^{\mathrm{op}} \longrightarrow \text{Sets}$. The set

$$\left\{n_{\le} \underset{\text{Sets}}{\longrightarrow} |I|\right\} = \{(t_1, ..., t_n) \in |I|^n\} = |I|^n$$

is equipped with the filter generated by sets of the form ε^n, $\varepsilon \in \mathfrak{F}$.

Let $I_\bullet^{\le,\mathfrak{F}}$ be the simplicial filter whose underlying simplicial set is represented by the linear order I, (fixme: better? corepresented by I considered as a linear

---

[15] A morphism $i : A \to B$ in a category has *the left lifting property* with respect to a morphism $p : X \to Y$, and $p : X \to Y$ also has *the right lifting property* with respect to $i : A \to B$, denoted $i \rtimes p$, iff for each $f : A \to X$ and $g : B \to Y$ such that $p \circ f = g \circ i$ there exists $h : B \to X$ such that $h \circ i = f$ and $p \circ h = g$. This notion is used to define properties of morphisms starting from an explicitly given class of morphisms, often a list of (counter)examples, and a useful intuition is to think that the property of left-lifting against a class C is a kind of negation of the property of being in C, and that right-lifting is also a kind of negation. See [Wikipedia,Lifting_property] for details and examples.

order?) i.e. the functor $\left\{- \xrightarrow[preorders]{} I^{\leq}\right\} : \Delta^{\mathrm{op}} \longrightarrow \mathrm{Sets}$. The set

$$\left\{n_{\leq} \xrightarrow[preorders]{} I^{\leq}\right\} = \{(t_1, ..., t_n) \in |I|^n \, : \, t_1 \leq ... \leq t_n\}$$

is equipped with the filter generated by sets of the form

$$\{(t_1, ..., t_n) \in \varepsilon^n \, : \, t_1 \leq ... \leq t_n\}, \quad \varepsilon \in \mathfrak{F}$$

Let $I^{\leq\mathrm{tails}}_\bullet := I^{\leq,\mathfrak{F}}_\bullet$, $|I|^{\mathrm{tails}}_\bullet := |I|^{\mathfrak{F}}_\bullet$ for $\mathfrak{F} := \{\{x : x \geq i\} : i \in I\}$ the filter generated by non-empty final segments of I.

5.1.2. Simplicial filters associated with structures (fixme: models?))— First we need some preliminary notation. For a formula $\phi(x_1, .., x_r)$ of arity r and a natural number $n > 0$, let $\phi^{n\text{-EM}}$ be the n-ary formula

$$\bigwedge_{1\leq i_1<...<i_r\leq n, 1\leq j_1<...<j_r\leq n} \left(\bigwedge_{1\leq s<t\leq r} x_{i_s} \neq x_{i_t} \& x_{j_s} \neq x_{j_t} \implies (\phi(x_{i_1}, .., x_{i_r}) \leftrightarrow \phi(x_{j_1}, .., x_{j_r})) \right)$$

The formula $\phi^{n\text{-EM}}(a_1, ..., a_n)$ says that each subsequence of $a_1, .., a_n$ with *distinct* elements is ϕ-indiscernible. In particular, $\phi^{n\text{-EM}}()$ belongs to the EM-type of each ϕ-indiscernible sequence.

Definition 5.1.2.1 **($M^{\{\phi\}}_\bullet : \boldsymbol{\wr\varphi}$, $M^{\Sigma}_\bullet : \boldsymbol{\wr\varphi}$).** — Let M be a model, and let Σ be a set of formulas in the language of M.

Let $M^{\Sigma}_\bullet : \Delta^{\mathrm{op}} \longrightarrow \boldsymbol{\varphi}$ be the simplicial filter whose underlying simplicial set is $|M|_\bullet$ represented by the set of elements of M, i.e. the functor $\left\{- \xrightarrow[\mathrm{Sets}]{} |M|\right\} : \Delta^{\mathrm{op}} \longrightarrow \mathrm{Sets}$. The set

$$\left\{n_{\leq} \xrightarrow[\mathrm{Sets}]{} |M|\right\} = \{(x_1, ..., x_n) \in |I|^n\} = |M|^n$$

is equipped with the filter generated by the sets $\phi^{n\text{-EM}}(M^n)$ of all tuples satisfying the formula $\phi^{n\text{-EM}}$, for $\phi \in \Sigma$.

Let $M_\bullet$ denote $M^{\Sigma}_\bullet$ for Σ the set of all parameter-free formulas of the language of M, and let $M^{L(A)}_\bullet$ denote $M^{\Sigma}_\bullet$ for Σ the set of all formulas of the language of M with parameters in A. Let $M^{\mathrm{qf}}_\bullet$ denote $M^{\Sigma}_\bullet$ for Σ the set of all quantifier-free formulas of the language of M.

The reader may wish to check that the forgetful functor $\boldsymbol{\wr\varphi} \longrightarrow \mathrm{Top}$ (see [7,§12.6.3], also [6,§2.2.4])) takes $M_\bullet$ into the set $|M|$ equipped with the topology generated by sets $\phi(M)$ for all the unary formulas ϕ of M; so to say, it is the 1-Stone space of M *before* it has been quotiented by the relation of having the same type.

To verify that $M^{\{\phi\}}_\bullet : \Delta^{\mathrm{op}} \longrightarrow \boldsymbol{\varphi}$ is indeed a functor to $\boldsymbol{\varphi}$ rather than just Sets, it is enough to verify that for each $\phi \in \Sigma$ $M^{\{\phi\}}_\bullet : \Delta^{\mathrm{op}} \longrightarrow \boldsymbol{\varphi}$ is indeed a functor to $\boldsymbol{\varphi}$ rather than just Sets. We only need to check that maps

$$[i_1 \leq ... \leq i_n] : M^m \longrightarrow M^n, \quad (x_1, .., x_m) \longmapsto (x_{i_1}, ..., x_{i_n})$$

are continuous for any $1 \leq i_1 \leq ... \leq i_n \leq m$, i.e. that for each $x_1, .., x_n \in M$

$$M \models \phi^{m\text{-EM}}(x_1, ..., x_m) \implies \phi^{n\text{-EM}}(x_{i_1}, ..., x_{i_n})$$

However, this is trivially true, as $\phi^{m\text{-EM}}(x_1, ..., x_m)$ says that each subsequence of $x_1, ..., x_m$ with distinct elements is ϕ-indiscernible, and $\phi^{n\text{-EM}}(x_{i_1}, ..., x_{i_n})$ says that each subsequence of $x_{i_1}, ..., x_{i_n}$ with distinct elements is ϕ-indiscernible. Note that it is essential in the argument that we talk about *distinct* elements, i.e. that the inequalities in $\phi^{n\text{-EM}}$ are essential.

5.1.3. Indiscernible sequences with repetitions. — Call a sequence *(totally) ϕ-indiscernible with repetitions* iff each subsequence with *distinct* elements is necessarily (totally) ϕ-indiscernible. By the definition of $\phi^{n\text{-EM}}$, a sequence is ϕ-indiscernible with repetitions iff $\phi^{n\text{-EM}}$ belongs to its EM-type for any $n > 0$ (equiv., for some $n > 2r$ where r is the arity of ϕ). Note that we allow that there are only finitely many distinct elements in a ϕ-indiscernible sequence with repetitions, e.g. $(a, b, a, b, a, b, ...)$ is ϕ-indiscernible with repetitions for $\{a, b\}$ an indiscernible set.

The following lemma is the key observation which started this paper.

Lemma 5.1.3.1. — *For any linear order I and any structure M the following holds.*

- *An eventually ϕ-indiscernible sequence $(a_i)_{i\in I}$ in M with repetitions induces an $\wp$-morphism $a_\bullet : I_\bullet^{\leq tails} \longrightarrow M_\bullet^{\{\phi\}}$, and, conversely, each morphism $I_\bullet^{\leq tails} \longrightarrow M_\bullet^{\{\phi\}}$ is induced by a unique such sequence.*
- *An eventually order ϕ-indiscernible sequence $(a_i)_{i\in I}$ with repetitions induces an $\wp$-morphism $a_\bullet : |I|_\bullet^{tails} \longrightarrow M_\bullet^{\{\phi\}}$, and, conversely, every morphism : $|I|_\bullet^{tails} \longrightarrow M_\bullet^{\{\phi\}}$ is induced by a unique such sequence.*

Proof. — First note that by Example 2.2.2 maps $|I| \longrightarrow |M|$ of sets are into one-to-one correspondence with morphisms $|I|_\bullet \longrightarrow |M|_\bullet$, i.e. the morphisms of the underlying simplicial sets $|I|_\bullet = ||I|_\bullet^{\text{tails}}|$ and $|M|_\bullet = |M_\bullet^{\{\phi\}}|$, and the same is true for $|I_\bullet^{\leq}| = |I_\bullet^{\leq\text{tails}}| \longrightarrow |M|_\bullet = |M_\bullet^{\{\phi\}}|$. Hence, at the level of the underlying simplicial sets, any sequence $(a_i)_{i\in I}$ gives rise to morphisms $|I_\bullet^{\leq\text{tails}}| \longrightarrow |\ M_\bullet^{\{\phi\}}|$ and $||I|_\bullet^{\text{tails}}| \longrightarrow |M_\bullet^{\{\phi\}}|$ of the underlying simplicial sets. Therefore, we only need to check what it means to be continuous for the following induced maps of filters for each $n > 0$:

$$|I_\bullet^{\leq\text{tails}}(n_\leq)| = \{(i_1, ..., i_n) \in |I|^n : i_1 \leq ... \leq i_n\} \longrightarrow M^n$$
$$||I|_\bullet^{\text{tails}}(n_\leq)| = |I|^n \longrightarrow M^n$$

where M^n is equipped with the ϕ-EM-filter, and $|I|^n$ and $(i_1, ..., i_n) \in |I|^n : i_1 \leq ... \leq i_n\}$ are equipped with the filter of tails.

Continuity of $\{(i_1, ..., i_n) \in |I|^n : i_1 \leq ... \leq i_n\} \longrightarrow M^n$ means that the preimage of any neighbourhood, i.e. of $\phi^{n\text{-EM}}(M^n)$, contains a neighbourhood, i.e. the subset $\{(i_1, ..., i_n) \in |I|^n : i_0 \leq i_1 \leq ... \leq i_n\}$ for some $i_0 \in I$. That is, for each $i_0 \leq i_1 \leq ... \leq i_n$ $M \vDash \phi^{n\text{-EM}}(a_{i_1}, ..., a_{i_n})$. By definition of $\phi^{n\text{-EM}}$, it means that for two any subsequences $a_{i_{l_l}}, ..., a_{i_{l_r}}$ and $a_{i_{k_1}}, ..., a_{i_{k_r}}$, $1 \leq l_1 < .. < l_r \leq n, 1 \leq k_1 < ... < k_r \leq n$, with *distinct* elements of the final segment $(a_i)_{i\geq i_0}$, it holds that $\phi(a_{i_{l_l}}, ..., a_{i_{l_r}}) \leftrightarrow \phi(a_{i_{k_1}}, ..., a_{i_{k_r}})$. That is, by definition it means that the sequence $(a_i)_{i\geq i_0}$ is ϕ-indiscernible with repetitions.

Continuity of $|I|^n \longrightarrow M^n$ means that the preimage of any neighbourhood, i.e. of $\phi^{n\text{-EM}}(M^n)$, contains a neighbourhood, i.e. the subset $\{(i_1, ..., i_n) \in |I|^n : i_0 \leq i_1, i_0 \leq i_1, ..., i_0 \leq i_n\}$ for some $i_0 \in I$. That is, for each $i_0 \leq i_1, ..., i_0 \leq i_n$ $M \vDash \phi^{n\text{-EM}}(a_{i_1}, ..., a_{i_n})$; note that now $i_1, ..., i_n$ are not necessarily either distinct or increasing. That is, for each $i_0 \leq i_1 \leq ... \leq i_n$ $M \vDash \phi^{n\text{-EM}}(a_{i_1}, ..., a_{i_n})$. By definition of $\phi^{n\text{-EM}}$, it means that for two subsequences $a_{i_{l_l}}, ..., a_{i_{l_r}}$ and $a_{i_{k_1}}, ..., a_{i_{k_r}}$ with *distinct* elements of the final segment $(a_i)_{i\geq i_0}$, it holds that $\phi(a_{i_{l_l}}, ..., a_{i_{l_r}}) \leftrightarrow \phi(a_{i_{k_1}}, ..., a_{i_{k_r}})$. This is exactly the definition of being order ϕ-indiscernible with repetitions. □

5.1.4. Cauchy sequences: a formal analogy to indiscernible sequences. — This is formally unnecessary but might help the reader's intuition.

Recall that with a metric space M we associate its simplicial filter $M_\bullet$ whose underlying simplicial set $|M|_\bullet$ is represented by the set of elements of M, and where we equip $|M|^n$ with *the filter of ε-neighbourhoods of the main diagonal* generated by the subsets, for $\epsilon > 0$

$$\{\,(x_1,...,x_n)\,:\,\mathrm{dist}(x_i,x_j) < \epsilon \text{ for } 1 \le i < j \le n\,\}$$

This lemma establishes a formal analogy between Cauchy sequences and indiscernible sequences with repetitions: they both are defined as in $\wr\mathcal{P}$ as morphisms from the same object associated with a linear order.

Lemma 5.1.4.1. — *For any linear order I and any metric space M the following holds.*

(i) *A sequence $(a_i)_{i\in I}$ in M induces an morphism $a_\bullet : |I|_\bullet \longrightarrow |M|_\bullet$ of* sSets, *and, conversely,every morphism $|I|_\bullet \longrightarrow |M|_\bullet$ corresponds to a unique such sequence*

(ii) *A Cauchy sequence $(a_i)_{i\in I}$ in M induces an $\wr\mathcal{P}$-morphism $a_\bullet : I_\bullet^{\le tails} \longrightarrow M_\bullet$, and, conversely, every morphism $I_\bullet^{\le tails} \longrightarrow M_\bullet^{\{\phi\}}$ corresponds to a unique such sequence.*

(iii) *A Cauchy sequence $(a_i)_{i\in I}$ in M induces an $\wr\mathcal{P}$-morphism $a_\bullet : |I|_\bullet^{tails} \longrightarrow M_\bullet^{\{\phi\}}$, and, conversely, every morphism $|I|_\bullet^{tails} \longrightarrow M_\bullet^{\{\phi\}}$ is induced by a unique such sequence.*

Proof. — (i): A map $a : |I| \longrightarrow |M|$ of sets induces a natural transformation of functors $f_\bullet : I_\bullet^\le \longrightarrow |M|_\bullet$: for each $n > 0$, a tuple $(i_1 \le ..,\le i_n) \in I_\bullet^\le(n_\le)$ goes into tuple $(a_{i_1},,...,a_{i_n}) \in |M|^n = |M|_\bullet(n_\le)$. Now let us show that every natural transformation $a_\bullet : I_\bullet^\le \longrightarrow |M|_\bullet$ is necessarily of this form, by the following easy argument. Let $(y_1,..,y_n) = f_n(x_1,..,x_n)$; by functoriality using maps $[i] : 1 \longrightarrow n, 1 \mapsto i$ we know that $y_i = (y_1,..,y_n)[i] = f_n(x_1,..,x_n)[i] = f_1(x_i)$. Finally, use that for any $y_1,..,y_n \in |M|_\bullet(1_\le) = |M|$ there is a unique element $\tilde{y} \in |M|_\bullet(n_\le) = |M|^n$ such that $y_i = \tilde{y}[i]$. In a more geometric language, we may say that we used the following property of the simplicial set $|M|_\bullet$: that each "(n-1)-simplex" $(y_1,..,y_n)$ is uniquely determined by its "0-dimensional faces" $y_1,...,y_n \in M$.

(ii): We need to check what it means to for the map $f_n : I_\bullet^\le(n_\le) \longrightarrow M^n$ of filters to be continuous. Consider $n = 2$; for $n > 2$ the argument is the same. Continuity means that for each $\epsilon > 0$ and thereby "ϵ-neighbourhood $\varepsilon := \{(x,y) : x,y \in M, \mathrm{dist}(x,y) < \epsilon\}$ of the main diagonal" there is $N > 0$ and thereby a "N-tail" neighbourhood $\delta := \{(i,j) \in \omega\times\omega : N \le i \le j\}$ such that $a(\delta) \subset \varepsilon$, i.e. for each $j \ge i > N$ it holds $\mathrm{dist}(a_i,a_j) < \epsilon$.

(iii): We need to check what it means to for the map $f_n : |I|^n \longrightarrow M^n$ of filters to be continuous. Consider $n = 2$; for $n > 2$ the argument is the same. Continuity means that for each $\epsilon > 0$ and thereby "ϵ-neighbourhood $\varepsilon := \{(x,y) : x,y \in M, \mathrm{dist}(x,y) < \epsilon\}$ of the main diagonal" there is $N > 0$ and thereby a "N-tail" neighbourhood $\delta := \{(i,j) \in \omega\times\omega : i,j > N\}$ such that $a(\delta) \subset \varepsilon$, i.e. for each $i,j > N$ it holds $\mathrm{dist}(a_i,a_j) < \epsilon$. □

5.1.5. Stability as Quillen negation of indiscernible sets (fixme: better subtitle..)— Let $\top = |\{pt\}|_\bullet$ denote the terminal object of the category $\wr\mathcal{P}$, i.e. "the simplicial set represented by a singleton equipped with indiscrete filters": for any $n > 0$ $\top(n_\le) := \{pt\}$, and the only big subset is $\{pt\}$ itself.

Proposition 5.1.5.1 **(Stability as Quillen negation)**

Let M be a model, and let ϕ be a formula in the language of M. Let I be a linear order. The following are equivalent:

(i) *in the model M, each infinite ϕ-indiscernible sequence is necessarily a ϕ-indiscernible set*
(ii) *in the model M, each eventually (possibly finite!) ϕ-indiscernible sequence is necessarily an eventually ϕ-indiscernible set*
(ii') *in the model M, each eventually (possibly finite!) ϕ-indiscernible sequence with repetitions is necessarily eventually order ϕ-indiscernible with repetitions*
(iii) *the following lifting property holds in the category $\wr\mathcal{P}$:*

$$I_\bullet^{\leq tails} \longrightarrow |I|_\bullet^{tails} \rightthreetimes M_\bullet^{\{\phi\}} \longrightarrow \top$$

i.e. the following diagram in $\wr\mathcal{P}$ holds:

$$\begin{array}{ccc} I_\bullet^{\leq tails} & \longrightarrow & M_\bullet^{\{\phi\}} \\ \downarrow & \nearrow & \downarrow \\ |I|_\bullet^{tails} & \longrightarrow & \top \end{array}$$

Proof. — (i)⇔(ii) is obvious and well-known.

(ii') $\Longrightarrow$ (ii) is trivial so we only need to prove (ii) $\Longrightarrow$ (ii): Consider an eventually ϕ-indiscernible sequence with repetitions. Take a maximal subsequence with distinct elements. First assume it is infinite. Then it is eventually ϕ-indiscernible, hence eventually order ϕ-indiscernible by (ii), hence the original sequence is eventually order ϕ-indiscernible with repetitions.

So what happens if there are only finitely many distinct elements?(16) Call this sequence $(a_i)_{i\in I}$. Pick an initial segment $(a_i)_{i\leq i_0}$ of the sequence which contains all the elements of $(a_i)_{i\in I}$ which occur only finitely many times; then in the corresponding final segment $(a_i)_{i>i_0}$ each element occurs infinitely many times whenever it occurs there at all. Therefore any finite subsequence of that final segment occurs there in an arbitrary order, hence $(a_i)_{i>i_0}$ is ϕ-indiscernible iff it is order ϕ-indiscernible.

(iii) $\Longrightarrow$ (ii'): Let $(a_i)_{i\in I}$ be an eventually ϕ-indiscernible sequence with repetitions. By Lemma 3.3.1.1 it induces an $\wr\mathcal{P}$-morphism $a_\bullet : I_\bullet^{\leq \mathrm{tails}} \longrightarrow M_\bullet^{\{\phi\}}$. By the lifting property it lifts to a $\wr\mathcal{P}$-morphism $a'_\bullet : |I|_\bullet^{\mathrm{tails}} \longrightarrow M_\bullet^{\{\phi\}}$. By commutativity they both correspond to the same map of the underlying simplicial sets, i.e. to the same sequence $(a_i)_{i\in I}$. Again by Lemma 3.3.1.1, this sequence is eventually order ϕ-indiscernible with repetitions.

(ii') $\Longrightarrow$ (iii): Let $a_\bullet : I_\bullet^{\leq \mathrm{tails}} \longrightarrow M_\bullet^{\{\phi\}}$ be the $\wr\mathcal{P}$-morphism corresponding to the lower horizontal arrow. By Lemma 3.3.1.1 it corresponds to an eventually ϕ-indiscernible sequence $(a_i)_{i\in I}$ with repetitions. By (ii') this sequence is also eventually order ϕ-indiscernible with repetitions. By Lemma 3.3.1.1 it corresponds to an $\wr\mathcal{P}$-morphism $|I|_\bullet^{\mathrm{tails}} \longrightarrow M_\bullet^{\{\phi\}}$, as required. □

(16) We remark that in category theory it is often important that things work in a "degenerate" case, such as, here, the case of a ϕ-indiscernible sequence which has only finitely many distinct elements. Note that we would not have been able to write the lifting property if not for this set-theoretic argument.

5.1.6. NIP as almost a lifting property. —

***Lemma 5.1.6.1* (NIP as almost a lifting property)**

Let M be a structure. Let I be a linear order. The following are equivalent:

(i) *in the model M, for each $b \in M$, each eventually indiscernible sequence (over $\varnothing$) is eventually $\phi(-,b)$-indiscernible*

(ii) *in $s\mathcal{F}$ each injective morphism $I_\bullet^{\leq tails} \longrightarrow M_\bullet$ factors as $I_\bullet^{\leq tails} \longrightarrow M_\bullet^{L(M)} \longrightarrow M_\bullet$, i.e. the following diagram holds:*

$$\begin{array}{ccc} \bot & \longrightarrow & M_\bullet^{L(M)} \\ \downarrow & \nearrow & \downarrow \\ I_\bullet^{\leq tails} & \text{-}(inj)\!\longrightarrow & M_\bullet \end{array}$$

(iii) *in $s\mathcal{F}$ each injective morphism $\omega_\bullet^{\leq tails} \longrightarrow M_\bullet$ factors as $\omega_\bullet^{\leq tails} \longrightarrow M_\bullet^{L(M)} \longrightarrow M_\bullet$, i.e. the following diagram holds:*

$$\begin{array}{ccc} \bot & \longrightarrow & M_\bullet^{L(M)} \\ \downarrow & \nearrow & \downarrow \\ \omega_\bullet^{\leq tails} & \text{-}(inj)\!\longrightarrow & M_\bullet \end{array}$$

Proof. — (i)↔(ii): Both the bottom arrow and the diagonal arrow $|I_\bullet^{\leq \text{tails}}| \longrightarrow |\, M_\bullet^{L(M)}| = M_\bullet|$ correspond to the same map $a : |I| \longrightarrow |M|$, i.e. a sequence $(a_i)_{i\in I}$ of elements of M. Thus, we only need to check continuity. Injectivity and continuity of the bottom arrow $I_\bullet^{\leq \text{tails}} \longrightarrow M_\bullet$ means it is an eventually indiscernible sequence with repetitions. Continuity of the diagonal arrow $I_\bullet^{\leq \text{tails}} \longrightarrow M_\bullet^{L(M)}$ means it is an eventually ϕ-indiscernible sequence with repetitions for any formula $\phi \in L(M)$ with parameters in M. This is exactly (i).

(ii)↔(iii): follows from compactness. □

However, note the diagram (ii) may fail for a non-injective morphism. Indeed, a sequence $(a,b,a,b,...)$ where (a,b) is an indiscernible set, represents a continuous map $I_\bullet^{\leq \text{tails}} \longrightarrow M_\bullet$. In Appendix §5.2 we slightly modify the notion of the EM-filter to take care of this "degenerate" case, and define NIP as a lifting property

$$\bot \longrightarrow I_\bullet^{\leq \text{tails}} \;\rightthreetimes\; M_\bullet^{L(M)} \longrightarrow M_\bullet$$

Later we also reformulate as a lifting property the characterisation of NIP using average/limit types.

5.1.7. Simplicial Stone spaces. — Let $M_\bullet/A$ denote the *simplicial Stone space of types over A*, i.e. the simplicial filter $M_\bullet$ quotiented by the relation of having the same type over A. Note that in all our characterisations above we could have used $M_\bullet/A$ rather than $M_\bullet$. We also note that from a certain category-theoretic point of view it is somewhat more interesting, as its underlying simplicial set is not represented.

We use these spaces to reformulate non-dividing in a diagram-chasing manner in Appendix §5.3.

5.2. NIP and limit types as Quillen negation. — To define NIP in terms of limit types, we need to introduce the shift endofunctors of Δ and thereby $s\mathcal{F}$, and shifted models. We also need to modify the notion of EM-type and make it not symmetric. We do so now.

5.2.1. The "shift" endofunctor $[+\infty] : \Delta \longrightarrow \Delta$ "forgetting the last coordinate". — Let us define the "shift" endofunctor of Δ^{op} "forgetting the last coordinate". Let $[+\infty] : \Delta \longrightarrow \Delta$ denote the shift by 1 adding a new maximal element, which is kept fixed by the morphisms. In notation, the endofunctor $[+\infty]$ sends the linear order $n_\leq \in \mathrm{Ob}\,\Delta^{op}$ into the linear order $(n+1)_\leq \in \mathrm{Ob}\,\Delta^{op}$, and a morphism $f : m_\leq \longrightarrow_\leq n_\leq$ into the morphism $f[+\infty] : (m+1)_\leq \longrightarrow (n+1)_\leq$ defined by $f[+\infty](m+1) = n+1$, and for all $1 \leq i \leq m$ $f[+\infty](i) = f(i)$. [17] The endofunctor is equipped with a natural transformation $[-\infty] : [+\infty] \Longrightarrow \mathrm{id} : \Delta^{op} \longrightarrow \Delta^{op}$, $[1 < 2 < .. < n] : (n+1)_\leq \longrightarrow n_\leq$.

For a $X_\bullet : \mathfrak{sF}$ a simplicial filter, the morphism $X_\bullet[+\infty] \xrightarrow{[-\infty]} X_\bullet$ will be particularly useful to us.

To gain intuition, one may want to consider the example of a represented set. The shift endofunctor takes a represented sset

$$\left\{- \underset{Sets}{\longrightarrow} M\right\} = (M, M \times M, ...)$$

into

$$\left\{- \underset{Sets}{\longrightarrow} M\right\} \circ [+1] = (M \times M, M \times M \times M, ...)$$

equipped with a natural transformation ""forgetting the first coordinate": $M \times M \longrightarrow M$, $(x_1, x_2) \longmapsto x_1$, and $M \times M \times M \longrightarrow M \times M$, $(x_1, x_2, x_3) \longmapsto (x_1, x_2)$,...

5.2.2. Completeness of a metric space in terms of the shift endofunctor. — Recall that, for a metric space M, its simplicial filter $M_\bullet$ is $|M|_\bullet$ where we equip $|M|^n$ with *the filter of ε-neighbourhoods of the main diagonal*: a subset is big (a neighbourhood) iff for some $\varepsilon > 0$ it contains all tuples of diameter $< \varepsilon$.

Proposition 5.2.2.1. — *A metric space is complete iff either of the following equivalent conditions holds:*

(iv') *the following lifting property holds:*

$$\bot \longrightarrow I_\bullet^{tails} \;\rightthreetimes\; M_\bullet \circ [+\infty] \longrightarrow M_\bullet$$

(v'') *the following lifting property holds:*

$$I_\bullet^{\leq tails} \longrightarrow (I \sqcup \{+\infty\})_\bullet^{\leq tails \sqcup \{+\infty\}} \;\rightthreetimes\; M_\bullet \longrightarrow \top$$

Proof. — (iv'): First consider the level of underlying simplicial sets. To give a map $|I_\bullet^{\mathrm{tails}}| \longrightarrow |M_\bullet|$ is to give a map $a : |I| \longrightarrow |M|$ of sets. As a simplicial set,

$$|M|_\bullet \circ [+\infty] = \bigsqcup_{a_\infty \in |M|} |M|_\bullet \times \{a_\infty\}$$

is the disjoint union of connected components $|M|_\bullet \times \{a_\infty\}$, $a_\infty \in |M|$. The sset $|I_\bullet^{\mathrm{tails}}|$ is connected, and thus any map $f_\bullet : |I_\bullet^{\mathrm{tails}}| \longrightarrow |M|_\bullet \circ [+\infty]$ maps it into the single connected component, as the following argument shows. For any $i \leq j \in |I|_\bullet(1_\leq)$ there is a simplex $(i < j) \in |I|_\bullet(2_\leq)$ such that $i = (i \leq j)[1]$ and $j = (i \leq j)[2]$. Consider the image $f_{2_\leq}(i \leq j) = (x, y, z) \in |M \times M \times M|$. By functoriality $f_{1_\leq}(i) =$

[17] FIXME: FIXME:, I find notation below with $+\infty$ somewhat more telling, but i suppose it is also confusing...

$$[+\infty] : (1 < ... < n) \longmapsto (1 < ... < n < +\infty)$$

and a morphism $f : m_\leq \longrightarrow_\leq n_\leq$ into the morphism $f[+\infty] : (m+1)_\leq \longrightarrow (n+1)_\leq$ defined by

$$;\quad ([+1]f)(+\infty) := +\infty;\quad ([+1]f)(i) := i$$

$f_{2_\le}(i \le j)[1] = (x, z)$ and $f_{1_\le}(j) = f_{2_\le}(i \le j)[2] = (y, z)$ lie in the same component of the disjoint union. Restricted to any component of the disjoint union, the map $|M|_\bullet \times \{a_\infty\} \longrightarrow |M|_\bullet$ is an isomorphism. Hence, the diagonal arrow

$$|I^{\text{tails}}_\bullet| \longrightarrow |M|_\bullet \circ [+\infty]$$

corresponds to a map $|I| \longrightarrow |M \times M|$, $i \longmapsto (a_i, a_\infty)$ for some $a_\infty \in |M|$.

Continuity of this map means that for each neighbourhood $\{(x, y) : \text{dist}(x, y) < \epsilon\}$, its preimage $\{i : \text{dist}(a_i, a_\infty)\}$ contains a final segment of $I^\le$. That is, a_∞ is the limit of a_i's.

Continuity of the bottom map $I^{\text{tails}}_\bullet \longrightarrow M_\bullet$ means the sequence is Cauchy. Indeed, it means that for each neighbourhood $\{(x, y) : \text{dist}(x, y) < \epsilon\}$, its preimage $\{(i \le j) : \text{dist}(a_i, a_j)\}$ contains all pairs with big enough elements.

(iv'): by a similar diagram chasing argument. □

Thus we saw that $[+\infty] : \wr\boldsymbol{\varphi} \longrightarrow \wr\boldsymbol{\varphi}$ endofunctors allows us to talk about limits in metric spaces. Unfortunately, adapting this reformulation to talk about average types and NIP requires some tweaks. We proceed to do so now.

5.2.3. "Shifted" structures. — With NIP working locally becomes a bit cumbersome, and so we state the definition globally for all the formulas together.

First we need some preliminary notation.(18) For a formula $\phi(x_1, .., x_r)$ of arity r and a natural number $n > 0$, let $\phi^{n\text{-EM'}}$ be the n-ary formula

$$\bigwedge_{1 \le i_1 < ... < i_r \le n, 1 \le j_1 < ... < j_r \le n} \left(\bigwedge_{1 \le s < r} x_{i_s} \ne x_{i_{s+1}} \& x_{j_s} \ne x_{j_{s+1}} \implies (\phi(x_{i_1}, .., x_{i_r}) \leftrightarrow \phi(x_{j_1}, .., x_{j_r})) \right)$$

The formula $\phi^{n\text{-EM'}}(a_1, ..., a_n)$ says that each subsequence of $a_1, .., a_n$ with consequitive elements being *distinct* is necessarily ϕ-indiscernible, and in particular, all elements of the subsequence are distinct. (This is the difference with $\phi^{n\text{-EM}}$: for distinct a, b, c the formula $\phi^{n\text{-EM'}}(a, b, a, c)$ necessarily fails for $\phi(x, y)$ being $x = y$, whereas $\phi^{n\text{-EM}}(a, b, a, c)$ is true.) In particular, $\phi^{n\text{-EM'}}()$ belongs to the EM-type of each ϕ-indiscernible sequence.

For a formula $\phi(x_1, .., x_{r+1})$ of arity $r + 1$ and natural numbers $n > 0$, let $\phi^{n,1\text{-EM'}}$ be the $(n + 1)$-ary formula

$$\bigwedge_{\substack{1 \le i_1 < ... < i_r \le n \\ 1 \le j_1 < ... < j_r \le n}} \left(\bigwedge_{1 \le s < r} x_{i_s} \ne x_{i_{s+1}} \& x_{j_s} \ne x_{j_{s+1}} \implies (\phi(x_{i_1}, .., x_{i_r}, x_{n+1}) \leftrightarrow \phi(x_{j_1}, .., x_{j_r}, x_{n+1})) \right)$$

Definition 5.2.3.1 ($M[+\infty]_\bullet : \wr\boldsymbol{\varphi}$). — Let M be a structure.

Let $M[+\infty]_\bullet$ denote the simplicial filter whose underlying simplicial set is $|M|_\bullet \circ [+\infty]$. The set

$$(|M|_\bullet \circ [+\infty])(n_\le) = |M|_\bullet((n + 1)_\le) = |M|^{n+1}$$

is equipped with the filter generated by sets $\phi^{n,1\text{-EM'}}(M^{n+1})$, for arbitrary formula ϕ in the language of M.

The verification that this functor $M[+\infty]_\bullet : \Delta^{\text{op}} \longrightarrow \boldsymbol{\varphi}$ is indeed well-defined is the same as in Definition 5.1.2.1.

(18)FIXME: this notation is rather poor and misleading. We do need two notions, though, for the characterisation of stability fails with these "unsymmetrical" filters... they talk about indiscernible sequences without consequitive repetitions.

5.2.4. NIP as Quillen negation. — Now we come to our main observation about NIP. Note that the filters on $M_\bullet$ we consider here are different from those used to characterise stability and lack their symmetry.

Let $I_\bullet^{\leq \text{tails}} := I_\bullet^{\leq,\mathfrak{F}}$, $|I|_\bullet^{\text{tails}} := |I|_\bullet^{\mathfrak{F}}$ for $\mathfrak{F} := \{\{x : x \geq i\} : i \in I\}$ the filter generated by non-empty final segments of I.

Let $I \sqcup \{+\infty\}$ denote the linear order I with a new maximal element $+\infty$ added;

Let $(I \sqcup \{+\infty\})_\bullet^{\leq I\text{-tails}} = I_\bullet^{\leq,\mathfrak{F}}$ with $\mathfrak{F} := \{\{x : x \geq i\} : i \in I\}$ the filter generated by non-empty final segments of I. Let $(I \sqcup \{+\infty\})_\bullet^{\leq\text{tails}} = I_\bullet^{\leq,\mathfrak{F}}$ with $\mathfrak{F} := \{\{x : x \geq i\} \sqcup \{+\infty\} : i \in I\}$ the filter generated by final segments of $I \sqcup \{+\infty\}$.

Lemma 5.2.4.1 **(NIP)**. — *Let M be a structure. Let I be a linear order. The following are equivalent:*

(i) *in the model M, for each $b \in M$, each eventually indiscernible sequence (over $\varnothing$) is eventually $\phi(-,b)$-indiscernible*

(ii) *in the model M, the filter of final segments of any indiscernible sequence has a complete average type (fixme: say correctly)*

(iii) *in $s\mathcal{P}$ the following lifting property holds:*

$$\bot \longrightarrow I_\bullet^{\leq tails} \rightthreetimes M_\bullet^{L(M)} \longrightarrow M_\bullet$$

i.e. in $s\mathcal{P}$ the following diagram holds:

$$\begin{array}{ccc} \bot & \longrightarrow & M_\bullet^{L(M)} \\ \downarrow & \nearrow & \downarrow \\ I_\bullet^{\leq tails} & \longrightarrow & M_\bullet \end{array}$$

(iv) *in $s\mathcal{P}$ the following lifting property holds:*

$$|\{1\}|_\bullet \longrightarrow I_\bullet^{\leq tails} \rightthreetimes M[+\infty]_\bullet \longrightarrow M_\bullet$$

$$\begin{array}{ccc} |\{1\}|_\bullet & \longrightarrow & M[+\infty]_\bullet \\ \downarrow & \nearrow & \downarrow \\ I_\bullet^{\leq tails} & \longrightarrow & M_\bullet \end{array}$$

(v) *in $s\mathcal{P}$ the following lifting property holds:* [19]

$$I^{\leq tails} \sqcup \{+\infty\}_\bullet \longrightarrow (I \sqcup \{+\infty\})_\bullet^{\leq tails \sqcup \{+\infty\}} \rightthreetimes M_\bullet \longrightarrow \top$$

$$\begin{array}{ccc} I \sqcup \{+\infty\})_\bullet^{\leq I-tails} & \longrightarrow & M_\bullet \\ \downarrow & \nearrow & \downarrow \\ (I_\bullet \sqcup \{+\infty\})^{\leq tails} & \longrightarrow & \top \end{array}$$

FIXME: FIXME:, I'm using slightly different notation in the diagram and the lifting property. which is better ?

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(19) FIXME:: the point of this older notation below was to show explicitly the underlying simplicial set.... I guess we should abandon it, it's not really helpful. equivalently in older notation,

$$\left\{-_\leq \xrightarrow[preorders]{} I \sqcup \{+\infty\}_\leq\right\}_{\text{cofinal in } I_\leq} \longrightarrow \left\{-_\leq \xrightarrow[preorders]{} I \sqcup \{+\infty\}_\leq\right\}_{\text{cofinal}} \rightthreetimes M_\bullet \longrightarrow \top$$

Proof. — (i)⇔(ii) is by definition of the average type.

(ii)⇔(iii): As $|M_\bullet^{L(M)}| = |M_\bullet|$, the diagram trivially holds at the level of the underlying simplicial sets. Recall by Fixme 2.2.2 that a morphism $I_\bullet^{\text{tails}}| \longrightarrow |M_\bullet|$ of sSets is the same as a map $|I| \longrightarrow |M|$ of sets. Thus the bottom horizontal arrow $I_\bullet^{\text{tails}} \longrightarrow M_\bullet$ is a sequence $(a_i)_{i\in I}$, and its continuity means it is an eventually indiscernible sequence over $\varnothing$. Similarly, the diagonal arrow $I_\bullet^{\text{tails}} \longrightarrow M_\bullet^{L(M)}$ is an eventually indiscernible sequence over M.

(ii)⇔(v): In (v), notice we may ignore arrows to $\top$. Now consider first the upper horizontal arrow $I_\bullet^{\leq\text{tails}} \sqcup \{+\infty\}_\bullet \longrightarrow M_\bullet$. It corresponds to a map $|I| \sqcup \{+\infty\} \longrightarrow |M|$ of sets, i.e. a sequence $(a_i)_{i\in i}, a_{+\infty}$. For clarity of exposition assume that all elements of the sequence are distinct. A neighbourhood in $I_\bullet^{\leq\text{tails}}$ is defined as a subset containing all increasing tuples of large enough elements of I; thus continuity of this map places no restrictions on $a_{+\infty}$ and just means that for every EM-formula is satisfied by such a subset, i.e. for each ϕ the sequence $(a_i)_{i\in I}$ is eventually ϕ-indiscernible. Now consider the diagonal arrow $(I \sqcup \{+\infty\})_\bullet^{\leq\text{tails}\sqcup\{+\infty\}} \longrightarrow M_\bullet$. By commutativity it corresponds to the same sequence $(a_i)_{i\in i}, a_{+\infty}$. However, now the neighbourhoods are defined differently: they are larger and have to include $+\infty$: namely, a neighbourhood in $(I \sqcup \{+\infty\})_\bullet^{\leq\text{tails}\sqcup\{+\infty\}}$ consists of all increasing tuples of large enough elements of $I \sqcup \{+\infty\}$. In particular, for each $(r+1)$-ary formula ϕ, for all elements $i_1 < ... < i_r$ of I large enough, $M \vDash \phi(a_{i_1}, ..., a_{i_r}, a_{+\infty}) \leftrightarrow \phi(a_{i_1}, ..., a_r, a_{+\infty})$. This means exactly that either ϕ or $\neg\phi$ belong to the limit type of the sequence. This finishes the proof that (v) $\Longrightarrow$ (ii). To check the converse, we still need to check that the lifting property holds if we drop the assumption that all elements of the sequence are distinct. However, by continuity we know that increasing tuples of large enough elements satisfy $\phi^{n\text{-EM'}}$ for phi being $x_1 = x_2 \,\&\, x_2 = x_3$, and this implies the sequence cannot have subsequences of the form a, b, a with $b \neq a$ by definition of $\phi^{n\text{-EM'}}$. An elementary combinatorial argument now finishes the proof, by showing the sequence is either eventually constant, or it reduces to the previous case of distinct elements. (fixme: say nicely..)

Let us analyse (iv).

The bottom horizontal arrow picks a sequence $(a_i)_{i\in I}$; by the argument above, we only need to consider the case when all elements are distinct. In this case it is eventually indiscernible over $\varnothing$, as we already know. (fixme: say nicely).

As a simplicial set, $M[+\infty]_\bullet$ is the disjoint union of copies of $M_\bullet$ indexed by M, i.e.

$$|M[+\infty]_\bullet| = \sqcup_{a_\infty \in M} |M_\bullet| \times \{a_\infty\}$$

The sset $I_\bullet$ is connected (as a simplicial set), thus it has to map into the same connected component, and thereby the diagonal arrow pick an "end" element $a_\infty \in M$ and a sequence $(a'_i)_{i\in I}$.

Commutativity of the lower triangle means it is the same sequence $(a_i)_{i\in I}$ as picked by the bottom horizontal arrow, i.e. $a'_i = a_i, i \in I$. Thus, on the level of simplicial sets, the diagonal arrow always exists, and we only need to check continuity.

Finally, notice that $\{a_\infty\}$ has the limit type of $(a_i)_{i\in I}$: for any formula $(r+1)$-ary $\phi()$ there is $i_0 \in I$ such that for any distinct elements $a_{i_1}, ..., a_{i_r}$, $i_0 < i_1 < ... < i_r$ and $a_{j_1}, ..., a_{j_r}$ $i_0 < j_1 < ... < j_r$, $\phi(a_{i_1}, ..., a_{i_r}, a_\infty) \leftrightarrow \phi(a_{j_1}, ..., a_{j_r}, a_\infty)$. i.e. for any distinct elements $a_{i_1}, ..., a_{i_n}$, $i_0 < i_1 < ... < i_n$, the sequence $a_{i_1}, ..., a_{i_n}$ is $\phi(-, a_{+\infty})$-indiscernible. □

5.2.5. NIP and completeness have somewhat similar definitions. — A lifting property reminiscent of Lemma 5.2.4.1(iv)-(v) defines completeness for metric spaces (and, more generally, uniform structures).

Recall that, for a metric space M, we equip M^n with *the filter of ε-neighbourhoods of the main diagonal*: a subset is big (a neighbourhood) iff for some $\varepsilon > 0$ it contains all tuples of diameter $< \varepsilon$.

Proposition 5.2.5.1. — *A metric space M is complete iff either of the following equivalent conditions holds:*

(iv') *the following lifting property holds:*

$$\bot \longrightarrow I_\bullet^{tails} \rightthreetimes M_\bullet[+\infty] \longrightarrow M_\bullet$$

(v'') *the following lifting property holds:*

$$I_\bullet^{\leq tails} \longrightarrow (I \sqcup \{+\infty\})_\bullet^{\leq tails \sqcup \{+\infty\}} \rightthreetimes M_\bullet \longrightarrow \top$$

Proof. — Indeed, both in (iv') and (v'), a map to $M_\bullet$ is a Cauchy sequence, and the diagonal map picks its limit. In (v'), the image of $+\infty$ is the limit. □

5.3. Non-dividing. — Simplicial Stone spaces allow to rewrite as a lifting square the reformulation of non-dividing via indiscernible sequences. This demonstrates an important difference between the simplicial Stone spaces and the objects $M_\bullet$ associated with models we have been using so far: the underlying simplicial set of a simplicial Stone space is non-trivial (as a simplicial set).

5.3.1. Simplicial Stone spaces. — The definitions of $M_\bullet$ and $M[+\infty]_\bullet$ we have to use now are slightly different from the ones used above, and follow Definition 3.1.1.1.

Recall that in Definition 3.1.1.1 $M_\bullet$ is defined as a simplicial set $|M|_\bullet$ where $| = M_\bullet(n_\leq)| = |M|^n$ is equipped with the filter generated by the sets of Σ-indiscernible n-sequences with repetitions which can be extended to an indiscernible sequence with repetitions with at least distinct N elements, where Σ varies through finite sets of formulas in the language of M and N varies through arbitary natural numbers.

Let $M[+\infty]_\bullet$ be defined as the simplicial set $|M|_\bullet\circ[+\infty]$ where $| = M_\bullet\circ[+\infty](n_\leq)| = |M|^(n+1)$ is equipped with the filter generated by the sets of $(n+1)$-sequences $(a_1,..,a_n,a_{+\infty})$ with repetitions such that there is a Σ-indiscernible sequence $(a_1,..,a_n,a_{n+1},..,a_{N+}$ with at least N distinct elements and Σ-indiscernible over $a_{+\infty}$ with repetitions.

Recall that by $M_\bullet/A$ we denote the simplicial Stone space of types over A, i.e. the simplicial filter $M_\bullet$ quotiented by the relation of having the same type over A. The meaning of $M[+\infty]_\bullet/A$ is similar.

First observe that

– to give a type $p = \mathrm{tp}(a/A)$ is to give a $s\mathcal{F}$-morphism $\{pt\}_\bullet \longrightarrow M_\bullet/A$.

– to give a type $p = \mathrm{tp}(a/Ab)$ is to give a $s\mathcal{F}$-morphism $\{pt\}_\bullet \longrightarrow M[+\infty]_\bullet/A$.

Indeed, a morphism $\{pt\} \longrightarrow M_\bullet/A$ is the same as an element of $M_\bullet/A(1_\leq)$ which is the Stone space of 1-types over A; a morphism $\{pt\} \longrightarrow M[+\infty]_\bullet/A$ is the same as an element of $M[+\infty]_\bullet/A(1_\leq)$ which is the Stone space of 2-types over A.

5.3.2. Non-dividing. — We now rewrite as a lifting square the reformulation of non-dividing via indiscernible sequences.

Proposition 5.3.2.1. — *The following are equivalent:*

1. *$tp(a/Ab)$ does not divide over A.*

2. *For any infinite sequence of A-indiscernible I starting with b, there exists some a' with $tp(a'/Ab) = tp(a/Ab)$ and such that I is indiscernible over Aa'.*
3. *For any infinite sequence of A-indiscernible I starting with b, there exists I' with $tp(I'/Ab) = tp(I/Ab)$ and such that I is indiscernible over Aa.*
4. *the following diagram holds:*

$$\begin{array}{ccc} \{1\}_\bullet & \xrightarrow{tp(a/Ab)} & M[+\infty]_\bullet/A \\ \downarrow & \overset{\exists I'}{\nearrow} & \downarrow{\scriptstyle [+\infty]} \\ I^{\leq}_\bullet & \xrightarrow{\quad I \quad} & M_\bullet/A \end{array}$$

Proof. — 1⇔2⇔3 is [Tent-Ziegler, Corollary 7.1.5].

Deciphering (4) gives (3), as follows. (4) $\Longrightarrow$ (3): Let $I = (b_i)_i$ be a sequence as in (3). It induces The bottom horizontal arrow $I^{\leq}_\bullet \longrightarrow M_\bullet/A$. The top horizontal arrow $\{(b,a)\}_\bullet \xrightarrow{\mathrm{tp}(a/Ab)} M[+\infty]_\bullet/A$ represents the type $\mathrm{tp}(a/Ab)$ and thus the only point $1 \in \{1\}(1_\leq)$ goes into the type of a pair $(b',a') \in M \times M = M[+\infty]_\bullet(1_\leq)$ of type $\mathrm{tp}(ba/A)$. By assumption I starts with b, hence the square commutes. The diagonal arrow $I^{\leq}_\bullet \longrightarrow M[\infty]_\bullet/A$ represents a sequence of pairs $(b''_i, a'')_i$ for some a''. Commutativity of the lower triangle means that $\mathrm{tp}(b''_i/A) = \mathrm{tp}(b_i/A)$ for all i, i.e. $\mathrm{tp}(I'/A) = \mathrm{tp}(I/A)$. Commutativity of the upper triangle means that b'', a'' and b, a have the same type over A, i.e. $\mathrm{tp}(b''/Aa'') = \mathrm{tp}(b/Aa)$. Continuity of $I^{\leq}_\bullet \longrightarrow M[\infty]_\bullet/A$ means the sequence $(b'')_i$ is indiscernible over Aa'', i.e. is required in (3).

(3) $\Longrightarrow$ (4): Indeed, the top horizontal arrow $\{(b,a)\}_\bullet \xrightarrow{\mathrm{tp}(a/Ab)} M[+\infty]_\bullet/A$ represents the type $\mathrm{tp}(a/Ab)$. The bottom horizontal arrow $I^{\leq}_\bullet \longrightarrow M_\bullet/A$ represents a sequence which extends to an infinite indiscernible sequence I over A. Commutativity of the square means I starts with an element of $\mathrm{tp}(b/A)$. Let $I' = (b'_i)_i$ and a' be as provided by (3). It means that the sequence $(b'_i, a')_i$ induce a diagonal arrow $I^{\leq}_\bullet \longrightarrow M[\infty]_\bullet/A$. Commutativity of the upper triangle means that (b'_0, a') and (b,a) have the same type over A, which holds by (3). Commutativity of the lower triangle means that $\mathrm{tp}((b')_i) = \mathrm{tp}((b_i)_i)$, again provided by (3). □

5.4. Order properties: NOP and NSOP. — We show how a particularly simple-minded transcribing of a definition of OP leads to a reformulation of NOP as a lifting property. We then discuss a simplification of this lifting property and its relationship to NSOP and how it reminds of compactness.

5.4.1. A simple-minded transcription of OP. — Now we use the Order Property to give an example of how to "transcribe" a definition into ɀ𝛗-language. Our "transcription" here is particularly simple minded and mechanistic, and it is rather a miracle that it works.

Let M be a model. Let us now show how to rewrite the order property if the language of ɀ𝛗. Recall a formula $\phi(-,-)$ has the *order property*[20] in a model M iff there exist an infinite sequence $a_i \in M, i \in \omega$ such that

$(\mathrm{OP}^1(\phi))$ $\phi(a_i, a_j)$ iff $i < j$

The definition mentions M and ϕ, thus we assume that an ɀ𝛗-reformulation should mention $M^{\{\phi\}}_\bullet$.

[20] FIXME: this is not a standard definition, should be a_i, b_j...

Now note that this definition considers a linear order $(\omega,<)$, and thus an $\wr\!\mathcal{P}$-reformulation should probably mention some $\wr\!\mathcal{P}$-object associated with a linear order: we already know three $\omega_\bullet^<, \omega_\bullet^{\leq\text{tails}}$ or $\omega_\bullet^{\{\leq\}}$. We should probably pick the latter as it is model-theoretic and is the object associated with the linear order $(\omega,<)$ as a structure.

Also note that $\mathrm{OP}^1(\phi)$ ignores elements of M not occurring in the sequence. This suggests us we should adjoin to ω a new element $\star$ and consider filters which "ignore" $\star$. This leads to the following modification of $\omega_\bullet^{\{\leq\}}$:

- the underlying simplicial set is $|\{\star\}\sqcup\omega|_\bullet$
- the filter on $(\{\star\}\sqcup\omega)^n = |\{\star\}\sqcup\omega|_\bullet(n_\leq)$ is generated the set of all sequences such that the elements of ω (i.e. not $\star$) occur in monotone order

$$\{(n_1,...,n_k)\in\{\{\star\}\sqcup\omega\} : n_i\neq\star\,\&\,n_j\neq\star \implies n_i\leq n_j \text{ for all } 1\leq i\leq j\leq n\}$$

Let us denote this simplicial filter as $(\{\star\}\sqcup\omega)_\bullet^{\{\leq\}}$.

Hence, we have two $\wr\!\mathcal{P}$-objects, and want to formulate (the full) OP. We consider the map $M\longrightarrow\{\star\}\sqcup\omega$ sending a_i to i for $i\in\omega$, and everything else into $\star$. This map is surjective, and $\mathrm{OP}^1(\phi)$ for any binary $\phi(-,-)$ implies this map is continuous. Therefore it occurs to us to reformulate (the full) OP as

- there is a surjective $\wr\!\mathcal{P}$-morphism $M_\bullet\longrightarrow(\{\star\}\sqcup\omega)_\bullet^{\{\leq\}}$.

As we will see in Proposition 5.4.2.1, this is indeed equivalent to OP, at least when say $M=M^{eq}$ has elimination of imaginaries. .

Remark 5.4.1.1. — We would like to make a meta-mathematical remark. We find it must amusing that such a simple-minded transcription produces the right result, and we think it calls for an explanation. At the very least one should collect and systematize examples where such a simple-minded, mechanical transcription produces correct results.

5.4.2. A less simple-minded transcription of OP. — Now let us to a less simple-minded transcription of $\mathrm{OP}^1(\phi)$ which would give us a proof of our simple-minded conjecture above.

We may assume that the sequence a_i is indiscernible (provided M is saturated enough), and thus rewrite $\mathrm{OP}^1(\phi)$ in terms of finite indiscernible sequences as:

$(\mathrm{OP}^1(\phi)_{EM})$ for each $i,j,k\in\omega$, the following implication holds:

- (i) if the 3-sequence (a_i,a_j,a_k) is ϕ-indiscernible then (ii) the 3-sequence (i,j,k) is $<$-indiscernible

This trivially holds if ϕ has the order property. In the other direction, by indiscernability of (a_i) assume that $\phi(a_i,a_j)$ whenever $i<j$. Pick $i<j<k$ and consider the tuple (a_j,a_i,a_k) with two first elements a_i,a_j permuted. By OP_{EM} it is not ϕ-indiscernible because (j,i,k) is not $<$-indiscernible. However, by assumption $\phi(a_i,a_k)$ and $\phi(a_j,a_k)$, and thus indiscernability may fail only if $\neg\phi(a_j,a_i)$, as required by the order property with respect to ϕ.

It can easily be checked that $(\mathrm{OP}^1(\phi)_{EM})$ merely states continuity of the map $M^3\longrightarrow(\{\star\}\sqcup\omega)^3$ sending a_i to i for $i\in\omega$, and everything else into $\star$, with respect to the appropriate filters, namely the ϕ-EM-filter on M^3 and the filter on $(\{\star\}\sqcup\omega)^3$ considered above.

Thus we obtain

Proposition 5.4.2.1. — *For a sufficiently saturated model M, say with elimination of imaginaries, the following are equivalent:*

– *M has NOP*
– *in* $\wr\mathcal{P}$ *there is no surjection* $M_\bullet \longrightarrow (\{\star\} \sqcup \omega)_\bullet^{\{\leq\}}$
– *the following lifting property holds:*

$$\bot \longrightarrow M_\bullet \;\rightthreetimes\; \bigsqcup_{n\in\omega}(\{\star\}\sqcup n)_\bullet^{\{\leq\}} \longrightarrow (\{\star\}\sqcup\omega)_\bullet^{\{\leq\}}$$

Proof. — Recall that an $\wr\mathcal{P}$-morphism $M_\bullet \longrightarrow \{\{\star\}\sqcup\omega\}_\bullet^{\{\leq\}}$ is necessarily induced by a map $|M| \longrightarrow \{\star\}\sqcup\omega$. The lifting property says that the image of any continuous such map is bounded in ω; this is equivalent to saying there is no such surjective continuous map. Arguments preceding the observation establish the equivalence of the latter to NOP. □

5.4.3. NOP and NSOP. — Let M be a model. Assume that M fails NSOP, i.e. there is a parameter-free formula $\phi(x,y)$ defining a linear order $\leq_\phi$ on M. Let $M_\bullet^{\{\leq_\phi\}}$ the simplicial set $|M|_\bullet$ equipped with the filter of the ϕ-indiscernible neighbourhood; this is very similar to the object associated with the model M considered in the language consisting only of the formula $\phi(-,-)$. Explicitly in terms of the linear order $\leq_\phi$ this filter is described as follows: a neighbourhood is a set contacting all the $\leq_\phi$-monotone sequences, both increasing and decreasing.

Evidently, definability of $\leq_\phi$ implies there is a surjective $\wr\mathcal{P}$-morphism

$$M_\bullet \longrightarrow M_\bullet^{\{\leq_\phi\}}$$

In the same simple-minded way as with OP this leads us to ask whether NSOP is equivalent existence of such a continuous surjective morphism for some linear order $\leq$.

However, we are only able to show that this condition implies OP and, as we just saw, is implied by NSOP.

Let us show it implies OP. Indeed, by continuity of

$$M_\bullet(3_\leq) \longrightarrow M_\bullet^{\{\leq_\phi\}}(3_\leq)$$

there is a finite set of 1- and 2-ary formulas Σ such that any Σ-indiscernible sequence of length 3 is monotone wrt $\leq$.

Let $(a_1,a_2,a_3)\in M$ be an indiscernible sequence in M. The sequence (a_2,a_1,a_3) is not $\leq$-indiscernible, hence by continuity it is not ϕ-indiscernible in M for some formula ϕ, necessarily of arity 2. That means that $\phi(a_1,a_2)\,\&\,\neg\phi(a_2,a_1)$. Now pick (a_1,a_2,a_3) to be a start of an infinite indiscernible sequence. This shows that $\phi(-,-)$ has the order property.

This leads to the following observation:

Proposition 5.4.3.1. — *Let M be an infinite model. The following implications hold:*

NOP $\Longrightarrow$ *(ii)*$\Leftrightarrow$*(ii)′* $\Leftrightarrow$ *(ii)″* $\Leftrightarrow$ *(ii)‴* $\Leftrightarrow$ *(iii)* $\Longrightarrow$ *NSOP*

FIXME: FIXME:: probably it is easy to prove all are equivalent to NSOP.

(i) *model M has NSOP.*
(ii) *there is no linear preorder* $\leq$ *on* M *with infinite chains such that the identity map* $id\colon M_\bullet \longrightarrow M_\bullet^{\{\leq\}}$ *is continuous*
(ii)′ *in* $\wr\mathcal{P}$ *there is no surjection* $id : M_\bullet \longrightarrow I_\bullet^{\{\leq\}}$ *to an infinite structure* $(I,\leq)$ *where* $\leq$ *is a linear preorder with infinite chains*
(ii)″ *in* $\wr\mathcal{P}$ *there is no surjection* $id\colon M_\bullet \longrightarrow \alpha_\bullet^{\{\leq\}}$ *for an infinite ordinal* α

(ii)‴ *in* $\wr\!\varphi$ *there is no infinite linear order* $(I,\le)$ *and a surjection* $id : M_\bullet \longrightarrow I_\bullet^{\gtrless}$ *where* $I_\bullet^{\gtrless}$ *is the simplicial set* $|I|_\bullet$ *equipped with the filter on* $|I|^n = I_\bullet(n_\le)$ *is generated by the subset of monotone sequences (fixme: skip this item, this explicit description is more confusing than helpful?)*

(iii) *the following lifting property holds for each limit cardinal* α*:*

$$\bot \longrightarrow M_\bullet \;\rightthreetimes\; \bigsqcup_{\beta<\alpha} \beta_\bullet^{\{\le\}} \longrightarrow \alpha_\bullet^{\{\le\}}$$

Proof. — The only non-trivial implication is NOP $\Longrightarrow$ (ii)′.

NOP $\Longrightarrow$ (ii)′: By Ramsey theory there is an infinite indiscernible sequence such that its image in I is infinite. Let $(a_1,a_2,a_3) \in M$ be a subsequence of such an indiscernible sequence in M. The sequence (a_2,a_1,a_3) is not $\le$-indiscernible, hence by continuity it is not ϕ-indiscernible in M for some formula ϕ, necessarily of arity 2. That means that $\phi(a_1,a_2)\,\&\,\neg\phi(a_2,a_1)$. This gives us the order property. This finishes the proof of this case. FIXME: TODO: ¿¿get strict order property if possible??

$\neg(i) \Longrightarrow \neg(ii)$: take $\le$ to be the definable linear order on M

$\neg(ii) \Longrightarrow \neg(ii)'$: take $(I,\le) = (M,\le)$

$\neg(ii)' \Longrightarrow \neg(ii)''$: take a surjection $I^\le \longrightarrow \alpha^\le$; it induces an $\wr\!\varphi$-surjection $I_\bullet^{\{\le\}} \longrightarrow \alpha_\bullet^{\{\le\}}$

$\neg(ii)'' \Leftrightarrow \neg(ii)'''$: this is just an explicit reformulation

$(ii)'' \Leftrightarrow (iii)$ the lifting property says that the image of any map $M_\bullet \longrightarrow \alpha_\bullet^{\{\le\}}$ is bounded by some $\beta < \alpha$. Any surjection as in $\neg(ii)''$ would fails this; in the other direction, such a map is surjective on its image, which is some infinite cardinal.

□

5.4.4. NSOP$_n$: questions. — Let $(I,\le)$ be a preorder and $n > 0$. Let $I_\bullet^{\{\le_n\}}$ be the simplicial set $|I|_\bullet$ equipped with the following filter:

- a neighbourhood is a subset containing all sequences which can be split into at most n monotone subsequences

Note that for $n = 1$ it holds $I_\bullet^{\{\le_n\}} = I_\bullet^{\{\le\}}$.

TODO 5.4.4.1. — What is the model theoretic meaning of the following analogue of (iii) above ? Is it related to NSOP$_n$? Find a similar lifting property related to NSOP$_n$.

(iii)$_n$ the following lifting property holds for each cardinal α:

$$\bot \longrightarrow M_\bullet \;\rightthreetimes\; \bigsqcup_{\beta<\alpha} \beta_\bullet^{\{\le_n\}} \longrightarrow \alpha_\bullet^{\{\le_n\}}$$

5.4.5. NSOP: analogy to compactness of topological spaces (finite subcover property). — Let $\alpha^>$ denote the ordinal α considered as a topological space with the initial interval topology (i.e. the open subsets are initial intervals $\{\gamma : \gamma < \beta\}$, $\beta < \alpha$). The object $\alpha_\bullet^{\ge} : \wr\!\varphi$ is described as follows: the underlying simplicial set $|\alpha|_\bullet$ is represented by the set $\{\beta : \beta < \alpha\} = \alpha$, and the filter on $|\alpha|_\bullet(n_\le) = \alpha^n$ is generated by the set of all weakly decreasing sequences

$$\{(\beta_1,...,\beta_n) \in \alpha^n : \beta_1 \ge .. \ge \beta_n\}$$

Note the filter on $\alpha_\bullet^{\{\le\}}$ is different, and generated by the set of all weakly decreasing or weakly increasing sequences

$$\{(\beta_1,...,\beta_n) \in \alpha^n : \beta_1 \ge .. \ge \beta_n \text{ or } \beta_1 \le .. \le \beta_n\}$$

Proposition 5.4.5.1. — *The following are equivalent for a connected topological space X:*

- *X is compact, i.e. each open cover of X has a finite subcover*
- *X cannot be represented as an increasing union of open subsets, i.e. there is no limit ordinal α, open subsets $U_\beta \subsetneq X$ such that $X = \bigcup_{\beta<\alpha} U_\beta$ and $U_\gamma \subsetneq U_\beta$ for $\gamma < \beta < \alpha$*
- *there is no surjection $X \longrightarrow \alpha^>$ for a limit ordinal α*
- *in the category of topological spaces, each map $X \longrightarrow \alpha^>$ factors via some $\beta^> \longrightarrow \alpha^>$, $\beta < \alpha$, i.e. the following lifting property holds:*

$$\bot \longrightarrow X \rightthreetimes \bigsqcup_{\beta<\alpha} \beta^> \longrightarrow \alpha^>$$

- *in the category ʅɸ, each map $X \longrightarrow \alpha^>_\bullet$ factors via some $\beta^>_\bullet \longrightarrow \alpha^>_\bullet$, $\beta < \alpha$, i.e. the following lifting property holds:*

$$\bot \longrightarrow X_\bullet \rightthreetimes \bigsqcup_{\beta<\alpha} \beta^>_\bullet \longrightarrow \alpha^>_\bullet$$

Proof. — Easy. To see (2)⇔(3), take the U_β's to be the preimages of the initial intervals in α. Note that the lifting property reformulation in Top but not in ʅɸ essentially uses that X is connected: in ʅɸ it is enough that the underlying simplicial set of X is connected (as a simplicial set). □

6. Appendix. Ramsey theory and indiscernible in category theoretic language

This exposition is intended for a category-theory minded reader. We formulate in terms of simplicial sets some of Ramsey theory and the definition of the ʅɸ-objects we associate with models.

6.1. Ramsey theory. — Ramsey theory admits a description in terms of ʅɸ.

6.1.1. c-homogeneous simplicies. — Let $X_\bullet$: sSet be a simplicial set, and let a "colouring" $c : X_\bullet(n_\leq) \longrightarrow C$ be an arbitrary map.

Say a simplex $x \in X_\bullet(m_\leq)$ is *c-homogeneous* iff all its *hereditary non-degenerate* faces have the same c-colour, where we say that a simplex $x \in X_\bullet(m_\leq)$ is *hereditarily non-degenerate* iff a face $x[i_1 < ... < i_n]$ is non-degenerate whenever $1 \leq i_1 < ... < i_n \leq m$.

The c-homogeneous simplicies form a subobject of $X_\bullet$. Call a subset of $X_\bullet(m_\leq)$ a *c-neighbourhood of the main diagonal in $X_\bullet(m_\leq)$* if it contains all the c-homogeneous simplicies in $X_\bullet(m_\leq)$. This notion of a neighbourhood defines the *simplicial filter of c-neighbourhoods of the main diagonal.*

Ramsey theory implies that if the set C of colours is finite and a simplicial set $X_\bullet$ has hereditarily non-degenerate simplicies of arbitrarily high dimension, then there are c-homogeneous simplicies of of arbitrarily high dimension.

Now consider the equivalence relations of having hereditarily non-degenerate faces being of the same colour:

- $x \approx_c y$ iff for arbitrary $1 \leq i_1 \leq ... \leq i_n \leq m$
  - $c(x[i_1 \leq ... \leq i_n]) = c(y[i_1 \leq ... \leq i_n])$ whenever both faces $x[i_1 \leq ... \leq i_n]$ and $y[i_1 \leq ... \leq i_n]$ are hereditarily non-degenerate
  - $x[i_1 \leq ... \leq i_n]$ is hereditarily non-degenerate iff $y[i_1 \leq ... \leq i_n]$ is hereditarily non-degenerate.

Consider the quotient

$$c_\bullet : X_\bullet \longrightarrow C_\bullet, \quad C_\bullet(n_\leq) := X_\bullet(n_\leq)/ \approx_c$$

by these equivalence relations. This gives a morphism to the filter of main diagonals on $C_\bullet$ from the simplicial filter of c-neighbourhoods of the main diagonal on $X_\bullet$.

Moreover, we can define the simplicial filter of c-neighbourhoods of the main diagonal on $X_\bullet$ as the filter pulled back by this factorisation map from the filter of main diagonals on $C_\bullet$.

6.1.2. Filters associated with models. — Let us not observe that we can apply this construction to formulas in the language of a structure.

Let M be a structure. Consider an formula $\phi(x_1, ..., x_r)$ of the language of M as a colouring $\varphi : M^r \longrightarrow \{\text{true}, \text{false}\}$, and equip the simplicial set $|M|_\bullet$ represented by the set of elements of M with the simplicial filter of ϕ-neighbourhoods of the main diagonal.

For a set of formulas Σ, let $M_\bullet^\Sigma$ denote the simplicial set $|M|_\bullet$ equipped with the filters generated by the ϕ-neighbourhoods of the main diagonal for all $\phi \in \Sigma$.

This is a brief description of our key construction, which is defined in Definition 3.1.1.1 in a set-theoretic language.

7. Appendix. Conclusions and Speculations.

6.1. Topological intuition/vision. — The category[(21)] ረቀ of simplicial filters we use carries the intuition of point-set topology: [6,§3] argues that the description of the intuition of general topology in the Introduction to the book of [Bourbaki] on General Topology can be understood to apply to ረቀ almost verbatim. ረቀ can be used as one of "structures which give a mathematical content to the intuitive notions of limit, continuity and neighbourhood", but is somewhat more flexible than the usual category of topological spaces: In ረቀ the notion of limit [6,§4.10, Ex.4.10.1.1,4.10.2.1,4.11.1.1], a Cauchy sequence [6.,Ex.4.10.1.2], and being locally trivial [6,§4.8] can be expressed in terms of diagram chasing; these diagrams use an endomorphism of ረቀ $= Func(\Delta^{\mathrm{op}}$, ቀ$)$ induced by an endomorphism of Δ^{op} which is not available in the category of topological spaces but does play a role in the category of simplicial sets. In terms of ረቀ, an indiscernible sequence in a model and a Cauchy sequence in a metric space are morphisms from the same object. This allows us to modify the ረቀ-diagram expressing expressing the notion of limit in topological or metric spaces to define average/limit types, and we use this to define NIP by a Quillen lifting property formally somewhat similar to the lifting property defining completeness.

Using a combination of simplicial methods and topological intuition one can write several definitions of spaces of morphisms from one object to another; [7,§3] gives an example of such a definition which can be used to define geometric realisation of a simplicial set. It is tempting to try to define a (model theoretically meaningful) notion of a space of maps between two models, or, perhaps more plausibly, the space of indiscernible sequences in a model.

(21) We suggest to pronounce ረቀ as sF as it is visually similar to $s\Phi$ standing for "*s*implicial ϕilters", even though it is unrelated to the actual pronunciation of these symbols coming from the Amharic script. A script rare in mathematics allows us to denote the category of filters by a single letter ቀ reminiscent of Φ, yet avoid overusing letters.

The definition of being locally trivial is a pull-back diagram in ɀφ which is formally meaningful for any object (nee [7,§3.4] for a sketch and for details [6,§2.2.5,§4.8]). Does it have model theoretic meaning for generalised Stone spaces ?

The relation of the simplicial Stone space to the usual Stone space of a model is similar to the relation of the uniform structure to the topological structure associated with a metric; there is a forgetful functor ɀφ $\longrightarrow Top$ (cf. [7,§12.6.3], also [6,§2.2.4]) which takes $M_\bullet$ as defined in Appendix 3.1.1.1 into the usual 1-Stone space of M (after taking the quotient). Hence one may ask whether the usual machinery of Stone spaces, e.g. Cantor-Benedixon ranks, meaningfully generalises to ɀφ.

6.2. Geometric vision. — Our approach shows a formal analogy between indiscernible and Cauchy sequences: both are morphisms from the same object, one to (the generalised Stone space of) a model and one to the uniform space (considered as an object of our category). Taking the limit of a sequence in a metric or uniform space corresponds to a certain lifting property involving an endomorphism of ɀφ; a modification of this lifting property allows to talk about limit types and the characterisation of NIP in terms of average types.

6.3. Homotopy vision: a speculation. — ɀφ has two full subcategories with homotopy theory (the category of topological spaces, and the category of simplicial sets), and thus one may hope it is possible to develop homotopy theory for ɀφ itself. It is possible to define a model structure on ɀφ, say by extending the model structures on these two subcategories ?

We now sketch a couple of analogies between homotopy theory and model theory produced by wishful thinking.

A path $\gamma : [0,1] \longrightarrow X$ in a metric space is also a morphism from a linear order (viewed as object of ɀφ in some way than for a Cauchy sequence ([7,§2.4.2], [6,§4.3-4])). Unfortunately, while it is formally correct to consider morphisms from the same object to a model, it does not seem a good notion. Still, it is tempting to speculate about the possibility of analogy between indiscernible sequences and paths. The following analogies come to mind but we do not know how to make sense of them.

What is a "homotopy" between indiscernible sequences ? A homotopy between paths is a family of paths "compatible" in some way. Arrays or trees of indiscernible sequences (FIXME: FIXME:, what's a correct way to phrase this?) in NTP-type (No Tree Properties) properties come to mind: perhaps to be thought of as families of "compatible" indiscernible sequences. Or perhaps, if one is to entertain the idea that the failure of a tree property is analogous to failure of having every loop contractible, then two branches of an indiscernible tree or array demonstrating failure of a tree property represent two paths which cannot be homotoped into each other.

What are the "ends" of an indiscernible sequence ? Nothing comes to mind but types of finite tuples in it.

What are "connected components" of a model ? There is a definition (§3.5.2, also [6,§4.7]) of π_0 in the category of topological spaces in terms of Quillen negations, and it can be interpreted in ɀφ. We would think this definition would be too straightforward to give interesting results for models.

8. Preliminary Appendix 8. Dechypering Question 3.5.1.1

With very limited success, below we try to dechyper the notation of Question 3.5.1.1 and describe as much as possible in the model theoretic language the class of (stable, as we will see) models M such that

– $M_\bullet \longrightarrow \top \in \{(\mathbb{C};+,*)_\bullet \longrightarrow \top\}^{\text{lr}}$.

Everything here is very preliminary.

8.1. Dechypering Question 3.5.1.1(i). — Call a model M stable iff in M any countable infinite indiscernible sequence is necessarily an indiscernible set.

Call a morphism $A_\bullet \longrightarrow B_\bullet$ *anti-M* iff each $\wr\mathfrak{P}$-morphism $A_\bullet \longrightarrow M_\bullet$ factors as $A_\bullet \longrightarrow B_\bullet \longrightarrow M_\bullet$, i.e. in notation the following lifting property holds:

$$A_\bullet \longrightarrow B_\bullet \rightthreetimes M_\bullet \longrightarrow \top$$

In this terminology, Proposition 3.3.2.1(iv) says that a model M is stable iff morphism $\wr\mathfrak{P}$-morphism $\omega_\bullet \longrightarrow |\omega|_\bullet$ is anti-M.

Lemma 8.1.0.1. — *The following are equivalent:*

(i) $M_\bullet \longrightarrow \top \in \{(\mathbb{C};+,*)_\bullet \longrightarrow \top\}^{lr}$

(ii) *M is stable, and in $\wr\mathfrak{P}$ each anti-$(\mathbb{C};+,*)$-morphism is anti-M*

At the moment I do not see how to simply (ii) further. What we can do, however, is to write from it means for (ii) to hold a particular class of morphisms.

8.1.1. Extending EM-representations. — Here I describe a class of $\wr\mathfrak{P}$-morphisms associated with structures and the notion of EM-representation. In the next subsection I describe a more general class of morphisms.

Let $\mathbf{J}$ contain (as substructure) a reduct of a substructure of $\mathbf{J}$. This corresponds to an $\wr\mathfrak{P}$-morphism $\mathbf{I}_\bullet^{\text{qf}} \longrightarrow \mathbf{J}_\bullet^{\text{qf}}$. Let us slightly generalise the notion of EM-representation:

> Let $\mathbf{I}$ and M be structures, and let $\Sigma_\mathbf{I}$ and Σ_M be sets of formulas in the language of $\mathbf{I}$ and of M, resp. We say that *a structure* $\mathbf{I}$ $(\Sigma_\mathbf{I},\Sigma_M)$ *EM-represents a model M with a function $f:|I|\longrightarrow|M|$* iff for each $n>0$, and each finite subset $\Upsilon\subset\Sigma_M$ of formulas in the language of M there exists a finite subset $\Delta\subset\Sigma_\mathbf{I}$ of formulas in the language of $\mathbf{I}$ such that
>
> – the image under $f:|I|\longrightarrow|M|$ of each Δ-indiscernible sequence in $\mathbf{I}$ is necessarily also Υ-indiscernible in M, i.e. for each Δ-indiscernible sequence (a_i) in $\mathbf{I}$ its image $(f(a_i)_i)$ is Υ-indiscernible in M

Lemma 8.1.1.1. — *It holds (i) $\Longrightarrow$ (ii)$_{EM}$ where*

(i) $M_\bullet \longrightarrow \top \in \{(\mathbb{C};+,*)_\bullet \longrightarrow \top\}^{lr}$ *(as above)*

(ii)$_{EM}$ *M is stable, and if any $(\Sigma_\mathbf{I}, L_{ACF})$EM-representation of $(\mathbb{C};+,*)$ by $\mathbf{I}$ extends to an $(\Sigma_\mathbf{J}, L_{ACF})$EM-representation by $\mathbf{J}$, then any $(\Sigma_\mathbf{I}, L_M)$EM-representation of M by $\mathbf{I}$ extends to an $(\Sigma_\mathbf{J}, L_M)$EM-representation by $\mathbf{J}$.*

8.1.2. Filters on Cartesian powers of a set. — Let A be a set, and for each $0<n\in\omega$ let $\mathfrak{F}_n$ be a filter on A^n such that

– for each $m,n>0$, weakly increasing sequence $1\le i_1\le ...\le i_n\le m$, for each neighbourhood $\varepsilon\in\mathfrak{F}_n$

$$\{(t_1,...,t_m)\in A^m : (t_{i_1},...,t_{i_n})\in\varepsilon\}\in\mathfrak{F}_m$$

(Such a sequence of filters gives rise to a simplicial filter $|A|_\bullet^{\mathfrak{F}} : \Delta^{\rm op} \longrightarrow \boldsymbol{\varphi}$ defined as follows:

- $|A|_\bullet^{\mathfrak{F}}(n_\leq) := \mathfrak{F}_n$
- for each weakly increasing sequence $1 \leq i_1 \leq ... \leq i_n \leq m$, the continuous map of filters $[i_1 \leq ... \leq i_n] : I_\bullet^\leq(m_\leq) \longrightarrow I_\bullet^\leq(n_\leq)$ is given by

$$(t_1, ..., t_m) \longmapsto (t_{i_1}, ..., t_{i_n})$$

The condition on the filters $\mathfrak{F}_n$ means exactly that these maps are continuous.)

Let $\mathfrak{G}_n$ on A^n, $n > 0$ be another sequence of filters with the same properties such that $\mathfrak{F}_n$ is finer than $\mathfrak{G}_n$ for each $n > 0$. (This means there $id : |A| \longrightarrow |A|$ induces a morphism $|A|_\bullet^{\mathfrak{F}} \longrightarrow |A|_\bullet^{\mathfrak{G}}$).

Say that $(A; \mathfrak{F}; \mathfrak{G})$ have *M-extension property* iff any function $f : A \longrightarrow |M|$ such that

- for each ϕ in the language of M for each $n > 0$ there is a neighbourhood $\varepsilon \in \mathfrak{F}_n$ such that for each $(a_1, ..., a_n) \in \varepsilon$ the sequence $f(a_1), ..., f(a_n)$ is ϕ-indiscernible(22) with repetitions

it holds that

- for each ϕ in the language of M for each $n > 0$ there is a neighbourhood $\varepsilon \in \mathfrak{G}_n$ (sic) such that for each $(a_1, ..., a_n) \in \varepsilon$ the sequence $f(a_1), ..., f(a_n)$ is ϕ-indiscernible with repetitions

Lemma 8.1.2.1. — *It holds (i)$\Longrightarrow$(ii') and that (iii)$\Longleftrightarrow$(iii') where*

(i) $M_\bullet \longrightarrow \top \in \{(\mathbb{C}; +, *)_\bullet \longrightarrow \top\}^{lr}$ *(as above)*

(iii) *M is stable, and for each $(A; \mathfrak{F}; \mathfrak{G})$ as described above*

$$A_\bullet^{\mathfrak{F}} \longrightarrow A_\bullet^{\mathfrak{G}} \rightthreetimes (\mathbb{C}; +, *)_\bullet \longrightarrow \top \text{ implies } A_\bullet^{\mathfrak{F}} \longrightarrow A_\bullet^{\mathfrak{G}} \rightthreetimes M_\bullet \longrightarrow \top$$

(iii') *M is stable, and for each $(A; \mathfrak{F}; \mathfrak{G})$ as described above, if $(A; \mathfrak{F}; \mathfrak{G})$ has the* $(\mathbb{C}; +, *)$-extension property*, then $(A; \mathfrak{F}; \mathfrak{G})$ has the* M-extension property

In fact in Lemma above there is no need to consider $\mathfrak{G}_n$ to be defined on the same set as $\mathfrak{F}_n$, $n > 0$.

Let B be a set, and let filter $\mathfrak{G}_n$ be on B^n satisfying the same properties as above. Let $\iota : A \longrightarrow B$ be a map of sets.

Say that $(A; B; \mathfrak{F}; \mathfrak{G})$ has *M-extension property* iff any function $f : A \longrightarrow |M|$ such that

- for each ϕ in the language of M for each $n > 0$ there is a neighbourhood $\varepsilon \in \mathfrak{F}_n$ such that for each $(a_1, ..., a_n) \in \varepsilon$ the sequence $f(a_1), ..., f(a_n)$ is ϕ-indiscernible with repetitions

there is a function $\tilde{f} : B \longrightarrow |M|$ such that $f \circ \iota = \tilde{f}$ and it holds that

- for each ϕ in the language of M for each $n > 0$ there is a neighbourhood $\varepsilon \in \mathfrak{G}_n$ (sic) such that for each $(a_1, ..., a_n) \in \varepsilon$ the sequence $\tilde{f}(a_1), ..., \tilde{f}(a_n)$ is ϕ-indiscernible with repetitions

In fact I believe in the lemma below (i)$\Longleftrightarrow$(iv)$\Longleftrightarrow$(iv') and this is shown by a simple diagram chasing argument, but I am not sure yet.

(22) The condition on indiscernability may vary slightly, as the definition of $M_\bullet$ may vary slightly. For example, here you may rather require that the sequence $f(a_1), ..., f(a_n)$ is part of an arbitrarily long ϕ-indiscernible sequence (ignoring the repetitions). Of course, this change should be made everywhere at the same time.

Lemma 8.1.2.2. — *It holds (i)$\Longrightarrow$(iv') and that (iv)$\Longleftrightarrow$(iv') where*

(i) $M_\bullet \longrightarrow \top \in \{(\mathbb{C};+,*)_\bullet \longrightarrow \top\}^{lr}$ *(as above)*

(iv) *M is stable. and for each $(A;\mathfrak{F};\mathfrak{G})$ as described above*

$$A_\bullet^{\mathfrak{F}} \longrightarrow A_\bullet^{\mathfrak{G}} \curlywedge (\mathbb{C};+,*)_\bullet \longrightarrow \top \text{ implies } A_\bullet^{\mathfrak{F}} \longrightarrow A_\bullet^{\mathfrak{G}} \curlywedge M_\bullet \longrightarrow \top$$

(iv') *M is stable, and for each $(A;B;\mathfrak{F};\mathfrak{G})$ as described above if $(A;B;\mathfrak{F};\mathfrak{G})$ has the* $(\mathbb{C};+,*)$-extension property*, then $(A;B;\mathfrak{F};\mathfrak{G})$ has the M-*extension property

9. Preliminary Unfinished Appendix 9. An attempt to answer Question on Simplicity 3.5.4.1

This Appendix is not finished yet. Help in proofreading welcome.

We follow the "android" approach of [GH] to transcribe to $\wr\!\!\varphi$ the definition of the tree property and the simple theory in [Tent-Ziegler].

9.1. A shorter explanation.

— We quote [Tent-Ziegler]:

Definition 7.2.1. 1. A formula $\varphi(x,y)$ has the *tree property* (with respect to k) if there is a tree of parameters $(a_s \mid \emptyset \neq s \in {}^{<\omega}\omega)$ such that:

a) For all $s \in {}^{<\omega}\omega$, $(\varphi(x,a_{si}) \mid i < \omega)$ is k-inconsistent.

b) For all $\sigma \in {}^{\omega}\omega$, $\{\varphi(x,a_s) \mid \emptyset \neq s \subseteq \sigma\}$ is consistent.

2. A theory T is *simple* if there is no formula $\varphi(x,y)$ with the tree property.

9.1.1. A reformulation in set theoretic language. — We now reformulate item a) in a form convenient for diagram chasing reformulation. It is easy to see that item a) can be replaced by either of

a^1) For each $s \in {}^{<\omega}\omega$ there is an infinite set $S \subset {}^{<\omega}\omega$ of descendants such that $s' \supseteq s$ for each $s' \in S$ and the set $(\varphi(x,a_{s'}) \mid s' \in S)$ is k-inconsistent.

a^2) For each $s \in {}^{<\omega}\omega$ there is an infinite set $S \subset {}^{<\omega}\omega$ of descendants such that $s' \supseteq s$ for each $s' \in S$, the set $(\varphi(x,a_{s'}) \mid s' \in S)$ is k-inconsistent, and S is linearly ordered by the lexicographic order.

a^3) There is an infinite subset $\sigma \subseteq {}^{<\omega}\omega$ such that for each $s \in \sigma$ there is an infinite set $S \subseteq \sigma$ of descendants such that $s' \supseteq s$ for each $s' \in S$ and the set $(\varphi(x,a_{s'}) \mid s' \in S)$ is k-inconsistent.

a^4) There is an infinite subset $\sigma \subseteq {}^{<\omega}\omega$ such that for each $s \in \sigma$ there is an infinite set $S \subseteq \sigma$ of descendants such that $s' \supseteq s$ for each $s' \in S$, the set $(\varphi(x,a_{s'}) \mid s' \in S)$ is k-inconsistent, and S is linearly ordered by the lexicographic order.

a^5) There is an infinite subset $\sigma \subseteq {}^{<\omega}\omega$ such that for each $s \in \sigma$ the set $S \subseteq \sigma$ of immediate descendants of s is infinite and the set $(\varphi(x,a_{s'}) \mid s' \in S)$ is k-inconsistent, and S is linearly ordered by the lexicographic order.

a^6) There is a subtree $\sigma \subseteq {}^{<\omega}\omega$ isomorphic to ${}^{<\omega}\omega$ satisfying a).

Proof. — Indeed, any of these items implies that there is a subtree of ${}^{<\omega}\omega$ isomorphic to ${}^{<\omega}\omega$ itself which satisfies a), and any subtree satisfies b). □

9.1.2. Inconsistent instances of $\varphi(x,-)$. — To be able to discuss in $\wr\!\!\varphi$ inconsistency of sets of form $(\varphi(x,a_{s'}) \mid s' \in S)$, equip $|M|^n$ with the filter generated by the set

$$\varepsilon_n := \{(b_1,\ldots,b_n) \in |M|^n : M \vDash \exists x \bigwedge_{1\le i\le n} \varphi(x,b_i)\}$$

This turns the simplicial set $\left\{- \to |M|\right\}$ represented by the set of elements of M into a simplicial filter which we denote by $M_\bullet^{\exists x\varphi(x,-)}$. Accordingly, call a tuple $(b_1,...,b_n) \in |M|^n$ *small* or $\exists x\varphi(x,-)$*-small* iff the set $\{\varphi(x,b_1),..,\varphi(x,b_n)\}$ is consistent, i.e. $M \models \exists x \bigwedge_{1\leq i\leq n} \varphi(x,b_i)$.

9.1.3. Item b). — Item b) just says that the morphism $({}^{<\omega}\omega)_\bullet^{\leq} \longrightarrow M_\bullet^{\exists x\varphi(x,-)}$ defined by the parameters $(a_s)_s$ is continuous when $({}^{<\omega}\omega)_\bullet^{\leq}$ is equipped with indiscrete filters. Recall that

$$({}^{<\omega}\omega)_\bullet^{\leq}(n_\leq) := \left\{(s_1,...,s_{i_n}) : s_1,..,s_n \in {}^{<\omega}\omega, s_1 \subseteq ... \subseteq s_n\right\}, n<\omega$$

is the set of all weakly increasing tuples in ${}^{<\omega}\omega$ (which we here consider with the prefix order $\subseteq$), i.e. tuples of elements lying on the same branch in a weakly increasing order.

9.1.4. Item a). — To talk about consistency of sets occuring in item a^1)-a^6), we consider the morphism

$$({}^{<\omega}\omega^{\mathrm{lex}})_\bullet^{\leq} \longrightarrow M_\bullet^{\exists x\varphi(x,-)}$$

where ${}^{<\omega}\omega_{\mathrm{lex}}$ is the *lexicographic* partial order on ${}^{<\omega}\omega$.

Recall that

$$({}^{<\omega}\omega^{\mathrm{lex}})_\bullet^{\leq}(n_\leq) := \left\{(s_1,...,s_n) : s_1,...,s_n \in {}^{<\omega}\omega, s_1 \leq_{lex} \cdots \leq_{lex} s_n\right\}, n<\omega$$

Note that

$$|({}^{<\omega}\omega)_\bullet^{\leq}(1_\leq)| = |({}^{<\omega}\omega_{\mathrm{lex}})_\bullet^{\leq}(1_\leq)| = |{}^{<\omega}\omega|$$

We have the morphism $|({}^{<\omega}\omega_{\mathrm{lex}})_\bullet^{\leq}| \longrightarrow |M_\bullet|$ of simplicial sets. Each of items a^2)-a^6) says that the preimage of $\varepsilon_k \subseteq |M|^k$ does not intersect sets of certain form, namely the set of k-tuples required to be inconsistent in a copy of ${}^{<\omega}\omega$ by the tree property; equivalently, $\neg$a^2)-$\neg$a^6) says that the preimage of $\varepsilon_k \subseteq |M|^k$, necessarily large under a continuous map, intersects sets of certain form. Thus we "read off" the definition of a filter on $|({}^{<\omega}\omega_{\mathrm{lex}})_\bullet^{\leq}(k_\leq)|$: a subset is defined to be large iff it intersects any subset of form described in, say, item a^5), i.e. in notation: a subset $\delta \subset |({}^{<\omega}\omega_{\mathrm{lex}})_\bullet^{\leq}(k_\leq)|$ is *large* iff

- For any infinite subset $\sigma \subseteq {}^{<\omega}\omega$ such that for each $s \in \sigma$ the set $S \subseteq \sigma$ of immediate descendants of s is infinite, there is $s' \in \sigma$ and distinct immediate descendants $s'_1,...,s'_k$ such that the tuple $(s'_1,...,s'_k) \in \delta$ is "δ-small".

9.1.5. The lifting property. — Thus we arrive at the following conjecture.

Conjecture 9.1.5.1. — *The formula $\varphi(x,y)$ does not have the tree property in M iff the following lifting property holds:*

$$({}^{<\omega}\omega)_\bullet^{\leq} \longrightarrow ({}^{<\omega}\omega)_\bullet^{\leq} \cup ({}^{<\omega}\omega_{lex})_\bullet^{\leq} \rightthreetimes M_\bullet^{\exists x\varphi(x,-)} \longrightarrow \top$$

Proof(sketch). — Assume $\varphi(x,y)$ has the tree property in M, and let $(a_s \,|\, s \in {}^{<\omega}\omega)$ be a tree of parameters as in Definition 7.2.1. By b) the induced morphism $({}^{<\omega}\omega)_\bullet^{\leq} \longrightarrow M_\bullet^{\exists x\varphi(x,-)}$ is continuous. Consider the induced diagonal morphism of the underlying simplicial sets. By a) the preimage of the large subset generating the filter on M^k does not intersect the set of non-degenerate simplicies in $({}^{<\omega}\omega_{\mathrm{llex}})_\bullet^{\leq}(k_\leq)$, hence is not large. That is, the induced diagonal map is not continuous, and thereby the lifting property fails.

Assume the lifting property fails. A morphism $({}^{<\omega}\omega)_\bullet^{\leq} \longrightarrow M_\bullet^{\exists x\varphi(x,-)}$ is a tree of parameters $(a_s \,|\, s \in {}^{<\omega}\omega)$ satisfying item b) of Definition 7.2.1. It induces a unique morphism $|({}^{<\omega}\omega)^{\leq}_{\bullet} \cup ({}^{<\omega}\omega_{\mathrm{lex}})^{\leq}_{\bullet}| \longrightarrow |M_{\bullet}^{\exists x\varphi(x,-)}|$ of simplicial sets. This morphism fails to be continuous iff for some k the preimage ε of the set of $\varphi(x,-)$-consistent tuples is not large, i.e. there is an isomorphic copy $\sigma \subseteq {}^{<\omega}\omega$ of ${}^{<\omega}\omega$ such that any simplex in $\varepsilon \cap \sigma(n_{\leq})$ is degenerate. That is, each k-tuple with distinct elements in $(\sigma_{\mathrm{lex}})_{\bullet}(n_{\leq})$ is $\varphi(x,-)$-inconsistent, i.e. σ satisfies item a). Hence, σ is a witness to the tree property. □

9.2. Transcribing the definition. — Now in a verbose manner we step-by-step follow the "android" approach of [GH] to transcribe to ɀɸ the definition of the tree property in [Tent-Ziegler].

9.2.1. Inconsistent instances of $\varphi(x,-)$. — Equip $|M|^n$ with the filter generated by the set

$$\{(b_1,...,b_n) \in |M|^n : M \vDash \exists x \bigwedge_{1\leq i\leq n} \varphi(x,b_i)\}$$

Motivation: The definition 7.2.1 talks about the formulas $\exists x \bigwedge_{1\leq i\leq n} \varphi(x,b_i)$ implicitly, or rather tuples satisfying these formulas.

This turns the simplicial set $\left\{- \to |M|\right\}$ represented by the set of elements of M into a simplicial filter which we denote by $M_{\bullet}^{\exists x\varphi(x,-)}$. Accordingly, call a tuple $(b_1,...,b_n) \in |M|^n$ *small* or $\exists x\varphi(x,-)$*-small* iff the set $\{\varphi(x,b_1),..,\varphi(x,b_n)\}$ is consistent, i.e. $M \vDash \exists x \bigwedge_{1\leq i\leq n} \varphi(x,b_i)$.

Motivation 2 (fixme:remove?): The definition speaks of consistency of formulas of form $\varphi(x,a_s)$. Which is what we use to express this as the property of continuity of a morphism in ɀɸ.

9.2.2. Item b). —

b) *For all $\sigma \in {}^{\omega}\omega$ $\{\varphi(x,a_s) \,|\, \emptyset \neq s \subseteq \sigma\}$ is consistent.*

This just says that the morphism $({}^{<\omega}\omega)^{\leq}_{\bullet} \longrightarrow M_{\bullet}^{\exists x\varphi(x,-)}$ defined by the parameters $(a_s)_s$ is continuous when $({}^{<\omega}\omega)^{\leq}_{\bullet}$ is equipped with indiscrete filters. Recall that

$$({}^{<\omega}\omega)^{\leq}_{\bullet}(n_{\leq}) := \{(s_1,...,s_{i_n}) : s_1,..,s_n \in {}^{<\omega}\omega, s_1 \subseteq ... \subseteq s_n\}, n < \omega$$

is the set of all weakly increasing tuples in ${}^{<\omega}\omega$ (which we here consider with the prefix order $\subseteq$).

9.2.3. Item a). —

a) *For all $s \in {}^{<\omega}\omega$, $(\varphi(x,a_{si}) \,|\, i < \omega)$ is k-inconsistent.*

Item a) considers consistency of tuples of formulas $(\varphi(x,a_{si})|i < \omega)$, and, implicitly, the linear orders $si \leq sj$ iff $i \leq j$, $s \in {}^{<\omega}\omega$. Hence, we consider a simplicial set containing these tuples:

$$({}^{<\omega}\omega_{\mathrm{fans}})^{\leq}_{\bullet}(n_{\leq}) = \{(si_1,...,si_n) : s \in {}^{<\omega}\omega, 1 \leq i_1 \leq ... \leq i_n < \omega\}, n < \omega$$

where ${}^{<\omega}\omega_{\mathrm{fans}}$ is the *fan* partial order defined by

$$a_{si} \leq a_{s'i'} \text{ iff } s = s' \text{ and } i \leq i'$$

Note that

$$({}^{<\omega}\omega)^{\leq}_{\bullet}(1_{\leq}) = ({}^{<\omega}\omega_{\mathrm{fans}})^{\leq}_{\bullet}(1_{\leq}) = {}^{<\omega}\omega$$

We have the morphism $|{}^{<\omega}\omega_{\mathrm{fans}\bullet}| \longrightarrow |M_{\bullet}|$ of simplicial sets. Item a) says that the tuples with distinct elements (=non-degenerate simplicies) in the image of the morphism $|{}^{<\omega}\omega_{\mathrm{fans}\bullet}(k_{\leq})| \longrightarrow |M_{\bullet}(k_{\leq})| = M^k$ lie outside of the large subset generating the filter on $M^{\exists x\varphi(x,-)} \bullet (k_{\leq})$, or, equivalently, outside of some large subset of that filter.

9.2.4. The lifting property: first attempt. — So we see that a formula $\varphi(x,y)$ with the tree property provides a counterexample to the following lifting property in a very strong sense:

$$({}^{<\omega}\omega)^{\leq}_{\bullet} \longrightarrow ({}^{<\omega}\omega)^{\leq}_{\bullet} \cup ({}^{<\omega}\omega_{\text{fans}})^{\leq}_{\bullet} \not\rightthreetimes M_{\bullet}^{\exists x\varphi(x,-)} \longrightarrow \top$$

The map $|({}^{<\omega}\omega)^{\leq}_{\bullet}| \longrightarrow |M_{\bullet}^{\exists x\varphi(x,-)}|$ of simplicial sets extends uniquely to a map of simplicial sets $|({}^{<\omega}\omega)^{\leq}_{\bullet} \cup ({}^{<\omega}\omega_{\text{fans}})^{\leq}_{\bullet}| \longrightarrow |M_{\bullet}^{\exists x\varphi(x,-)}|$.

Item b) says the map $({}^{<\omega}\omega)^{\leq}_{\bullet} \longrightarrow M_{\bullet}^{\exists x\varphi(x,-)}$ is continuous when the source is equipped with the indiscrete filter.

Item a) says the map $({}^{<\omega}\omega)^{\leq}_{\bullet}(k_{\leq}) \cup ({}^{<\omega}\omega_{\text{fans}})^{\leq}_{\bullet}(k_{\leq}) \longrightarrow M_{\bullet}^{\exists x\varphi(x,-)}(k_{\leq})$ is *not* continuous if we equip with *any* filter which contains a large set extending $({}^{<\omega}\omega)^{\leq}_{\bullet}(k_{\leq})$ by a non-degenerate simplex (=tuple with distinct elements).

9.2.5. The filter appropriate to express item a). — Thus we'd like to equip the source of the latter map with the finest filter containing $({}^{<\omega}\omega)^{\leq}_{\bullet}(k_{\leq})$ defined by a property which can be "read off" from Definition 7.2.1. If item a) is satisfied not by all the vertices but still by enough vertices to form a isomorphic copy $\sigma \subseteq {}^{<\omega}\omega$ of ${}^{<\omega}\omega$, then the tree property still fails, and there is no harm in a large set intersecting $(\sigma_{\text{fans}})^{\leq}_{\bullet}(k_{\leq})$ non-trivially. This suggests the following definition: we define a subset to be *large* iff it contains a tuple with distinct elements in $(\sigma_{\text{fans}})^{\leq}_{\bullet}(k_{\leq})$ for each isomorphic copy $\sigma \subseteq {}^{<\omega}\omega$ of ${}^{<\omega}\omega$ in ${}^{<\omega}\omega$.

Finally, note that $\sigma \subseteq {}^{<\omega}\omega$ does not imply that $(\sigma_{\text{fans}})^{\leq}_{\bullet}(k_{\leq}) \subseteq ({}^{<\omega}\omega_{\text{fans}})^{\leq}_{\bullet}(k_{\leq})$ unless the immediate children in σ are necessarily immediate children in ${}^{<\omega}\omega$, and there is no reason to assume this.

Hence, we modify the definition of $({}^{<\omega}\omega_{\text{fans}})^{\leq}_{\bullet}(k_{\leq})$ so that it talks about arbitrary descendants rather than the si's:

$$|({}^{<\omega}\omega_{\text{antichains}})^{\leq}_{\bullet}(n_{\leq})| := \{(s_1,...,s_n) : s_i \leq_{lex} s_j \forall 1 \leq i \leq j \leq n, \text{ and } s_i \not\subseteq s_j \forall 1 \leq i \neq j \leq n\}, n < \omega$$

where $\leq_{lex}$ is the lexicographic order on ${}^{<\omega}\omega$.

A subset $\varepsilon \subseteq ({}^{<\omega}\omega_{\text{antichains}})^{\leq}_{\bullet}(n_{\leq})$ is *large* iff for each isomorphic copy $\sigma \subseteq {}^{<\omega}\omega$ of ${}^{<\omega}\omega$ the set $\varepsilon \cap (\sigma_{\text{antichains}})_{\bullet}(n_{\leq})$ contains a non-degenerate simplex, i.e. a tuple with all elements distinct.

9.2.6. The lifting property. — Thus we arrive at the following conjecture.

Conjecture 9.2.6.1. — *The formula $\varphi(x,y)$ does not have the tree property in M iff the following lifting property holds:*

$$({}^{<\omega}\omega)^{\leq}_{\bullet} \longrightarrow ({}^{<\omega}\omega)^{\leq}_{\bullet} \cup ({}^{<\omega}\omega_{antichains})^{\leq}_{\bullet} \rightthreetimes M_{\bullet}^{\exists x\varphi(x,-)} \longrightarrow \top$$

Proof(sketch). — Assume $\varphi(x,y)$ has the tree property in M, and let $(a_s \,|\, s \in {}^{<\omega}\omega)$ be a tree of parameters as in Definition 7.2.1. By b) the induced morphism $({}^{<\omega}\omega)^{\leq}_{\bullet} \longrightarrow M_{\bullet}^{\exists x\varphi(x,-)}$ is continuous. Consider the induced diagonal morphism of the underlying simplicial sets. By a) the preimage of the large subset generating the filter on M^k intersects $({}^{<\omega}\omega_{\text{antichains}})^{\leq}_{\bullet}(k_{\leq})$ only by degenerate simplicies, hence is not large. That is, the induced diagonal map is not continuous, and thereby the lifting property fails.

Assume the lifting property fails. A morphism $({}^{<\omega}\omega)^{\leq}_{\bullet} \longrightarrow M_{\bullet}^{\exists x\varphi(x,-)}$ is a tree of parameters $(a_s \,|\, s \in {}^{<\omega}\omega)$ satisfying item a) of Definition 7.2.1. It induces a unique morphism $|({}^{<\omega}\omega)^{\leq}_{\bullet} \cup ({}^{<\omega}\omega_{\text{antichains}})^{\leq}_{\bullet}| \longrightarrow |M_{\bullet}^{\exists x\varphi(x,-)}|$ of simplicial sets. This morphism fails to be continuous iff for some k the preimage ε of set of $\varphi(x,-)$-consistent tuples is not large, i.e. there is an isomorphic copy $\sigma \subseteq {}^{<\omega}\omega$ of ${}^{<\omega}\omega$ such that $\varepsilon \cap \sigma(n_{\leq})$

consists only of degenerate tuples. That is, each k-tuple in $(\sigma_{\text{antichains}})_\bullet(n_\leq)$ is $\varphi(x,-)$-inconsistent, i.e. σ satisfies item a). Hence, σ is a witness to the tree property. □

9.2.7. The lifting property of stability. — Notice that the lifting property Proposition 3.3.2.1(iv) of stability holds trivially for $M_\bullet^{\exists x\varphi(x,-)} \longrightarrow \top$ for any model M

$$\omega_\bullet \longrightarrow |\omega|_\bullet \rightthreetimes M_\bullet^{\exists x\varphi(x,-)} \longrightarrow \top$$

Also check whether it holds that

$$({}^{<\omega}\omega)_\bullet^\leq \longrightarrow ({}^{<\omega}\omega)_\bullet^\leq \cup ({}^{<\omega}\omega_{\text{antichains}})_\bullet^\leq \rightthreetimes M_\bullet^{\{\varphi\}} \longrightarrow \top$$

where $M_\bullet$ is defined as in Proposition 3.3.2.1.

9.2.8. The homotopy intuition $\mathbb{S}^k \longrightarrow \mathbb{D}^k$ and $A \longrightarrow A \times I$. — Recall our motivation in transcribing the tree property was to get a clue towards homotopy theory for model theory and ⅄. The formal analogy is to the lifting properties of topological spaces

$$\mathbb{S}^k \longrightarrow \mathbb{D}^k \rightthreetimes X \longrightarrow \top \text{ and } A \longrightarrow A \times I \rightthreetimes X \longrightarrow \top.$$

The first says that each sphere $\mathbb{S}^k$ can be "filled in" with a disk $\mathbb{D}^k$ and thus is contractible, suggests an intuition that "$({}^{<\omega}\omega_{\text{antichains}})_\bullet^\leq$ fills in in a hole in $({}^{<\omega}\omega)_\bullet^\leq$", and that $M_\bullet^{\exists x\varphi(x,-)}$ has trivial π_1 or $M_\bullet^{\exists x\varphi(x,-)} \longrightarrow \top$ is an acyclic fibration. The second is a path lifting property and homotopy extension property, and suggest an intuition that $({}^{<\omega}\omega_{\text{antichains}})_\bullet^\leq$ is a homotopy (of branches?) of $({}^{<\omega}\omega)_\bullet^\leq$, and that $M_\bullet^{\exists x\varphi(x,-)} \longrightarrow \top$ is a fibration.

We feel that making sense of this intuition is one of the most pressing needs in our approach to model theory.

Contents

MISHA GAVRILOVICH , Appendices unfinished. • Corrections to be sent to here or mishap@sdf.org • https://mishap.sdf.org/yetanothernotanobfuscatedstudy.pdf
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