
EXTREMALLY DISCONNECTED SPACES AS
 $\{\{u \rightarrow a, b \leftarrow u\} \longrightarrow \{u \rightarrow a = b \leftarrow v\}\}^l$, **AND BEING PROPER AS**
 $(\{\{o\} \longrightarrow \{o \rightarrow c\}\}^r_{<4})^{lr}$

Abstract. — We observe that the notions of a topological space being extremally disconnected, and of a continuous map of compact Hausdorff spaces being proper, can each be defined in terms of the Quillen lifting property with respect to a surjective proper morphism of finite topological spaces, i.e. in terms of a monotone map of finite preorders. This reveals the preorders implicit in the statement of the Gleason theorem that extremally disconnected spaces are projective in the category of compact Hausdorff topological spaces, and interprets it as an instance of a weak factorisation system generated by an explicitly given morphism.

1. Introduction

We observe that the notions of a topological space being extremally disconnected, having closed points, and a continuous map of compact Hausdorff spaces being proper, and being surjective proper, can each be defined in terms of the Quillen lifting property with respect to a surjective proper morphism of finite topological spaces, i.e. in terms of a monotone map of finite preorders (see Fig. 1). Based on this, we introduce a concise, and, in a sense intuitive, combinatorial notation expressing these notions via simplest (counter)examples, and often closely following the standard definitions. We hope our results suggest that this notation can be used to formalise these properties.

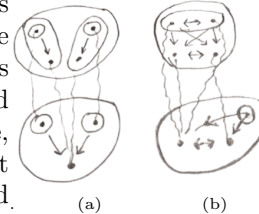


Fig.1. Maps of preorders for
 (a) extr. disconnected
 (b) proper

This allows us to write a couple of facts in general topology mentioned in the lecture notes on condensed mathematics by P.Scholze in a category-theoretic language hopefully closer to the spirit of the notes. The theorem of [Gleason], cf. [Condensed, Def.2.4], that extremally disconnected spaces are projective in the category of compact Hausdorff spaces, can then be seen as saying that there is a weak factorisation system generated by a certain surjective proper morphism of finite topological spaces. Our reformulation allows us to see that it is important for this theorem that the map of preorders implicit in the definition of extremally disconnected, is both surjective and proper. The fact that a surjective proper map is necessarily a quotient in our notation is expressed as an inclusion of orthogonals of morphisms of finite spaces:

$$\{\{u \rightarrow a, b \leftarrow v\} \longrightarrow \{u \rightarrow a = b \leftarrow v\}, \{a \leftrightarrow b \rightarrow c \leftrightarrow d\} \longrightarrow \{a \leftrightarrow b = c = d\}\}^{\wedge lr} \subset \{\{o \rightarrow c\} \longrightarrow \{o \leftrightarrow c\}\}^{\wedge l}$$

Die Mathematiker sind eine Art Franzosen: Redet man zu ihnen, so übersetzen sie es in ihre Sprache, und dann ist es alsobald ganz etwas anderes.—Johann Wolfgang von Goethe. Maximen und Reflexionen. Aphorismen und Aufzeichnungen. Nach den Handschriften des Goethe- und Schiller-Archivs hg. von Max Hecker, Verlag der Goethe-Gesellschaft, Weimar 1907, Aus dem Nachlass, Nr. 1005, Über Natur und Naturwissenschaft.

This fact is "the key point" [Analytic, p.7] in the sheaf condition holding for condensed sets represented by topological spaces.

Lemma 2.1.2 and Lemma 2.1.3 state our reformulations; §2 introduces necessary notation and reformulations in terms of lifting properties. Appendix A gives a list of reformulations of elementary notions in general topology in terms of lifting properties. In an unfinished appendix B we attempt to present a diagram chasing rendering of the proof of Lemma 1.3 [Analytic] stating that product commutes with filtered colimits in the category of compact Hausdorff spaces whenever all maps are (closed) inclusions, to help the reader ponder whether our notation can be used in an efficient theorem prover for elementary topology.

1.1. Further questions. — Our reformulations reveal combinatorics of finite preorders implicit, perhaps surprisingly, in many standard definitions in elementary general topology. It may be interesting to understand this combinatorics or make use of it.

How much of elementary topology could be developed entirely combinatorially? Say, could one prove the Gleason theorem entirely by a combinatorial diagram chasing calculation? Or the combinatorial expression above representing the fact that surjective proper maps are necessarily quotient.

Could these combinatorial expressions, say for quotients or properness, be interpreted in larger categories of spaces, say of condensed sets or the category of simplicial objects in the category of filters [situs]?

Formalisation of condensed mathematics may perhaps need the notions of extremally disjoint spaces or proper maps. Could our reformulation suggest an efficient theorem prover/tactic for elementary claims about these notions? Note that notions defined by lifting properties behave nicely with respect to limits and colimits.

1.2. Explaining notation in the title. — The notation and necessary definitions are introduced in the next section. Here we give a brief sketch assuming familiarity with lifting properties.

1.2.1. *Extremally disconnected.* — For a class of morphisms P , let $P^{\leftarrow r}$ and $P^{\leftarrow l}$ denote the class of morphisms having the left, resp. right, lifting property with respect to each morphism in P . Let $P^{\leftarrow r l} := (P^{\leftarrow r})^{\leftarrow l}$, $P^{\leftarrow l r} := (P^{\leftarrow l})^{\leftarrow r}$. Recall a topology on a finite set may be regarded as a preorder or, equivalently, as a category with unique morphisms: $x \leq y$, resp. $x \searrow y$, iff y lies in the closure of x .

Thus $\{u \rightarrow a, b \leftarrow v\}$ denotes the topological space with two open points u and v , two closed points a and b , split into two connected components $\{u \rightarrow a\}$ and $\{b \leftarrow v\}$. The expression $\{u \rightarrow a, b \leftarrow v\} \rightarrow \{u \rightarrow a = b \leftarrow v\}$ denotes the morphism gluing together points a and b . In terms of categories, we think that this morphism is the functor "adding an identity morphism between objects a and b ". Hence, $\{\{u \rightarrow a, b \leftarrow v\} \rightarrow \{u \rightarrow a = b \leftarrow v\}\}^{\leftarrow l}$ denotes the class of morphisms having the left lifting property with respect to $\{u \rightarrow a, b \leftarrow v\} \rightarrow \{u \rightarrow a = b \leftarrow v\}$. To give a map to $\{u \rightarrow a = b \leftarrow v\}$ is the same as to give two disjoint open subsets of E (the preimages of u and v). It lifts to

$\{u \rightarrow a, b \leftarrow v\}$ iff U and V lie in disjoint closed and open subsets. This is one of the equivalent definitions of E being extremally disconnected.

Note that the morphism $\{u \rightarrow a, b \leftarrow v\} \rightarrow \{u \rightarrow a = b \leftarrow v\}$ is surjective, closed and thereby proper, being a morphism of finite spaces. Also note that being surjective and proper are right lifting properties, and that this means that each map in $\{\{u \rightarrow a, b \leftarrow v\} \rightarrow \{u \rightarrow a = b \leftarrow v\}\}^{\wedge lr}$ is surjective and proper. Hence, the existence of the weak factorisation system generated by this morphism implies that each map $\emptyset \rightarrow E$ decomposes as $\emptyset \rightarrow X \in \{\{u \rightarrow a, b \leftarrow v\} \rightarrow \{u \rightarrow a = b \leftarrow v\}\}^{\wedge l}$ and a surjective proper map $E \rightarrow X \in \{\{u \rightarrow a, b \leftarrow v\} \rightarrow \{u \rightarrow a = b \leftarrow v\}\}^{\wedge lr}$. That is, each space admits a surjection from an extremally disconnected space, and a compact space admits a surjection from a compact extremally disconnected space. A similar argument shows that the extremally disconnected space can also be assumed Hausdorff, based on a reformulation of Axiom T1 (having closed points) as by a right lifting property, namely with respect to the morphism $\{o \searrow c\} \rightarrow \{o = c\}$ gluing the Sierpinski space into a single point.

1.2.2. *Proper.* — The expression $\{o\} \rightarrow \{o \rightarrow c\}$ denotes perhaps the simplest example of a non-proper (actually, not closed) map: the map sending a point into the open point of the Sierpinski space. It is easy to check that a map of finite spaces is in $\{\{o\} \rightarrow \{o \rightarrow c\}\}^r$, i.e. has the right lifting property with respect to this map, iff it is closed. By $\{\{o\} \rightarrow \{o \rightarrow c\}\}_{<4}^r$ we denote the subclass of $\{\{o\} \rightarrow \{o \rightarrow c\}\}^r$ consisting of maps of spaces with less than 4 points. By [Bourbaki, General Topology, I§10.2, Th.1(d)] (see Lemma 2.1.3(1)) being proper is a left lifting property, thus lr -orthogonal of any class of proper morphisms is a class of proper morphisms. Hence, each morphism in $(\{\{o\} \rightarrow \{o \rightarrow c\}\}_{<4}^r)^{lr}$ is proper, and by Engelking-Taimanov theorem (see Lemma 2.1.3(3)) any morphism of compact Hausdorff spaces is in this class.

2. Extremally disconnected sets being projective as a weak factorisation system

2.1. **Preliminary lemmas.** — A number of definitions in general topology can be expressed by applying several times the Quillen lifting property to a morphism of finite topological spaces, thereby leading to a concise notation based on finite preorders and their maps; Appendix A gives a list of such reformulations. In this subsection we introduce notations and state facts necessarily to express in this way the definitions of extremally disconnected and being proper.

2.1.1. *Quillen lifting property.* — Recall that a morphism i in a category has the *left lifting property* with respect to a morphism p , and p also has the *right lifting property* with respect to i , iff for each $f : A \rightarrow X$ and $g : B \rightarrow Y$ such that $p \circ f = g \circ i$ there exists $h : B \rightarrow X$ such that $h \circ i = f$ and $p \circ h = g$.

For a class P of morphisms in a category, its *left orthogonal* $P^{\wedge l}$ with respect to the lifting property, respectively its *right orthogonal* $P^{\wedge r}$, is the class of all morphisms which have the left, respectively right, lifting property with respect to each morphism

in the class P . In notation,

$$P^{\wedge l} := \{i : \forall p \in P \ i \prec p\}, P^{\wedge r} := \{p : \forall i \in P \ i \prec p\}, P^{\wedge lr} := (P^{\wedge l})^{\wedge r}, \dots$$

Taking the orthogonal of a class P is a simple way to define a class of morphisms excluding non-isomorphisms from P , in a way which is useful in a diagram chasing computation, and is often used to define properties of morphisms starting from an explicitly given class of (counter)examples. For this reason, it is convenient to refer to $P^{\wedge l}$ and $P^{\wedge r}$ as *left*, *resp. right*, *Quillen negation* of property P .

2.1.2. Notation for finite topological spaces and their morphisms. — A topological space comes with a *specialisation preorder* on its points: for points $x, y \in X$, $x \leq y$ iff $y \in clx$ (y is in the closure of x), or equivalently. The resulting preordered set may be regarded as a category whose objects are the points of X and where there is a unique morphism $x \searrow y$ iff $y \in clx$.

For a finite topological space X , the specialisation preorder or equivalently the corresponding category uniquely determines the space: a subset of X is closed iff it is downward closed, or equivalently, there are no morphisms going outside the subset.

The monotone maps (i.e. functors) are the continuous maps for this topology.

We denote a finite topological space by a list of the arrows (morphisms) in the corresponding category; arrows \rightarrow and \searrow are interchangeable and denote a morphism; ' \leftrightarrow ' denotes an isomorphism and '=' denotes the identity morphism. An arrow between two such lists denotes a continuous map (a functor) which sends each point to the correspondingly labelled point, but possibly turning some morphisms into identity morphisms, thus gluing some points. This notation leads to a formal syntax defining morphisms of finite topological space, and, to emphasize this, we sometimes typeset these expressions as code: $\{\mathbf{a} \leftarrow \mathbf{b}\}$ for $\{a \leftrightarrow b\}$, and $\{\mathbf{a} \rightarrow \mathbf{b}\}$ for $\{a \searrow b\}$ etc.

With this notation, we may display continuous functions for instance between the discrete space on two points, the Sierpinski space, the antidiscrete space and the point space as follows (where each point is understood to be mapped to the point of the same name in the next space):

$$\begin{array}{ccccccc} \{a, b\} & \longrightarrow & \{a \searrow b\} & \longrightarrow & \{a \leftrightarrow b\} & \longrightarrow & \{a = b\} \\ \{\mathbf{a}, \mathbf{b}\} & \dashrightarrow & \{\mathbf{a} \rightarrow \mathbf{b}\} & \dashrightarrow & \{\mathbf{a} \leftarrow \mathbf{b}\} & \dashrightarrow & \{\mathbf{a} = \mathbf{b}\} \\ \text{(discrete space)} & \longrightarrow & \text{(Sierpinski space)} & \longrightarrow & \text{(antidiscrete space)} & \longrightarrow & \text{(single point)} \end{array}$$

In $\{a \searrow b\}$, the point a is open and point b is closed.

Remark 2.1.1. — In $A \rightarrow B$, each object and each morphism in A necessarily appears in B as well. We may extend the notation to avoid listing the same object or morphism twice, to make it more concise and easier to read, although at a cost of getting used to. For example, we may wish to shorten $\{\mathbf{u} \rightarrow \mathbf{a}, \mathbf{b} \leftarrow \mathbf{v}\} \dashrightarrow \{\mathbf{u} \rightarrow \mathbf{a} = \mathbf{b} \leftarrow \mathbf{v}\}$ to $\{\mathbf{u} \rightarrow \mathbf{a}, \mathbf{b} \leftarrow \mathbf{v}\} \dashrightarrow \{\mathbf{a} = \mathbf{b}\}$ or even $\{u \searrow \overset{=}{a}, b \prec v\}$ using red and placing symbols above each other to indicate morphisms and objects added. Or perhaps to write $\{a\} \rightarrow \{a, b\}$ denoting the map from a single point to the discrete space with two points, as $\{a\} \rightarrow \{b\}$ or $\{a, b\}$.

Tricks like this can be useful in an actual implementation of this notation in a theorem prover.

2.1.3. *Preliminary results.* — We will use the following reformulations of properties of spaces and continuous maps in terms of lifting properties with respect to morphisms of finite topological spaces.

Note that each notion is defined with help of a counterexample, often the simplest or archetypal one. A concise convenient way to express e.g. items 3-5 is to say that *surjectivity, quotient, and injectivity are right Quillen negations of $\{a\} \dashrightarrow \{a \leftarrow b\}$, $\{o \rightarrow c\} \dashrightarrow \{o \leftarrow c\}$, and $\{a \leftarrow b\} \dashrightarrow \{a=b\}$, resp.*

Lemma 2.1.2. — *In the category of (all) topological spaces, the following holds.*

1. A map $X \rightarrow Y$ is surjective iff $\{\} \dashrightarrow \{o\} / _ X \dashrightarrow Y$
2. Points are closed within each fibre of a map $X \rightarrow Y$, i.e. the fibres (as subspaces with induced topology) satisfy separation axiom T1, iff $\{o \rightarrow c\} \dashrightarrow \{o=c\} / _ X \dashrightarrow Y$
3. A map $X \rightarrow Y$ is surjective iff $X \dashrightarrow Y / _ \{a\} \dashrightarrow \{a \leftarrow b\}$
4. A map $X \rightarrow Y$ is a quotient iff $X \dashrightarrow Y / _ \{o \rightarrow c\} \dashrightarrow \{o \leftarrow c\}$
5. A map $X \rightarrow Y$ is injective iff $X \dashrightarrow Y / _ \{a \leftarrow b\} \dashrightarrow \{a=b\}$
6. For every pair of disjoint closed subsets of X , the closures of their images of Y do not intersect, iff $X \dashrightarrow Y / _ \{x \leftarrow o \rightarrow y\} \dashrightarrow \{x=o=y\}$
7. A topological space X is extremally disconnected iff $\{\} \dashrightarrow X / _ \{u \rightarrow a, b \leftarrow v\} \dashrightarrow \{u \rightarrow a=b \leftarrow v\}$
8. The topology on X is induced from Y along the map $X \rightarrow Y$ iff $X \dashrightarrow Y / _ \{o \rightarrow c\} \dashrightarrow \{o=c\}$
9. The map $X \rightarrow Y$ has dense image iff $X \dashrightarrow Y / _ \{c\} \dashrightarrow \{o \rightarrow c\}$
10. The topology on X is induced from Y along the map $X \rightarrow Y$ and this map is open iff $X \dashrightarrow Y / _ \{a \leftarrow b \leftarrow c\} \dashrightarrow \{a \leftarrow b=c\}$
11. The topology on X is induced from Y along the map $X \rightarrow Y$ and this map is closed iff $X \dashrightarrow Y / _ \{a \leftarrow b \rightarrow c\} \dashrightarrow \{a \leftarrow b=c\}$

Proof. — Verification is a matter of expanding the definitions. We do only a few of the items. 1. This lifting property says that each point of Y (the image of o in $\{o\} \dashrightarrow Y$) has a preimage. 2. Let $x, y \in X$ be arbitrary. The map sending o to x and c to y , is continuous iff the closure of x contains y . The commutativity of the square means x and y lie in the same fibre. The commutativity of the triangles means $x = y$. 3. This lifting property says that each decomposition of $X = A \cup B$ (the preimages of points a and b in $X \dashrightarrow \{a \leftarrow b\}$) induces a decomposition $Y = p(A) \cup p(B)$ of Y where we denote $p : X \rightarrow Y$. This is evidently equivalent to injectivity. 6. Recall that a topological space X is extremally disconnected iff the closure of an open subset is closed. This lifting property says for every two disjoint open subsets U and V of X (the preimages of u and v in $X \dashrightarrow \{u \rightarrow a=b \leftarrow v\}$) there is a decomposition of X into two closed and open subsets U' and V' (the preimages of subspaces $\{u \rightarrow a\}$ and $b \leftarrow v$ in $X \dashrightarrow \{u \rightarrow a, b \leftarrow v\}$) such that $U \subset U'$ and $V \subset V'$. If X is extremally disconnected, then taking U' to be the closure of U gives the decomposition.

Alternatively but equivalently, the lifting property says that each $U \subset A \subset X$ where U is open and A is closed (the preimages of subspaces u and $u \rightarrow a$ in $X \dashrightarrow \{u \rightarrow a, b \leftarrow v\}$) is separated by a closed and open subset A' such that $U \subset A' \subset A$. Take $A := \bar{U}$ to be the closure of U ; then necessarily $A' = A$. Hence, the lifting property implies that the the closure of an open subset is open, i.e. X is extremally disconnected. The rest are analogous. \square

Following [Bourbaki, I§6.5, Definition 5, Example], given an ultrafilter \mathcal{U} on the set of points of a space B , define $B \sqcup_{\mathcal{U}} \{\infty\}$ to be the space B adjoined with a new closed point ∞ such that a subset is open iff it is either an open subset of B , or a union of $\{\infty\}$ and a \mathcal{U} -big open subset of B .

Lemma 2.1.3. — *In the category of (all) topological spaces, the following holds.*

1. *A map $X \rightarrow Y$ is proper iff for each set A viewed as a discrete topological space, each ultrafilter \mathcal{U} on A it holds*

$$B \rightarrow B \sqcup_{\mathcal{U}} \{\infty\} \times X \rightarrow Y$$

2. *The class $\{\mathbf{u} \rightarrow \mathbf{a}, \mathbf{b} \leftarrow \mathbf{v} \dashrightarrow \mathbf{u} \rightarrow \mathbf{a} = \mathbf{b} \leftarrow \mathbf{v}\}^{\wedge lr}$ is contained in the class of surjective proper morphisms, and, moreover,*
 - $\emptyset \rightarrow E \in \{\mathbf{u} \rightarrow \mathbf{a}, \mathbf{b} \leftarrow \mathbf{v} \dashrightarrow \mathbf{u} \rightarrow \mathbf{a} = \mathbf{b} \leftarrow \mathbf{v}\}^{\wedge l}$ iff E is extremally disconnected
 - if $X \rightarrow Y \in \{\mathbf{u} \rightarrow \mathbf{a}, \mathbf{b} \leftarrow \mathbf{v} \dashrightarrow \mathbf{u} \rightarrow \mathbf{a} = \mathbf{b} \leftarrow \mathbf{v}\}^{\wedge lr}$ and X is compact Hausdorff, so is Y
3. *Let P be the set of all proper morphisms of finite topological spaces mentioned in the right lifting properties of Lemma 2.1.2 (i.e. items 5-10), with or without item 7 (extremally disjoint). The class $(P)^{\wedge lr}$ consists of proper morphisms, and contains each proper morphism of compact Hausdorff spaces.*
4. *Let P' be the set of surjective proper morphisms of finite topological spaces mentioned in the right lifting properties of Lemma 2.1.2 (i.e. items 5-8), with or without item 6 (disjoint closures) or item 10 (open map and induced topology). The class $(P')^{\wedge lr}$ consists of surjective proper morphisms, and contains each surjective proper morphism of compact Hausdorff spaces.*

Proof. — 1. [Bourbaki, General Topology, I§10.2, Th.1(d)] almost states this lifting property: they take $A = |X|$ to be the set of points of X with discrete topology, and the horizontal map $A \rightarrow X$ in the square to be identity on points. An elementary argument shows that only the image in X would matter, and thus shows the equivalence of the statement by Bourbaki and this lifting property. See [mintsGE, §2.2.2] for details.

2. The map $\{\mathbf{u} \rightarrow \mathbf{a}, \mathbf{b} \leftarrow \mathbf{v} \dashrightarrow \mathbf{u} \rightarrow \mathbf{a} = \mathbf{b} \leftarrow \mathbf{v}\}$ is both surjective and closed, which is the same as proper for maps of finite topological spaces. By Lemma 2.1.2 both being surjective and being proper are right Quillen negations. Hence, each map in $(P_0)^{\wedge lr}$ is both surjective and proper, where $P_0 := \{\{\mathbf{u} \rightarrow \mathbf{a}, \mathbf{b} \leftarrow \mathbf{v} \dashrightarrow \mathbf{u} \rightarrow \mathbf{a} = \mathbf{b} \leftarrow \mathbf{v}\}\}$ is the class consisting of a single morphism $\{\mathbf{u} \rightarrow \mathbf{a}, \mathbf{b} \leftarrow \mathbf{v} \dashrightarrow \mathbf{u} \rightarrow \mathbf{a} = \mathbf{b} \leftarrow \mathbf{v}\}$. Lemma 2.1.2(6) states that the lifting property defines extremally disconnectedness. Moreover, note $\{\mathbf{o} \rightarrow \mathbf{c}\} \dashrightarrow \{\mathbf{o} = \mathbf{c}\} /_{_} \{\mathbf{u} \rightarrow \mathbf{a}, \mathbf{b} \leftarrow \mathbf{v} \dashrightarrow \mathbf{u} \rightarrow \mathbf{a} = \mathbf{b} \leftarrow \mathbf{v}\}$. By Lemma 2.1.2(2) this implies that if each point Y is closed (i.e. Y satisfies separation axiom $T1$) and $X \rightarrow Y$ is in $(P_0)^{\wedge lr}$, then each point of X is closed. For compact spaces, axiom $T1$ implies $T2$ (being Hausdorff). Hence, if $X \rightarrow Y$ is in $(P_0)^{\wedge lr}$ and if Y is compact Hausdorff, so is X .

3. Each morphism in P is proper, hence the morphisms mentioned in item 1 are in $P^{\wedge l}$, hence, again by item 1, each morphism in $P^{\wedge lr}$ is proper. Lemma 2.1.2(5-9)

imply that $P^{\wedge l}$ consists of inclusions $A \rightarrow B$ where A is a dense subset of B . A classic theorem in topology known as Engelking or Taimanov theorem says that a map to a compact Hausdorff space K always extends from a dense subset A to the whole domain B , i.e. $A \rightarrow B \times K \rightarrow \{o\}$; in fact the proof of this theorem also gives that $A \rightarrow B \times K_1 \rightarrow K_2$ holds for any proper map $K_1 \rightarrow K_2$ of normal Hausdorff spaces. See [mintsGE, §2.2] for a discussion.

4. Each map in (P) is both surjective and proper, and both being surjective and being proper are defined by right lifting properties. This implies that each map in (P) is both surjective and proper.

Now let $X \rightarrow Y$ be a surjective proper map of compact Hausdorff spaces. We need to show that it is in $(P)^{\wedge lr}$, i.e. that for each $A \rightarrow B \in (P)^{\wedge l}$ it holds $A \rightarrow B \times X \rightarrow Y$.

We know that $A \rightarrow B$ lifts with respect to each map in P , hence by Lemma 2.1.2 we may assume that A is an open subset of B , and the map is the inclusion.

Let $\bar{A} = \text{ClIm}_B(A)$ be the closure of A in B .

Now consider the lifting property $A \rightarrow B /_{-} \{u \rightarrow a, b \leftarrow v\} \rightarrow \{u \rightarrow a = b \leftarrow v\}$ defining extremally disconnected. Take $A \rightarrow \{u \rightarrow a, b \leftarrow v\}$ taking A to u , and $B \rightarrow \{u \rightarrow a = b \leftarrow v\}$ sending A to u , and $\bar{A} \setminus A$ to $a = b$, and $B \setminus \bar{A}$ to v . If A is non-empty, the lifting property implies that \bar{A} is open. Hence, both \bar{A} and $B \setminus \bar{A}$ are closed open subsets, and to construct the diagonal map, it is enough to construct it separately on \bar{A} and $B \setminus \bar{A}$. As A is dense in \bar{A} , by Lemma above $A \rightarrow \bar{A}$ lifts with respect to any proper map of compact Hausdorff spaces; this implies the former.

Note that $A \rightarrow B /_{-} \{u \rightarrow a, b \leftarrow v\} \rightarrow \{u \rightarrow a = b \leftarrow v\}$ implies that

$$\emptyset \rightarrow B \setminus \bar{A} \times \{u \rightarrow a, b \leftarrow v\} \rightarrow \{u \rightarrow a = b \leftarrow v\}$$

i.e. that $B \setminus \bar{A}$ is extremally disconnected. Finally, the theorem of Gleason that extremally disconnected sets are projective in the subcategory of compact Hausdorff spaces with proper maps, says precisely that this lifting property holds for each surjective proper map of compact Hausdorff spaces. \square

Remark 2.1.4. — We rely on the Gleason theorem rather than reproduce its proof. Probably a careful reformulation of Lemmas 2.1 and 2.4 of [Gleason] shall turn the proof there into a diagram chasing calculation with finite preorders.

2.2. Being a surjective image of a compact extremally disconnected space. —

We start with the observation that the map $\{u \rightarrow a, b \leftarrow v\} \rightarrow \{u \rightarrow a = b \leftarrow v\}$ appearing in the definition of extremally disconnected, is surjective and proper, and that being surjective and being proper are right Quillen negations. Hence, each map in $(P_0)^{\wedge lr}$ is both surjective and proper, where $P_0 := \{\{u \rightarrow a, b \leftarrow v\} \rightarrow \{u \rightarrow a = b \leftarrow v\}\}$ is the class consisting of a single morphism $\{u \rightarrow a, b \leftarrow v\} \rightarrow \{u \rightarrow a = b \leftarrow v\}$. Moreover, note $\{o \rightarrow c\} \rightarrow \{o = c\} /_{-} \{u \rightarrow a, b \leftarrow v\} \rightarrow \{u \rightarrow a = b \leftarrow v\}$. This implies that if each point of Y is closed (i.e. Y satisfies separation axiom $T1$) and $X \rightarrow Y$ is in $(P_0)^{\wedge lr}$, then each point of X is closed. For compact spaces, axiom $T1$ implies $T2$ (being Hausdorff). Hence, if $X \rightarrow Y$ is in $(P_0)^{\wedge lr}$ and if Y is compact Hausdorff, so is X .

Hence:

Observation 2.2.1. — *The fact that each topological space admits a surjection from an extremally disconnected space, and, moreover, each compact Hausdorff topological space admits a surjection from an compact Hausdorff extremally disconnected space, is implied by the following.*

Each morphism $\emptyset \rightarrow X$ decomposes as $\emptyset \xrightarrow{(P_0)^{\triangleleft l}} E \xrightarrow{(P_0)^{\triangleleft lr}} X$ where $P_0 := \{\{u \rightarrow a, b \leftarrow v\} \dashrightarrow \{u \rightarrow a = b \leftarrow v\}\}$ is a class of morphisms consisting of a single morphism which is both surjective and proper (and hence so is any map in $(P_0)^{\triangleleft lr}$).

In fact, this decomposition (weak factorisation system) implies that the extremally disconnected subspaces are projective in the (not full!) subcategory of topological spaces with morphisms in $(P_0)^{\triangleleft lr}$, and that subcategory has enough projectives.

Unfortunately, not each surjective proper map of compact Hausdorff spaces is in $(P_0)^{\triangleleft lr}$. Indeed, if the domain is connected, then it maps to one of the connected components $\{u \rightarrow a\}$ or $\{b \leftarrow v\}$, and by surjectivity the codomain does as well. Hence, any surjective map from a connected space is in $(P_0)^{\triangleleft l}$ and thus not in $(P_0)^{\triangleleft lr}$ unless is an isomorphism.

2.3. Extremally disconnected spaces being projective. — Let P' denote the class of all the closed (necessarily proper) surjective maps mentioned in right lifting properties in Lemma 2.1.2 (we give the list of morphisms in various notations):

$$\begin{array}{cccc} \{u \rightarrow a, b \leftarrow u\} & \{a \overset{=}{\leftrightarrow} b\} & \{a \overset{=}{\rightarrow} b\} & \{a \leftrightarrow b \overset{=}{\rightarrow} c\} \\ \{u \rightarrow a, b \leftarrow u\} \longrightarrow \{u \rightarrow a = b \leftarrow v\} & \{a \leftrightarrow b\} \longrightarrow \{a = b\} & \{o \triangleright c\} \longrightarrow \{o = c\} & \{a \leftrightarrow b \triangleright c\} \longrightarrow \{a \leftrightarrow b = c\} \\ \{u \rightarrow a, b \leftarrow v\} \dashrightarrow \{u \rightarrow a = b \leftarrow v\} & \{a \leftarrow b\} \dashrightarrow \{a = b\} & \{o \rightarrow c\} \dashrightarrow \{o = c\} & \{a \leftarrow b \rightarrow c\} \dashrightarrow \{a \leftarrow b = c\} \\ \text{(extremally disconnected)} & \text{(injective)} & \text{(pullback topology)} & \text{(closed map and pullback topology)} \end{array}$$

We can combine together the latter three morphisms and take instead e.g.

$$P'' := \{ \{u \rightarrow a, b \leftarrow v\} \dashrightarrow \{u \rightarrow a = b \leftarrow v\}, \{a \leftarrow b \rightarrow c \leftarrow d\} \dashrightarrow \{a \leftarrow b = c = d\} \}$$

We summarise the considerations above as

Observation 2.3.1. — *The fact that extremally disconnected space are projective in the category of compact Hausdorff spaces with proper maps, and this category has enough projectives, is implied by the decomposition above and the following.*

Each morphism $\emptyset \rightarrow X$ decomposes as $\emptyset \xrightarrow{(P')^{\triangleleft l}} E \xrightarrow{(P')^{\triangleleft lr}} X$ where

$$P' := \{\{u \rightarrow a, b \leftarrow v\} \dashrightarrow \{u \rightarrow a = b \leftarrow v\}, \{a \leftarrow b\} \dashrightarrow \{a = b\}, \{a \leftarrow b \rightarrow c\} \dashrightarrow \{a \leftarrow b = c\}\}$$

is a class of morphisms consisting of surjective proper morphisms.

Proof. — Use Lemma 2.1.3(3). Use the observation above to construct a compact Hausdorff E fitting the decomposition. We may omit $\{o \rightarrow c\} \dashrightarrow \{o = c\}$ because the map gluing together a and b in $\{a \leftarrow b \rightarrow c\} \dashrightarrow \{a \leftarrow b = c\}$ gives $\{o \rightarrow c\} \dashrightarrow \{o = c\}$. \square

Remark 2.3.2. — [Analytic,p7] writes “for part (2) [the sheaf condition on condensed sets represented by topological spaces] the key point is that any surjective

map of profinite sets is a quotient". In fact any surjective proper map is a quotient, and Lemma 2.1.2 and 2.1.3 allows to express this as:

$$P'^{\wedge lr} \subset \{o \rightarrow c\} \dashrightarrow \{o \leftarrow c\}^{\wedge l}$$

Explicitly,

$$\begin{aligned} & \{u \rightarrow a, b \leftarrow v\} \dashrightarrow \{u \rightarrow a = b \leftarrow v\}, \{a \leftarrow b\} \dashrightarrow \{a = b\}, \{a \leftarrow b \rightarrow c\} \dashrightarrow \{a \leftarrow b = c\}^{\wedge lr} \subset \{o \rightarrow c\} \dashrightarrow \{o \leftarrow c\}^{\wedge l} \\ & \{u \rightarrow a, b \leftarrow v\} \longrightarrow \{u \rightarrow a = b \leftarrow v\}, \{a \leftrightarrow b \rightarrow c \leftrightarrow d\} \longrightarrow \{a \leftrightarrow b = c = d\}^{\wedge lr} \subset \{o \rightarrow c\} \longrightarrow \{o \leftrightarrow c\}^{\wedge l} \end{aligned}$$

3. Appendix A. A list of reformulations of topological definitions

Here we give a list of examples of iterated lifting properties (negations) written in our notation. Sometimes we skip \wedge for readability. Most of the list below is taken from [mintsGE, §5.2].

3.1. Examples of iterated orthogonals obtained from maps between finite topological spaces. — Here we give a list of examples of iterated orthogonals starting from maps between finite topological spaces defining well-known properties of topological spaces.

In the category of topological spaces,

1. $(\emptyset \longrightarrow \{o\})^r$ is the class of surjections
2. $(\emptyset \longrightarrow \{o\})^r$ is the class of maps $A \longrightarrow B$ where $A \neq \emptyset$ or $A = B$
3. $(\emptyset \longrightarrow \{o\})^{rr}$ is the class of subsets, i.e. injective maps $A \hookrightarrow B$ where the topology on A is induced from B
4. $(\emptyset \longrightarrow \{o\})^{lr}$ is the class of maps $\emptyset \longrightarrow B$, B arbitrary
5. $(\emptyset \longrightarrow \{o\})^{lrr}$ is the class of maps $A \longrightarrow B$ which admit a section
6. $(\emptyset \longrightarrow \{o\})^l$ consists of maps $f: A \longrightarrow B$ such that either $A \neq \emptyset$ or $A = B = \emptyset$
7. $(\emptyset \longrightarrow \{o\})^{rl}$ is the class of maps of form $A \longrightarrow A \sqcup D$ where D is discrete
8. $\{\{z \leftrightarrow x \leftrightarrow y \searrow c\} \longrightarrow \{z = x \leftrightarrow y = c\}\}^{\wedge l} = \{\{c\} \longrightarrow \{o \searrow c\}\}^{\wedge lr}$ is the class of closed inclusions $A \subset B$ where A is closed
9. $\{\{z \leftrightarrow x \leftrightarrow y \swarrow c\} \longrightarrow \{z = x \leftrightarrow y = c\}\}^{\wedge l}$ is the class of open inclusions $A \subset B$ where A is open
10. $\{\{x \leftrightarrow y \searrow c\} \longrightarrow \{x \leftrightarrow y = c\}\}^{\wedge l}$ is the class of closed maps $A \longrightarrow B$ where the topology on A is pulled back from B
11. $\{\{x \leftrightarrow y \swarrow c\} \longrightarrow \{x \leftrightarrow y = c\}\}^{\wedge l}$ is the class of open maps $A \longrightarrow B$ where the topology on A is pulled back from B
12. $(\{b\} \longrightarrow \{a \searrow b\})^l$ is the class of maps with dense image
13. $(\{b\} \longrightarrow \{a \searrow b\})^{lr}$ is the class of closed subsets $A \subset X$, A a closed subset of X
14. $(\{a\} \longrightarrow \{a \searrow b\})^{lr}$ is the class of subsets $A \subset X$ such that A is the intersection of open subsets containing A
15. $(\{a \searrow b\} \longrightarrow \{a = b\})^l$ is the class of injections
16. $(\{a\} \longrightarrow \{a \searrow b\})_{\leq 5}^{r}{}^{lr}$ is roughly the class of proper maps

3.2. Examples of properties of topological spaces expressed as iterated orthogonals of maps between finite topological spaces. — Here give a list of examples of well-known properties defined by iterated orthogonals starting from maps between finite topological spaces, often with less than 5 elements.

1. $\{\bullet\} \longrightarrow A$ is in $(\emptyset \longrightarrow \{o\})^{rll}$ iff A is connected
2. Y is totally disconnected iff $\{\bullet\} \xrightarrow{y} Y$ is in $(\emptyset \longrightarrow \{o\})^{rllr}$ for each map $\{\bullet\} \xrightarrow{y} Y$ (or, in other words, each point $y \in Y$).
3. a Hausdorff space K is compact iff $K \longrightarrow \{o\}$ is in $((\{o\} \longrightarrow \{o \searrow c\})_{<5}^r)^{lr}$
4. a Hausdorff space K is compact iff $K \longrightarrow \{o\}$ is in $\{\{a \leftrightarrow b\} \longrightarrow \{a = b\}, \{o \searrow c\} \longrightarrow \{o = c\}, \{c\} \longrightarrow \{o \searrow c\}, \{a \not\leftarrow o \searrow b\} \longrightarrow \{a = o = b\}\}^{lr}$
5. a space D is discrete iff $\emptyset \longrightarrow D$ is in $(\emptyset \longrightarrow \{o\})^{rl}$
6. a space D is antidiscrete iff $D \longrightarrow \{o\}$ is in $(\{a, b\} \longrightarrow \{a = b\})^{rr} = (\{a \leftrightarrow b\} \longrightarrow \{a = b\})^{lr}$
7. a space K is connected or empty iff $K \longrightarrow \{o\}$ is in $(\{a, b\} \longrightarrow \{a = b\})^l$
8. a space K is totally disconnected and non-empty iff $K \longrightarrow \{o\}$ is in $(\{a, b\} \longrightarrow \{a = b\})^{lr}$
9. a space K is connected and non-empty iff for some arrow $\{o\} \longrightarrow K$ $\{o\} \longrightarrow K$ is in $(\emptyset \longrightarrow \{o\})^{rll} = (\{a\} \longrightarrow \{a, b\})^l$
10. a space K is non-empty iff $K \longrightarrow \{o\}$ is in $(\emptyset \longrightarrow \{o\})^l$
11. a space K is empty iff $K \longrightarrow \{o\}$ is in $(\emptyset \longrightarrow \{o\})^{ll}$
12. a space K is T_0 iff $K \longrightarrow \{o\}$ is in $(\{a \leftrightarrow b\} \longrightarrow \{a = b\})^r$
13. a space K is T_1 iff $K \longrightarrow \{o\}$ is in $(\{a \searrow b\} \longrightarrow \{a = b\})^r$
14. a space X is Hausdorff iff for each injective map $\{x, y\} \hookrightarrow X$ it holds $\{x, y\} \hookrightarrow X \times \{x \searrow o \not\leftarrow y\} \longrightarrow \{x = o = y\}$
15. a non-empty space X is regular (T3) iff for each arrow $\{x\} \longrightarrow X$ it holds $\{x\} \longrightarrow X \times \{x \searrow X \not\leftarrow U \searrow F\} \longrightarrow \{x = X = U \searrow F\}$
16. a space X is normal (T4) iff $\emptyset \longrightarrow X \times \{a \not\leftarrow U \searrow x \not\leftarrow V \searrow b\} \longrightarrow \{a \not\leftarrow U = x = V \searrow b\}$
17. a space X is completely normal iff $\emptyset \longrightarrow X \times [0, 1] \longrightarrow \{0 \not\leftarrow x \searrow 1\}$ where the map $[0, 1] \longrightarrow \{0 \not\leftarrow x \searrow 1\}$ sends 0 to 0, 1 to 1, and the rest $(0, 1)$ to x
18. a space X is hereditary normal iff $\emptyset \rightarrow X \times \{x \not\leftarrow au \leftrightarrow u' \not\leftarrow u \not\leftarrow uv \searrow v \searrow v' \leftrightarrow bv \searrow x\} \longrightarrow \{x \not\leftarrow au \leftrightarrow u' = u \not\leftarrow uv \searrow v = v' \leftrightarrow bv \searrow x\}$
19. a space X is path-connected iff $\{0, 1\} \longrightarrow [0, 1] \times X \longrightarrow \{o\}$
20. a space X is path-connected iff for each Hausdorff compact space K and each injective map $\{x, y\} \hookrightarrow K$ it holds $\{x, y\} \hookrightarrow K \times X \longrightarrow \{o\}$

3.3. A sample of a computer syntax. — Here we rewrite some of the examples above in a computer syntax. ASCII art on the right attempts to represent graphically the maps of preorders involved.

```
compactness:  { {o}-->{o->c} }^r_{<5}^lr ; {o}-->{o->c} is a non-proper map
dense image:  { {c}-->{o->c} }^l1 ; the image of {c}-->{o->c} is not dense
injection:    { {x,y}-->{x=y} }^r == { {x<->y}-->{x=y} }^l1 '~~' ( == .-. ).
surjection:   { {}-->{o} }^r == { {}-->{o} }^rrl1 simplest non-surjection {}-->{o}
connected:    { {}-->{o} }^rll1 { {x,y}-->{x=y} }^l1 simplest non-connected space {x,y}
```

discrete: $\{ \{ \} \rightarrow \{o\} \}^r \text{rl}$)(.
subset: $\{ \{ \} \rightarrow \{o\} \}^r \text{rr} == \{ \{x \leftarrow y \rightarrow c\} \rightarrow \{x=y=c\} \}^l$)). == $\sim \setminus (.$
closed subset: $\{ \{z \leftarrow x \leftarrow y \rightarrow c\} \rightarrow \{z=x \leftarrow y=c\} \}^l == \{ \{c\} \rightarrow \{o \rightarrow c\} \}^l \text{lr}$)'. == '
open subset: $\{ \{z \leftarrow x \leftarrow y \leftarrow c\} \rightarrow \{z=x \leftarrow y=c\} \}^l$)'.
intersection of open subsets: $\{ \{o\} \rightarrow \{o \rightarrow c\} \}^l \text{lr}$)'.
normal (T4): $\{ \{a \leftarrow b \rightarrow c \leftarrow d \rightarrow e\} \rightarrow \{b=c=d\} \}^l$ /V\(/
Tietze lemma (not quite) : $R \rightarrow \{o\}$ (- $\{ \{a \leftarrow b \rightarrow c \leftarrow d \rightarrow e\} \rightarrow \{b=c=d\}, \{a \leftarrow b \rightarrow c\} \rightarrow \{a=b=c\} \}^l \text{lr}$
Urysohn lemma (not quite): $R \rightarrow \{a \leftarrow b \rightarrow c\}$ (- $\{ \{a \leftarrow b \rightarrow c \leftarrow d \rightarrow e\} \rightarrow \{b=c=d\} \}^l \text{lr}$
Hausdorff: $\{u, v\} \text{--(inj)--} X /_ \{u \rightarrow x \leftarrow v\} \rightarrow \{u=x=v\}$
i.e. any injection $\{a, b\} \rightarrow X$ lifts wrt $\{u \rightarrow x \leftarrow v\} \rightarrow \{u=x=v\}$

Avoiding repetitions. — The reader would notice that the syntax above repeats almost everything twice: indeed, almost the same preorder appears on both sides of \rightarrow arrow. Below we give a sample of possible notations avoiding this repetition, hence the notation below is intentionally not consistent.

compactness: $\{ \{o \rightarrow c\} \}^r \text{r}_{\leq 5} \text{lr}$; $\{o\} \rightarrow \{o \rightarrow c\}$ is a non-proper map
dense image: $\{ \{o \rightarrow c\} \}^l$; the image of $\{c\} \rightarrow \{o \rightarrow c\}$ is not dense
injection: $\{ \{x, \cdot = y\} \}^r == \{ \{x \leftarrow y\} \rightarrow \{x=y\} \}^l$)'. == '
surjection: $\{ \{o\} \}^r == \{ \{ \} \rightarrow \{o\} \}^l \text{rrl}$ simplest non-surjection $\{ \} \rightarrow \{o\}$
connected: $\{ \{o\} \}^l \text{rll}$ $\{ \{x, \cdot = y\} \}^l$ simplest non-connected space $\{x, y\}$
discrete: $\{ \{o\} \}^r$)(.
subset: $\{ \{o\} \}^r$) == $\{ \{x \leftarrow y \rightarrow c\} \}^l$)). == $\sim \setminus (.$
closed subset: $\{ \{z \leftarrow x \leftarrow y \rightarrow c\} \}^l == \{ \{o \rightarrow c\} \}^l \text{lr}$)'. == '
open subset: $\{ \{z \leftarrow x \leftarrow y \leftarrow c\} \}^l$)'.
intersection of open subsets: $\{ \{o \rightarrow c\} \}^l \text{lr}$)'.
dense image: $\{ \{o \rightarrow c\} \}^l$; the image of $\{c\} \rightarrow \{o \rightarrow c\}$ is not dense
normal (T4): $\{ \{a \leftarrow b \rightarrow c \leftarrow d \rightarrow e\} \}^l$ /V\(/
Tietze lemma (not quite) : $R \rightarrow \{o\}$ (- $\{ \{a \leftarrow b \rightarrow c \leftarrow d \rightarrow e\}, \{a \leftarrow b \rightarrow c\} \}^l$)
Urysohn lemma (not quite): $R \rightarrow \{a \leftarrow b \rightarrow c\}$ (- $\{ \{a \leftarrow b \rightarrow c \leftarrow d \rightarrow e\} \}^l \text{lr}$
Hausdorff: $\{u, v\} \text{--(inj)--} X /_ \{u \rightarrow x \leftarrow v\}$
i.e. any injection $\{a, b\} \rightarrow X$ lifts wrt $\{u \rightarrow x \leftarrow v\}$

4. Appendix B (unfinished)

In this appendix (not indented for publication) we experiment with notation for diagram chasing calculations. We present an incomplete(!) diagram chasing calculation representing the proof of Lemma 1.3 [Analytic]. We hope our calculations give some evidence that it may be possible to use diagram chasing with preorders in an efficient formalisation of general topology.

4.1. Statement and proof of Lemma 1.3. — We quote [Analytic]:

Lemma 1.3. *Let $X_0 \rightarrow X_1 \dots$ and $Y_0 \rightarrow Y_1 \rightarrow \dots$ be two sequences of compact Hausdorff spaces with closed immersions. Then, inside the*

category of topological spaces, the natural map

$$\bigcup_n X_n \times Y_n \longrightarrow \left(\bigcup_n X_n\right) \times \left(\bigcup_n Y_n\right)$$

is a homeomorphism; i.e. the product on the right is equipped with its compactly generated topology.

Proof. — The map is clearly a continuous bijection. In general, for a union like $\bigcup_n X_n$, open subsets U are the subsets of the form $\bigcup_n U_n$ where each $U_n \subset X_n$ is open. Thus, let $U \subset \bigcup_n X_n \times Y_n$ be any open subset, written as a union of open subset $U_n \subset X_n \times Y_n$, and pick any point $(x, y) \in U$. Then for any large enough n (so that $(x, y) \in X_n \times Y_n$), we can find open neighborhoods $V_n \subseteq X_n$ of x in X_n and $W_n \subseteq Y_n$ of y in Y_n , such that $V_n \times W_n \subseteq U_n$. In fact, we can ensure that even $\bar{V}_n \times \bar{W}_n \subseteq U_n$ by shrinking V_n and W_n . Constructing the V_n and W_n inductively, we may then moreover ensure $V_n \subseteq V_{n+1}$ and $W_n \subseteq W_{n+1}$. Then $V = \bigcup_n V_n \subseteq \bigcup_n X_n$ and $W = \bigcup_n W_n \subseteq \bigcup_n Y_n$ are open, and $V \times W = \bigcup_n V_n \times W_n \subseteq U$ contains (x, y) , showing that U is open in the product topology. \square

4.1.1. *Partially commutative diagrams: $\textcircled{\{o\}}$.* — In a computation it is useful to consider partially commutative diagrams and we extend our lifting property notation accordingly. Given a diagram, and a letter o , possibly occurring in notation of one of the finite preorders (topological spaces), and an arrow $X \longrightarrow Y$ in the diagram, we label it by $\textcircled{\{o\}}$ as $X \xrightarrow{\textcircled{\{o\}}} Y$ to indicate that we only care about commutativity requirements with respect to elements denoted by o . In notation, we say that two paths $X = X_1 \xrightarrow{f_1} \dots \xrightarrow{f_{k-1}} X_k \xrightarrow[\textcircled{\{o\}}]{f_k} X_{k+1} \xrightarrow{f_{k+1}} X_{k+2} \xrightarrow{f_{k+3}} \dots \xrightarrow{f_{k'}} X_{k'+1} = Y$ and $Y = Y_1 \xrightarrow{g_1} \dots \xrightarrow{g_{l-1}} Y_l \xrightarrow{g_l} Y_{l+1} \xrightarrow{g_{l+1}} Y_{l+2} \xrightarrow{f_{l+3}} \dots \xrightarrow{f_{l'}} Y_{l'+1} = Y$ commute iff both $f_{k'}(f_{k'-1}(\dots f_1(o)\dots)) = g_{l'}(g_{l'-1}(\dots g_1(o)\dots))$ whenever X has a point denoted by o , and $f_{k'}(f_{k'-1}(\dots f_1(x)\dots)) = g_{l'}(g_{l'-1}(\dots g_1(x)\dots)) = o$ whenever Y has a point denoted by o and $f_{k'}(f_{k'-1}(\dots f_1(x)\dots)) = o$.

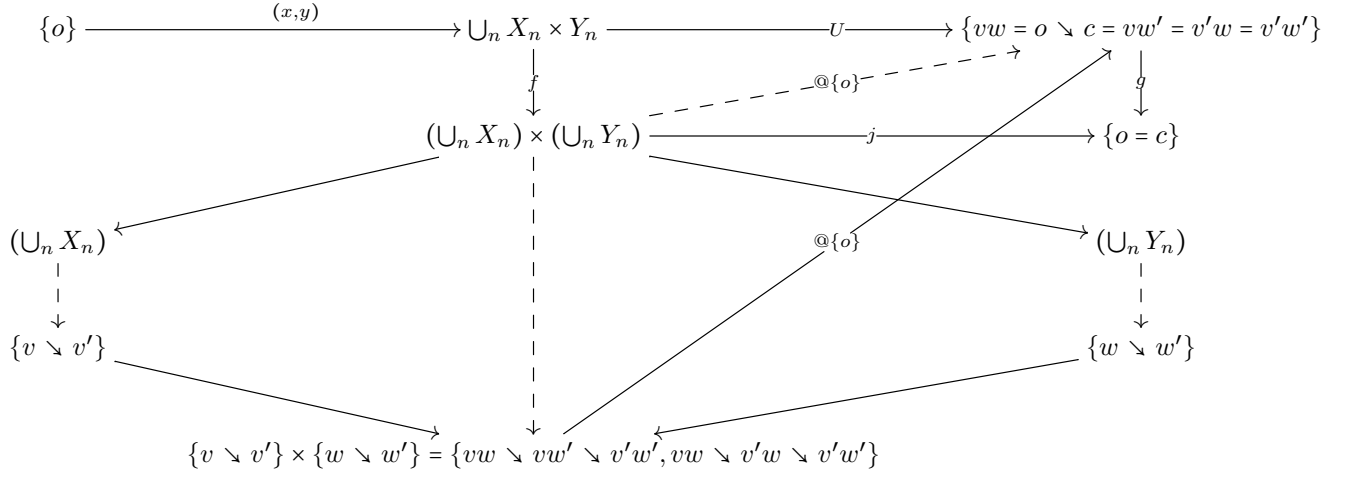
4.2. **Expanding the colimits.** — The first step in the proof is to “expand” the colimits and get a diagram without colimits. (1) We need to show that the topology on $\bigcup_n X_n \times Y_n$ is induced from $(\bigcup_n X_n) \times (\bigcup_n Y_n)$, i.e. the lifting property

$$\bigcup_n X_n \times Y_n \longrightarrow \left(\bigcup_n X_n\right) \times \left(\bigcup_n Y_n\right) \times \{o \rightarrow c\} \dashrightarrow \{o = c\}$$

(2) Being open and being commutative is defined pointwise, hence it is enough to construct the following diagram for each point $\{o\} \xrightarrow{(x,y)} \bigcup_n X_n \times Y_n$

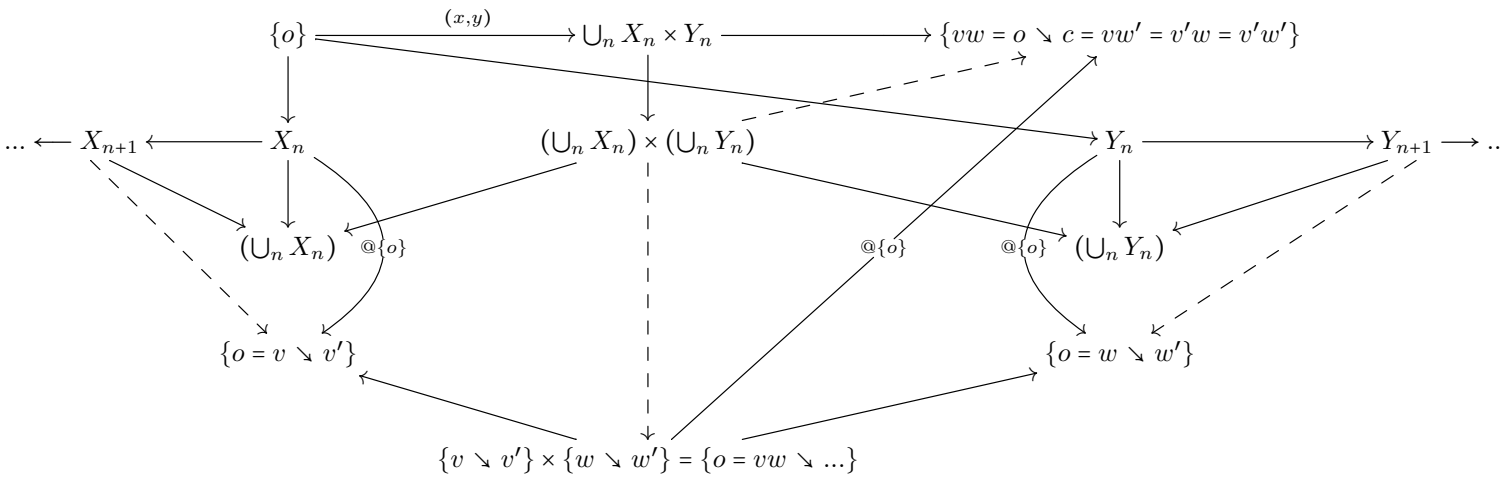
$$\begin{array}{ccc} \{o\} & \xrightarrow{(x,y)} \bigcup_n X_n \times Y_n & \longrightarrow \{o \searrow c\} \\ & \downarrow & \dashrightarrow \textcircled{\{o\}} \nearrow \\ & (\bigcup_n X_n) \times (\bigcup_n Y_n) & \longrightarrow \{o = c\} \end{array}$$

(3) Expanding the definition of product topology we see it is enough to construct the following diagram

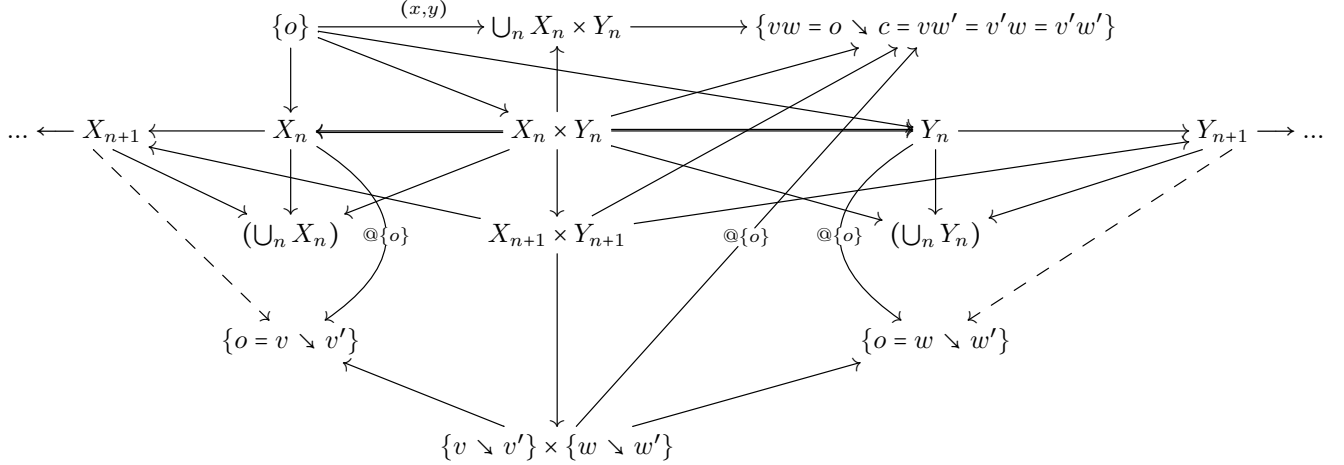


(4) To construct an arrow from the union/colimit one needs to construct compatible arrows from each X_n and Y_n . In a diagram chasing computation, we may do so by showing the inductive step that given an arrow from X_n , you can always add an arrow from X_{n+1} fitting into the same diagram, and the same for the Y 's. In fact we may label the arrows from X_n and Y_n by $@\{o\}$: in natural language this means we are constructing an increasing sequence of open subsets. We may assume $(x, y) \in X_n \times Y_n$.

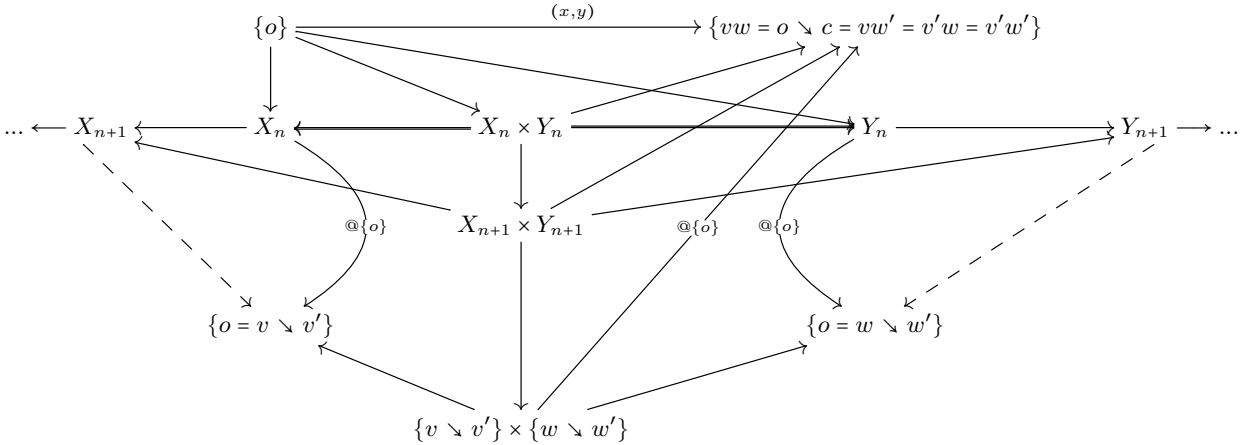
It is sufficient to construct the following diagram.



(4) We now add to the diagram the products $X_n \times Y_n$ and $X_{n+1} \times Y_{n+1}$, and remove the product of colimits.



(6) Now remove more vertices no longer needed:



4.2.1. *Preliminary lemmas.* — Recall that a topological space X is normal (T4) if any two disjoint closed subsets of X are separated by neighbourhoods, or, equivalently by Urysohn lemma, by a continuous function to \mathbb{R} .

Recall that a topological space X is regular (T3) if, given any point $x \in X$ and closed set B in X such that x does not belong to B , they are separated by neighbourhoods, or, equivalently, by closed neighbourhoods.

We do use the next Lemma and give it only for context.

Lemma 4.2.1. — *A topological space X is normal (T4) iff either of the following equivalent conditions holds:*

- $\emptyset \longrightarrow X \times \{a \leftarrow v \rightarrow x \leftarrow w \rightarrow b\} \longrightarrow \{a \leftarrow v = x = w \rightarrow b\}$

$$- \emptyset \longrightarrow X \times \{a \leftarrow v \rightarrow v' \leftarrow x \rightarrow w' \leftarrow w \rightarrow b\} \longrightarrow \{a \leftarrow v = v' = x = w' = w \rightarrow b\}$$

A topological space X is regular (T3) iff $\{v\} \longrightarrow X \times \{v \rightarrow a \leftarrow w \rightarrow b\} \longrightarrow \{v = a = w \rightarrow b\}$

Proof. — The preimages of a and b are disjoint closed subsets of X ; the preimages of $\{a \leftarrow v\}$ and $\{w \rightarrow b\}$ are open neighbourhoods separating A and B .

Let us consider the only interesting case is when v maps to v in $\{v = a = w \rightarrow b\}$ by the top horizontal arrow. The image of v is a point $v \in X$; the preimage of b is a closed subset $B \ni v$ of X not containing v , or, equivalently, the preimage of $v = a = w$ is an open neighbourhood of v . The preimage V of $v \in \{v \rightarrow a \leftarrow w \rightarrow b\}$ is an open neighbourhood of v in X disjoint from B . \square

Recall that a neighbourhood of a point is a set containing an open subset containing the point.

Lemma 4.2.2. — In a compact Hausdorff space, a neighbourhood of a point contains a closed neighbourhood of the point, and this is expressed by the following lifting property:

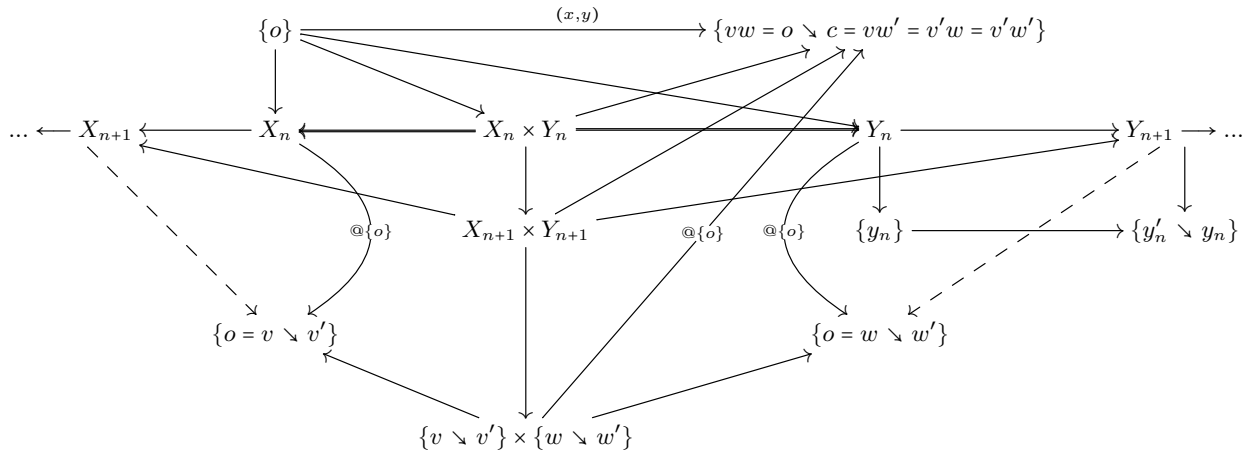
$$- \{o\} \longrightarrow X \text{ / } _ \{o \rightarrow a \leftarrow u \rightarrow c\} \longrightarrow \{o = a = u \rightarrow c\}$$

Proof. — Indeed, the lifting property holds trivially if the top horizontal arrow maps o into u or a , so consider the case it maps to o . The preimage of $o = a = u$ is an open neighbourhood U of $o \in X$. The preimage of $o \rightarrow a$ by the diagonal arrow is a closed neighbourhood of o contained in U . \square

Lemma 4.2.3. — A map $X \longrightarrow Y$ is a closed inclusion iff it can be obtained as a basechange from $\{c\} \longrightarrow \{o \rightarrow c\}$ along some map $Y \longrightarrow \{o \rightarrow c\}$.

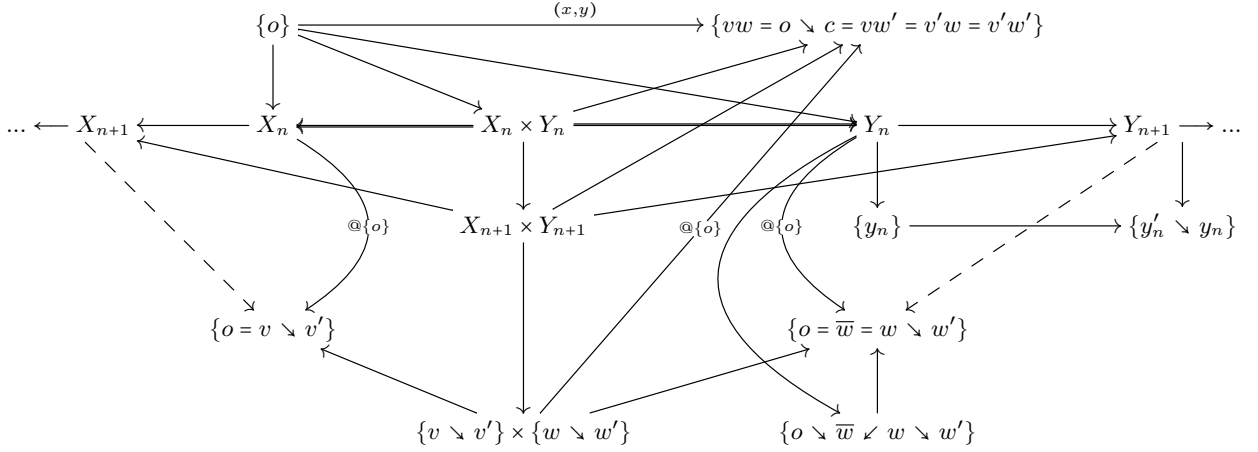
Proof. — Obvious. \square

4.2.2. *Diagram chasing proof continued.* — (7) Use that Y_n is a closed subset of Y_{n+1} by representing it as a pullback of y_n in $\{y'_n \rightrightarrows y_n\}$.

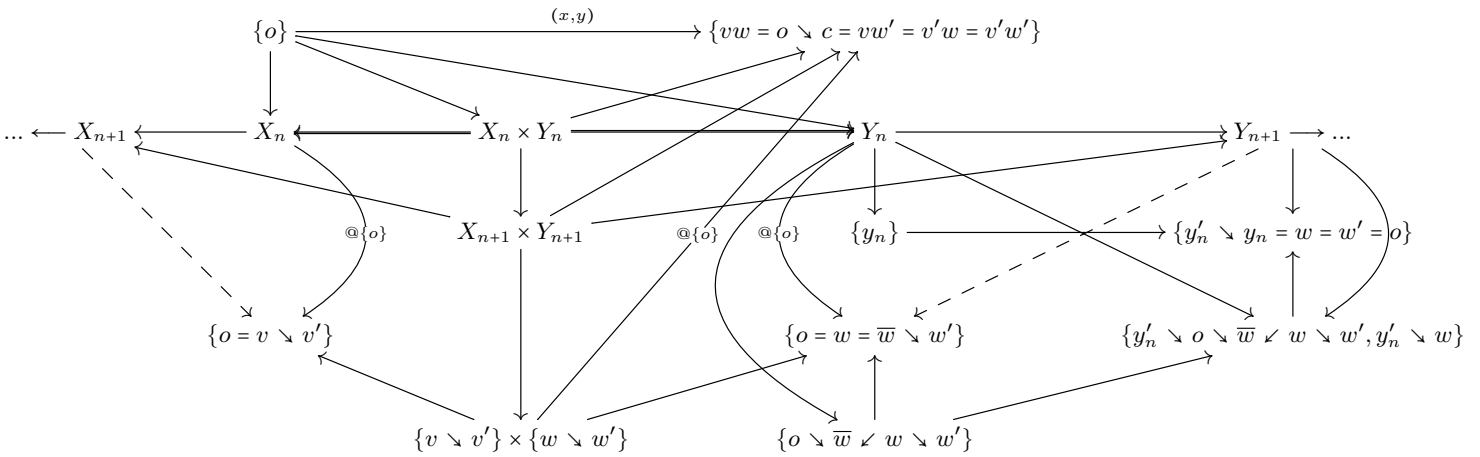


(8) By normality of Y_n construct $Y_n \longrightarrow \{o \rightrightarrows \bar{w} \leftarrow w \rightrightarrows w'\}$, i.e. an closed subset $y \in \bar{W}_n \subset W_n$ containing an open neighbourhood (the preimage of o) of y . Here the

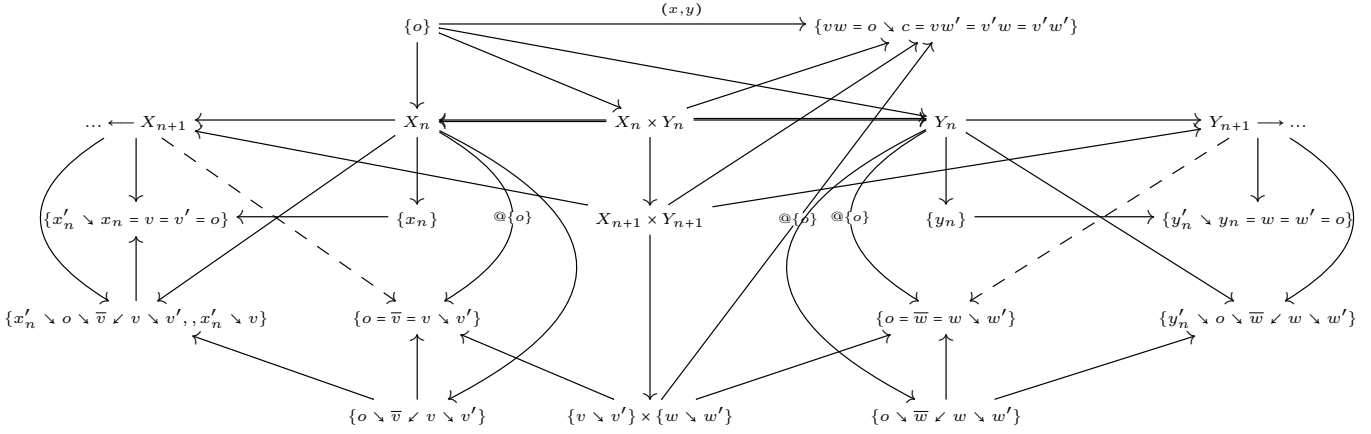
subset W_n (open in Y_n) is the preimage of $\{o \searrow \bar{w} \swarrow w\}$, and the closed subset \bar{W}_n is the preimage of $\{o \searrow \bar{w}\}$.



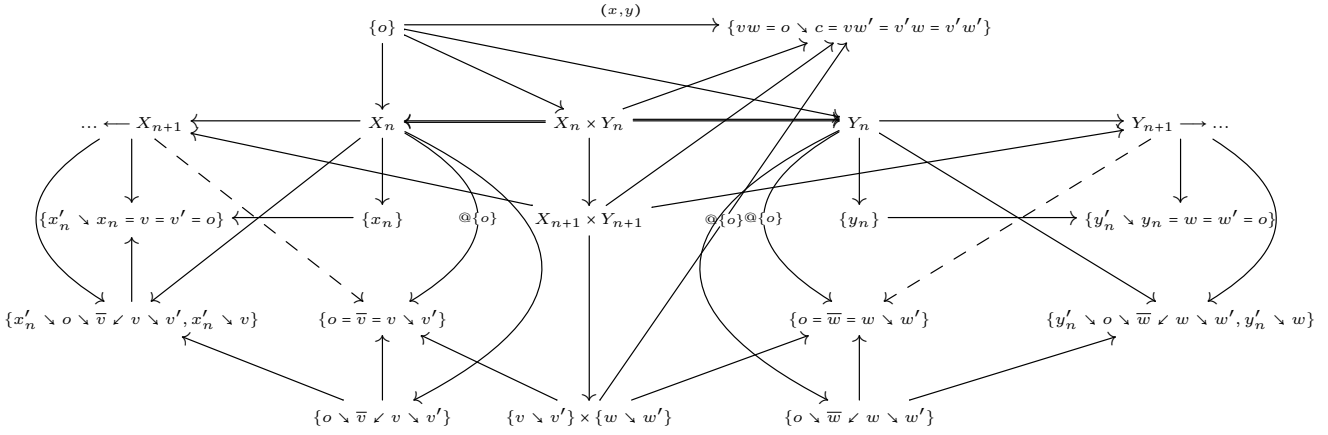
(9) Use that Y_n is a closed subset to construct a map $Y_{n+1} \rightarrow \{y'_n \searrow o \searrow \bar{w} \swarrow w \searrow w', y'_n \searrow w\}$, i.e. consider $\bar{W}_n \subset W_n$ as subsets of Y_{n+1} ; here $Y_{n+1} \setminus Y_n$ is the preimage of y'_n .



(10) By symmetry do the same for X_{n+1} , i.e. find a closed subset $x \in \bar{W}_n \subset W_n$ of X_n containing an open neighbourhood of x , and consider it as a subset of X_{n+1} .

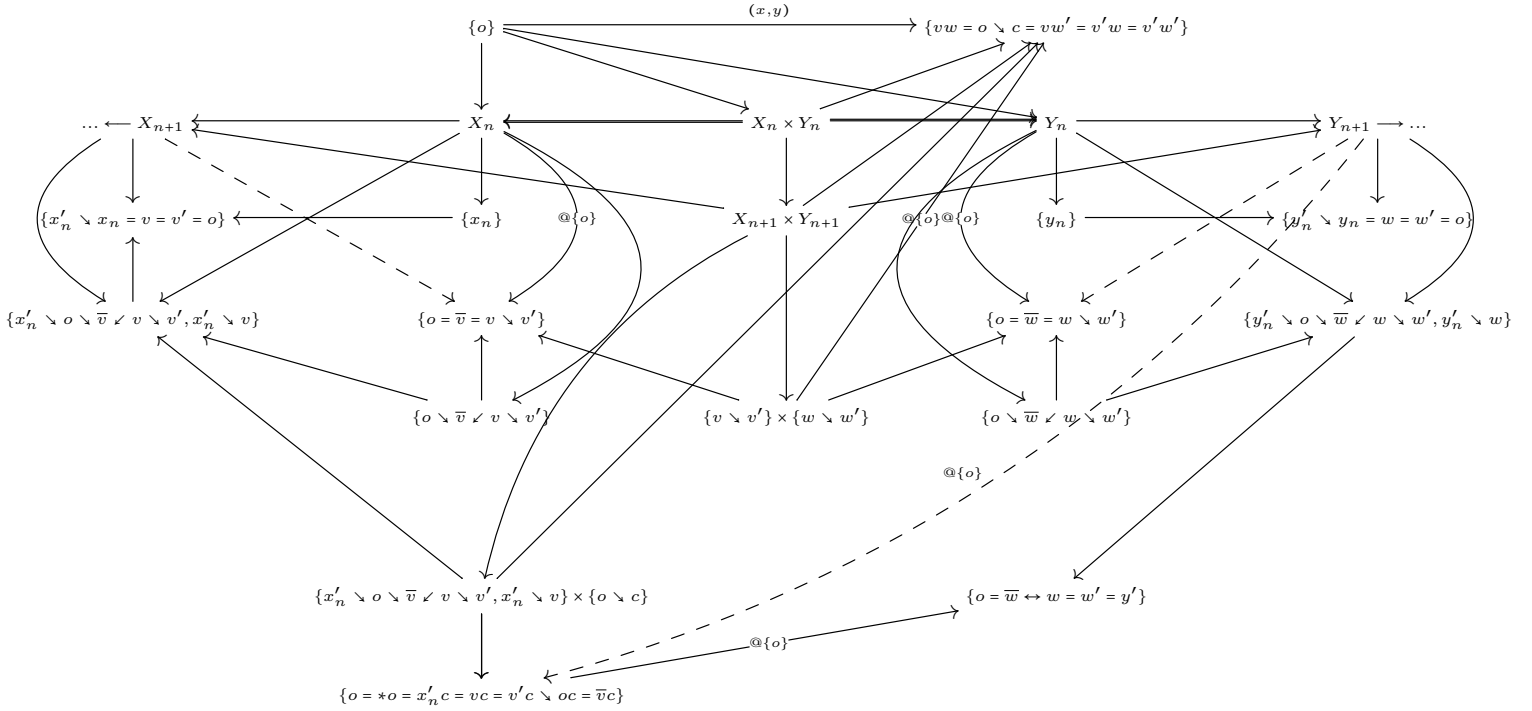


(11) Finally, we constructed $(x, y) \in V_n \times W_n \subset \bar{V}_n \times \bar{W}_n \subset X_n \times Y_n \subset X_{n+1} \times Y_{n+1}$ and $\bar{W}_n \times \bar{W}_n \subset U$ where V_n is open in X_n , and W_n is open in Y_n , and \bar{V}_n is closed in both X_n and X_{n+1} , and so is \bar{W}_n in Y_n and Y_{n+1} .



(12) Now we shall use that $X_{n+1} \times Y_{n+1} \rightarrow Y_{n+1}$ is closed. We have a closed subset $\bar{W}_n \subset X_{n+1}$, and an open subset of $U \subset X_{n+1} \times Y_{n+1}$, and we know that $\bar{V}_n \times \bar{W}_n \subset U$. Using that the projection $X_{n+1} \times Y_{n+1} \rightarrow Y_{n+1}$ is closed, we find an open subset $\bar{W}_n \subset W_{n+1} \subset Y_{n+1}$ (the preimage of o in $Y_{n+1} \rightarrow \{o = *o = x'_n c = vc = v'c \searrow oc = \bar{v}c\}$) such that $\bar{V}_n \times \bar{W}_{n+1} \subset U$. In this diagram, we use yet another extension of our notation: \textcircled{o} above the arrow means (in this picture) that it relates only to the

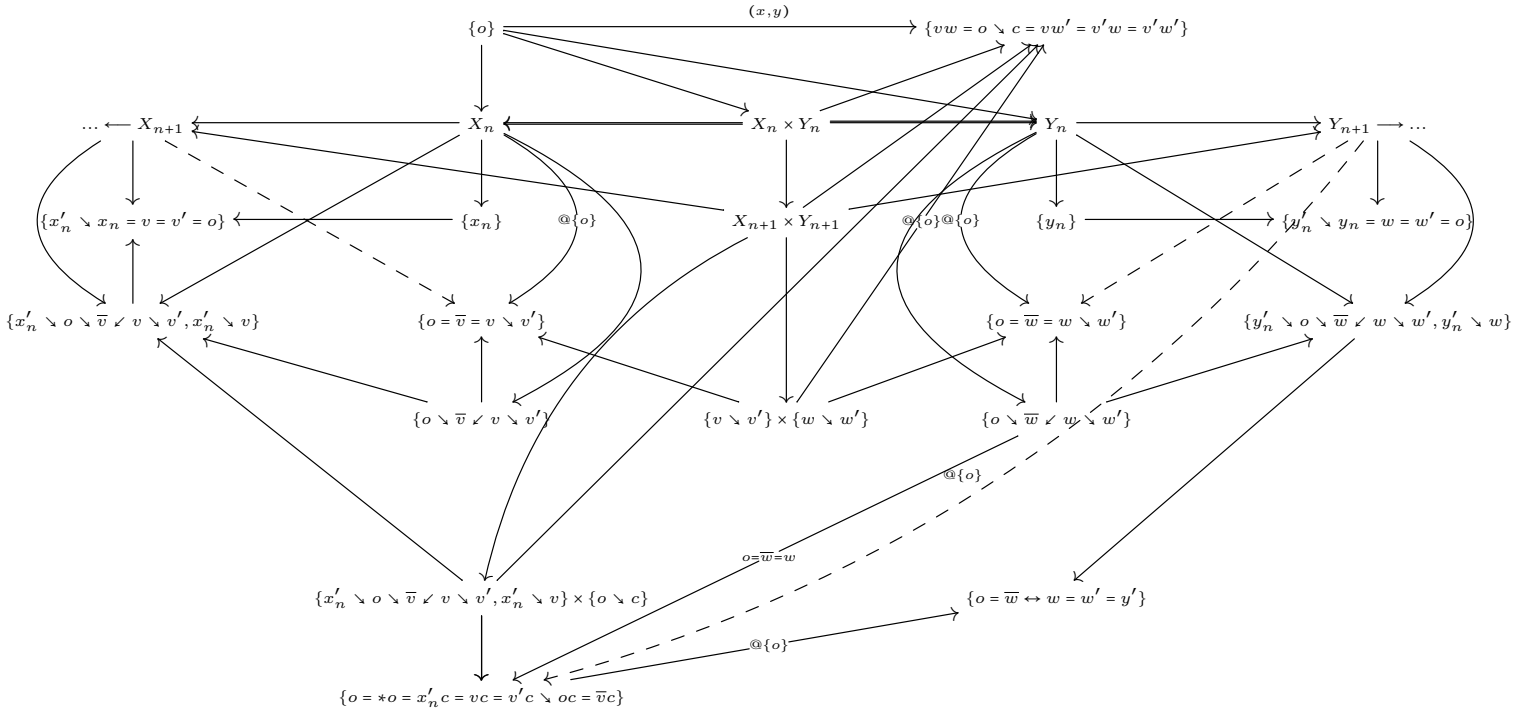
triangle above and not below.



(13) Unfortunately, the last step is somewhat vague, as it requires a careful handling of the inductive step, which we are not able to do. Neither do we carefully specify the commutativity conditions, unfortunately.

Finally, by normality of Y_{n+1} add an arrow $\{o = *o = x'_n c = vc = v'c \searrow oc = \bar{v}c\} \xrightarrow{@\{o\}} \{o \searrow \bar{w} \searrow w \searrow w'\}$. We now see that our calculation shows how to add an arrow $Y_{n+1} \xrightarrow{@\{o\}} \{o \searrow \bar{w} \searrow w \searrow w'\}$ given an arrow $Y_n \xrightarrow{@\{o\}} \{o \searrow \bar{w} \searrow w \searrow w'\}$, fitting the same diagram. By symmetry we may do the same for X_n and X_{n+1} , and this would complete the inductive step, and thereby the argument. Note that we modified the

inductive step assumption from (7).



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